Optimal Mechanism Design with Speculation and Resale

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Abstract

This paper examines the optimal auction design problem with one regular buyer and one speculator, when inter-buyer resale can not be prohibited. The resale is a stochastic ultimatum bargaining game. In the optimal mechanism, the winner in the initial market, if any, is always the speculator and the seller reveals no information to the resale market. In general, this optimal mechanism generates weakly less revenue than in the case resale is not permitted. However, the revenue is the same when the winner has all the bargaining power in the resale market. When the loser has all the bargaining power in the resale market, an efficient outcome is optimal. The chance of resale makes the seller sometimes hold back the object, which is never optimal in our model if the seller can prohibit resale.

Key words: Auctions, Mechanism Design, Resale

JEL classification: C72, D44, D82, D83, L12

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1 Introduction

When a seller has full control power and can prohibit inter-buyer resale, Myerson (1981) characterizes the optimal selling mechanism. The buyer with the highest non-negative virtual valuation should be awarded the object.[1] In many instances, however, resale is permitted or can not be banned. When inter-buyer resale is possible, Myerson's revenue may not be achievable all the time. This is indeed what happens in the model we set up in this paper.

Most of the current literature considers standard auctions with the possibility of resale. Hafalir and Krishna (2008) [5] examine first and second price auctions with resale. Either the winner or the loser in an auction has the possibility to offer a take-it-or-leave-it offer. Revenue rankings among standard auctions are discussed. Cheng and Tan (2008) [3] study asymmetric common-value auctions and apply the results to the revenue ranking in independent value auctions with resale. They generalize Hafalir and Krishna (2008) [5] by dropping the regularity assumption. Garratt and Troger (2006)[4] consider first and second price auctions with many symmetric bidders and an additional speculator whose valuation is commonly known as zero. In the resale market, the winner proposes a mechanism to the losers. In those papers, resale arises due to the inefficiencies in auctions. Haile (2003) [6] considers a different source of resale. After auctions, there is new information flow which affects buyers' valuations. He considers standard auctions with the winner proposing a selling mechanism to the losers.

However, in the above papers, one important question is how should the seller control the information revealed to the resale market. The common assumption in the literature is that only the transaction price (highest bid in first-price auctions and second highest bid in second-price auctions) is announced. Whether this is optimal for the seller to do is an issue. Potentially, the seller has many different options on the announcement rules: concealing all the information, revealing all the information, revealing the information stochastically, etc. It is quite difficult to formulate all the possible rules and find the optimal one. Even if we can find the optimal announcement rule, a new question is whether using a first or second price auction is still optimal.

To answer these questions, we need to take another approach and examine the optimal mechanism design. However, the difficulty here is that when the seller can not control buyers' actions in the resale market, the seller is going to design a mechanism with hidden information, hidden actions and multiple agents. Although Myerson (1982) [8] proves the revelation principle and formulates the problem under this general setting, solving the problem is nontrivial. The only paper has all these three features simultaneously is McAfee and McMillan (1991) [7] who examine the optimal team incentives.

In this paper, we will adopt the setup in Garratt and Troger (2006) [4] and examine the optimal mechanism design with one regular buyer and one speculator. The regular buyer has private information about his valuation and the speculator has a commonly known valuation. In the initial market, the seller can choose any mechanism to sell the object. In the resale market, the buyers engage in a stochastic ultimatum bargaining game.

[1] In this paper, we normalize the seller's reservation valuation to zero.

We find that in the optimal mechanism, the seller allocates the object to the speculator if his resale augmented virtual valuation is higher than the seller's reservation value, which is normalized to zero. The seller never allocates the object to the regular buyer directly in the initial market. The seller charges the speculator all the possible revenue to be raised in the resale market. Although the seller does not allocate the object to the regular buyer directly, she also charges him the amount equal to the gain in the resale market of a regular buyer with the lowest valuation. In addition, the seller reveals no information to the resale market. Therefore, in the resale market, buyers only know how the object is allocated in the initial market. In general, this optimal mechanism generates weakly less revenue than Myerson's revenue. However, the exact revenue can be achieved when the winner has all the bargaining power in the resale market. When the loser has all the bargaining power in the resale market, the efficient allocation is optimal. Furthermore, in Myerson's allocation, it is never the seller's interest to hold back the object in our model, since the speculator's virtual valuation is always positive. In contrast, when resale is possible, it is optimal for the seller to hold back the object if the regular buyer's valuation is low but still higher than the speculator's valuation.

The optimality of this mechanism is easier to understand when the winner has all the bargaining power in the resale market and the regular's virtual valuation is always greater than zero. Under this situation, the seller always allocates the object to the speculator. Since no information is revealed to the resale market, the speculator's belief about the regular buyer's valuation remains unchanged. Therefore, the speculator offers a price at which his valuation is equal to the regular buyer's virtual valuation to the regular buyer in the resale market. As a result, the final allocation coincides with Myerson's allocation. The revenue equivalent theorem then implies that our mechanism achieves Myerson's revenue. Since when she cannot prohibit resale, the seller can not generate more revenue than Myerson's revenue, the mechanism described above is optimal.

The rest of the paper is organized as follows. In Section 2, we present the literature review. In Section 3, we describe the model and characterize the optimal mechanism. In Section 4, we conclude.

2 Related Literature

In the optimal auction design literature, Ausubel and Cramton (1999) [1] characterize the optimal mechanism when the resale market is perfect in the sense that any inefficiency will be corrected in the resale market. It is not surprising that the seller should induce an efficient allocation directly in the primary market. Although this is an interesting benchmark, it is not clear how a perfect resale market can be achieved.[2] Our paper actually provides a situation supporting the efficient mechanism as the optimal mechanism. In this paper, when the loser has all the bargaining power in the resale market, it is optimal to induce an efficient outcome in the initial market.

Zheng (2002) [10] provides the condition under which Myerson's revenue can be achieved when inter-bidder resale is allowed. His method is an indirect one. He constructs an auction with resale

[2] Myerson and Satterthwaite (1983) [9] actually show that inefficiency can not necessarily be corrected in the secondary market.

to achieve Myerson's revenue for the seller. Since the seller can not achieve more revenue when she could not prohibit resale than that when she could, Myerson's revenue is an upper bound revenue when she can not prohibit resale. As a result, the auction with resale he constructs is optimal when the seller can not prohibit resale. However, his method works only when Myerson's revenue can be achieved. In his paper, it is required that in the resale market the winner has all the control and has the ability to select the optimal mechanism. However, in some instances, a loser may also have some bargaining power. We show that as long the winner does not have perfect bargaining power, Myerson's revenue is not achievable when the seller can not prohibit resale. In this case, Zheng's indirect method cannot be applied directly.

In contrast, our approach is a direct method. By formulating all possible mechanisms, we can characterize the optimal mechanism even when the Myerson's revenue can not be achieved. Calzolari and Pavan (2006) [2]consider optimal mechanism design with resale with two types. They assess that "the generalization to the continuum poses nontrivial problems". Without solving for the optimal mechanism, they assert that any deterministic mechanism can not be optimal.[3]

Garratt and Troger (2006) [4] characterize the equilibria in standard auctions in which the speculator plays an active role, i.e., he wins with positive probability. In this paper, we find that in the optimal mechanism the role of the speculator is not only active but also essential. The winner should always be the winner if any.

3 The model

One seller (she) with one indivisible object faces two buyers. Buyer 1 has commonly known valuation $v_1 \geq 0$, and is called the speculator in this paper. Buyer 2's valuation v_2 is his private information, with the *c.d.f* $F(\cdot)$ and the associate *p.d.f* $f(\cdot)$ on the support $[a, b]$. We call buyer 1 the speculator and buyer 2 the regular buyer.[4] We maintain the common assumption in the literature that the hazard ratio $\frac{f(v_2)}{1-F(v_2)}$ is increasing to simplify the characterization of the optimal mechanism. We normalize the reservation value of the seller to zero.

When the seller has all the controlling power and can prohibit inter-bidder resale, Myerson's optimal auction yields the most revenue for the seller. Let $J_1(v_1) = v_1$ and $J_2(v_2) = v_2 - \frac{1-F(v_2)}{f(v_2)}$ denote buyer 1 and buyer 2's virtual valuations, respectively. Note that buyer 1's valuation is also his virtual valuation. In Myerson's auction, the seller should allocate the object to the buyer with the higher nonnegative virtual valuation. If buyer 2's virtual valuation is always greater than buyer 1's virtual valuation, i.e. $J_2(a) \geq v_1$, then the seller allocates the object to buyer 2 all the time. If buyer 2's virtual valuation is always less than buyer 1's valuation, i.e., $J_2(b) \leq v_1$, the seller allocates the object to buyer 1 all the time. In those two cases, the outcomes are efficient and resale will not take place even if allowed, and Myerson's optimal revenue can be achieved. Since the seller can not generate more revenue than Myerson's revenue when she can not prohibit inter-buyer resale, implementing Myerson's allocation directly is optimal. It is of interest to note

[3]In the online appendix, they provide the characterization of the optimal mechanism for two polar cases: the winner has full bargaining power and the loser has full bargaining power.

[4]In Garratt and Troger (2006), the speculator's valuation is equal to zero, which is a special case of our setup.

that, since the speculator's virtual valuation is greater than zero, in Myerson's auction, the seller has no interest to hold back the object.

To exclude these two cases, we assume that buyer 1's valuation lies in the range of buyer 2's virtual valuation, i.e., $J_2(a) < v_1 < J_2(b)$. As a result, whether the seller can prohibit inter-buyer resale or not matters, since Myerson's allocation is not efficient. Our goal is to find the optimal mechanism in this case.

There are two markets in the model: the initial market and the resale market. In the initial market, the seller can use any mechanism to sell the object. She can decide how the object is allocated and how monetary transfers are made. In the resale market, although the seller can not control buyers behavior, she can reveal information to the resale market. Any communication is allowed and costless. The resale is a stochastic ultimatum bargaining game. With probability λ, the winner (i.e. the buyer who is allocated the object in the initial market), if any, proposes a take-it-or-leave-it offer; and with probability $1-\lambda$, the loser (i.e. the buyer who is not allocated the object in the initial market), if any, proposes a take-it-or-leave-it offer.[5]

Given that resale can happen in the resale market, the seller has to design a mechanism in the initial market to maximize her revenue. Since buyers' behaviors in the resale market is not contractible, we are dealing with mechanism design problem with hidden information, hidden actions and multiple agents. Many difficulty arise since not only the seller has to decide how to allocate the object and how to make monetary transfers, but also she has to control what information she wants to reveal to the resale market.

Thanks to the revelation principle in Myerson (1982) [8], we can restrict our search of the optimal mechanism to the following direct mechanisms. In direct mechanisms, buyer 2 first announces his valuation $\tilde{v}_2$ confidentially to the seller, then with probability $x_1(\tilde{v}_2)$ and $x_2(\tilde{v}_2)$, the object is allocated to buyer 1 and buyer 2, respectively.[6] In addition, the seller needs to send private recommendations to buyers in the resale market conditional on $\tilde{v}_2$. There are 4 different situations in the resale market.

Case 1: buyer 1 is the winner and buyer 1 makes the take-it-or-leave-it offer. The seller needs to recommend buyer 1's price offer $p_1(\tilde{v}_2) \in \mathcal{R}$, and buyer 2's acceptance decision conditional on buyer 1's price offer $A_1(\tilde{v}_2; p_1) \in \{Accept,\ Reject\}$. Note that the seller has to send buyer 2 acceptance recommendation for any realized price offer by buyer 1.

Case 2: buyer 1 is the winner and buyer 2 makes the take-it-or-leave-it offer. The seller needs to recommend buyer 2's price offer $p_2(\tilde{v}_2) \in \mathcal{R}$ and buyer 1's acceptance decision conditional on buyer 2's price offer $A_2(\tilde{v}_2; p_2) \in \{Accept,\ Reject\}$.

Case 3: buyer 2 is the winner and buyer 2 makes the take-it-or-leave-it offer. The seller needs to recommend buyer 2's price offer $p_3(\tilde{v}_2) \in \mathcal{R}$ and buyer 1's acceptance decision conditional on buyer 2's price offer $A_3(\tilde{v}_2; p_3) \in \{Accept,\ Reject\}$.

Case 4: buyer 2 is the winner and buyer 1 makes the take-it-or-leave-it offer. The seller needs

[5]When the seller holds back the object, there is no winner or loser.

[6]Since buyer 1 does not have private information, he does not need to report.

to recommend buyer 1's price offer $p_4(\tilde{v}_2) \in \mathcal{R}$ and buyer 2's acceptance decision conditional on buyer 1's price offer $A_4(\tilde{v}_2; p_4) \in \{Accept,\ Reject\}$.

When buyer 1 wins, buyer 1 privately learns of the recommendations $p_1(\tilde{v}_2)$ and $A_2(\tilde{v}_2; p_2)$, and buyer 2 privately learns of the recommendations $A_1(\tilde{v}_2; p_1)$ and $p_2(\tilde{v}_2)$ prior to the resale market. When buyer 2 wins, buyer 1 privately learns of the recommendations $A_3(\tilde{v}_2; p_3)$ and $p_4(\tilde{v}_2)$, and buyer 2 privately learns of the recommendations $p_3(\tilde{v}_2)$ and $A_2(\tilde{v}_2; p_4)$ prior to the resale market. The recommendations are functions of $\tilde{v}_2$. Thus, upon receiving the recommendations, buyer 1 should update his belief about buyer 2's valuation, since the recommendations contain information about buyer 2's private information in equilibrium.

To prohibit any information conveyed by the monetary transfers, $t_1(\tilde{v}_2)$ and $t_2(\tilde{v}_2)$ for buyer 1 and 2, respectively, are made privately and should be collected at the very end. However, if a buyer's monetary transfer does not depend on the other buyer's valuation, it does not matter whether the monetary transfer is made at the very end or before the resale market. The mechanism needs to satisfy the incentive compatibility constraints and participation constraints in both markets. In addition, since there is only one object to allocated, the allocation probabilities must satisfy

$$x_1(v_2) + x_2(v_2) \le 1,\ \forall v_2. \tag{1}$$

Since only buyer 2 has private information, incentive compatibility constraints imply the following. For buyer 1, given that buyer 2 is honest in the initial market and obedient in the resale market, he will be obedient in the resale market. For buyer 2, given that buyer 1 is obedient in the resale market, he has no incentive to manipulate his report in the initial market or disobey the seller's recommendations in the resale market. Buyer 2's incentive compatibility constraints contain two parts given the set up in the model. First, if buyer 2 has truthfully reported his valuation in the initial market, it is optimal for him to follow the seller's recommendation in the resale market. Second, buyer 2 will truthfully report his valuation in the initial market.

Participation constraints imply that participation in the mechanism is better than the outside option in the beginning, which is normalized to zero. We will examine the incentive compatibility constraints, starting from the resale stage.

3.1 Resale market

For the recommendations to be incentive compatible, the acceptance recommendation should be:

$$A_1(v_2; p_1) = \begin{cases} Accept & if\ p_1 \le v_2 \\ Reject & if\ p_1 > v_2 \end{cases} \tag{2}$$

$$A_2(v_2; p_2) = \begin{cases} Accept & if\ p_2 \ge v_1 \\ Reject & if\ p_2 < v_1 \end{cases} \tag{3}$$

$$A_3(v_2; p_3) = \begin{cases} Accept & if\ p_3 \leq v_1 \\ Reject & if\ p_3 > v_1 \end{cases} \tag{4}$$

$$A_4(v_2; p_4) = \begin{cases} Accept & if\ p_4 \geq v_2 \\ Reject & if\ p_4 < v_2 \end{cases} \tag{5}$$

The observation is that buyer 1's acceptance recommendations $A_2(v_2; p_1)$ and $A_3(v_2; p_1)$ do not depend on v_2. Thus, upon seeing those acceptance recommendations, buyer 1's belief about buyer 2's valuation does not change. The price offers are much more complicated, and we examine them case by case.

Case 1: buyer 1 is the winner and he makes the take-it-or-leave-it offer. Buyer 1 believes that buyer 2 is honest and obedient, i.e., $\tilde{v}_2 = v_2$ and

$$A_1(v_2; p_1) = \begin{cases} Accept & if\ p_1 \leq v_2 \\ Reject & if\ p_1 > v_2 \end{cases} . \tag{6}$$

The information buyer 1 has when deciding the price to offer is that he knows that he has won and the recommendations $p_1(v_2) = p_1^*$ and $A_2(v_2; p_2) = A_2^*(p_2)$. Thus, his problem is:

$$\max_{\tilde{p}} \quad v_1 Prob\{v_2 < \tilde{p} | buyer\ 1\ wins,\ p_1(v_2) = p_1^*\ and\ A_2(v_2; p_2) = A_2^*(p_2)\} \tag{7}$$

$$+ \quad \tilde{p} Prob\{v_2 \geq \tilde{p} | buyer\ 1\ wins,\ p_1(v_2) = p_1^*\ and\ A_2(v_2; p_2) = A_2^*(p_2)\} \tag{8}$$

$$= \max_{\tilde{p}} \quad v_1 Prob\{v_2 < \tilde{p} | buyer\ 1\ wins,\ p_1(v_2) = p_1^*\} \tag{9}$$

$$+ \quad \tilde{p} Prob\{v_2 \geq \tilde{p} | buyer\ 1\ wins,\ p_1(v_2) = p_1^*\} \tag{10}$$

The equality follows from the fact that $A_2(v_2; p_2)$ does not depend on v_2. Let

$$G_1(v_2) = F(v_2 | buyer\ 1\ wins, p_1(v_2) = p_1^*), \tag{11}$$

and

$$g_1(v_2) = f(v_2 | buyer\ 1\ wins, p_1(v_2) = p_1^*). \tag{12}$$

Then the above maximization problem is equivalent to

$$\max_{\tilde{p}} \qquad \Pi_1 = v_1 G_1(\tilde{p}) + \tilde{p}\,[1 - G_1(\tilde{p})] \tag{13}$$

We assume that the first order approach is valid, and then prove that it is indeed valid in the optimal mechanism. FOC gives us:

$$\frac{d\Pi_1}{d\tilde{p}} = \quad v_1 g_1(\tilde{p}) - \tilde{p} g_1(\tilde{p}) + [1 - G_1(\tilde{p})] = 0 \tag{14}$$

$$\Rightarrow \qquad v_1 = \tilde{p} - \tfrac{1-G_1(\tilde{p})}{g_1(\tilde{p})} \tag{15}$$

In equilibrium, buyer 1 should be obedient and offer the price $\tilde{p} = p_1^*$. This means the seller can only choose how to pool together his knowledge about buyer 2's valuation by sending same recommendations, and the value of the recommendations will be determined by the incentive compatible constraints in the resale market. For example, suppose the seller wants to fully pool buyer 2's valuation and the object is always allocated to buyer 1. In this case, $G_1(v_2) = F(v_2)$. From (13), we obtain $v_1 = \tilde{p} - \frac{1-F(\tilde{p})}{f(\tilde{p})}$. Denote the solution as p_1^*. Therefore, the recommendation should be $p_1(v_2) = p_1^*$. Note that buyer 1 will never offer a price less than his own valuation v_1. The induced outcome in equilibrium is that trade happens at price $p_1(v_2)$ if $p_1(v_2) \leq v_2$ and not happen if $p_1(v_2) > v_2$.

Case 2: buyer 1 is the winner and the loser, buyer 2, makes the take-it-or-leave-it offer. Buyer 2 believes that buyer 1 is obedient, i.e.,

$$A_2(v_2; p_2) = \begin{cases} Accept & if\ p_2 \geq v_1 \\ Reject & if\ p_2 < v_1 \end{cases} . \tag{16}$$

The information buyer 2 has when deciding the price to offer is that he knows that he has lost and the recommendations $p_2(v_2) = p_2^*$ and $A_1(v_2; p_1) = A_1^*(p_1)$. Since buyer 1 does not have any private information, the price offer is incentive compatible if and any if

$$p_2(v_2) = \begin{cases} v_1 & if\ v_2 \geq v_1 \\ any\ price\ lower\ than\ v_1 & if\ v_2 < v_1 \end{cases} . \tag{17}$$

The induced outcome is that buyer 1 sells the object to buyer 2 at price v_1 when $v_2 \geq v_1$, and buyer 1 keeps the object when $v_2 < v_1$.

Case 3: buyer 2 is the winner and the winner, buyer 2, makes the take-it-or-leave-it offer. Buyer 2 believes that buyer 1 is obedient, i.e.,

$$A_3(v_2; p_3) = \begin{cases} Accept & if\ p_3 \leq v_1 \\ Reject & if\ p_3 > v_1 \end{cases} . \tag{18}$$

The information buyer 2 has when deciding the price to offer is that he knows that he has won and the recommendations $p_3(v_2) = p_3^*$ and $A_4(v_2; p_4) = A_4^*(p_4)$. Since buyer 1 does not have any private information, the price offer is incentive compatible if and any if

$$p_3(v_2) = \begin{cases} v_1 & if\ v_2 \leq v_1 \\ any\ price\ higher\ than\ v_1 & if\ v_2 > v_1 \end{cases} \tag{19}$$

and buyer 1 takes the offer if the price is v_1 and rejects if it is higher than v_2. The induced outcome is that buyer 2 sells the object to buyer 1 at price v_1 if $v_2 \leq v_1$ and buyer 2 keeps the object if $v_2 > v_1$.

Case 4: buyer 2 is the winner and the loser, buyer 1, makes the take-it-or-leave-it offer. Buyer

1 believes that buyer 2 is honest and obedient, i.e., $\tilde{v}_2 = v_2$ and

$$A_4(v_2; p_4) = \begin{cases} Accept & if\ p_4 \geq v_2 \\ Reject & if\ p_4 < v_2 \end{cases} . \tag{20}$$

The information buyer 1 has when deciding the price to offer is that he knows that he has lost and the recommendations $p_4(v_2) = p_4^*$ and $A_3(v_2; p_3) = A_3^*(p_3)$. Thus, his problem is:

$$\max_{\tilde{p}} \quad (v_1 - \tilde{p}) Prob\{v_2 \leq \tilde{p} | buyer\ 2\ wins,\ p_4(v_2) = p_4^*,\ A_3(v_2; p_3) = A_3^*(p_3)\} \tag{21}$$

$$\Rightarrow \qquad \max_{\tilde{p}} \quad (v_1 - \tilde{p}) Prob\{v_2 \leq \tilde{p} | buyer\ 2\ wins,\ p_4(v_2) = p_4^*\} \tag{22}$$

The equality follows from the fact that $A_3(v_2; p_3)$ does not depend on v_2. Let

$$G_4(v_2) = F(v_2 | buyer\ 2\ wins, p_4(v_2) = p_4^*), \tag{23}$$

and

$$g_4(v_2) = f(v_2 | buyer\ 2\ wins, p_4(v_2) = p_4^*). \tag{24}$$

Then the above maximization problem is equivalent to

$$\max_{\tilde{p}} \quad \Pi_4 = (v_1 - \tilde{p}) G_4(\tilde{p}) \tag{25}$$

We assume that the first order approach is valid, and then prove that it is indeed valid in the optimal mechanism. FOC gives us:

$$\frac{d\Pi_4}{d\tilde{p}} = -G_4(\tilde{p}) + (v_1 - \tilde{p}) g_4(\tilde{p}) = 0 \tag{26}$$

$$\Rightarrow v_1 = \tilde{p} + \frac{G_4(\tilde{p})}{g_4(\tilde{p})} \tag{27}$$

From here, we can solve for the optimal reaction $\tilde{p}$ as a function of the recommendation. In equilibrium, buyer 1 should be obedient and offer the price $p_4(v_2) = p_4^*$.

In order to examine the incentive compatibility constraints in the initial market, we also need to know when buyer 2 lies about his valuation as $\tilde{v}_2$, how would happen in the resale market.

Case 1: Buyer 2 know that buyer 1 will be obedient to offer price $p_1(\tilde{v}_2)$. Thus, Buyer 2 accepts if $p_1(\tilde{v}_2) \leq v_2$, and reject if $p_1(\tilde{v}_2) > v_2$.

Case 2: Buyer 2 will offer price v_1 if $v_2 \geq v_1$, and offer any price lower than v_1 if $v_2 < v_1$.

Case 3: Buyer 2 will offer price v_1 if $v_2 \leq v_1$, and offer any price higher than v_1 if $v_2 > v_1$.

Case 4, Buyer 2 know that buyer 1 will be obedient to offer price $p_4(\tilde{v}_2)$. Thus, buyer 2 accepts if $p_4(\tilde{v}_2) \leq v_2$, and reject if $p_4(\tilde{v}_2) > v_2$.

3.2 Initial market

The above section characterizes the recommendations that are incentive compatible in the resale market. We will substitute those recommendations in the calculations below. However, we will leave $p_1(v_2)$ and $p_4(v_2)$ implicitly, since we do not have explicit solutions for them. The above section also illustrates what would happen in the resale market if buyer 2 lies about his valuation in the initial market.

Given buyer 2 is honest and obedient, buyer 1's payoff if he acts optimally in the resale market is given by:

$$U_1 = \int_a^b \Big(x_1(v_2) \Big\{ \lambda \Big[v_1 I_{\{v_2 < p_1(v_2)\}} + p_1(v_2) I_{\{v_2 \geq p_1(v_2)\}} \Big] + (1-\lambda) v_1 \Big\} \tag{28}$$

$$+x_2(v_2) \Big\{ (1-\lambda)(v_1 - p_4(v_2)) I_{\{v_2 \leq p_4(v_2)\}} \Big\} - t_1(v_2) \Big) dF(v_2) \tag{29}$$

The formula follows directly from the outcomes in the four cases in the resale market. Since buyer 1 does not have private information, there is no incentive compatibility constraint for him in the initial market. We normalize the payoff of not participating to zero. Therefore, buyer 1's participating constraint implies that $U_1 \geq 0$. It should be binding, otherwise, the seller can get higher revenue by increase $t_1(v_2)$ only. Thus,

$$\int_a^b t_1(v_2) dF(v_2) = \int_a^b \Big(x_1(v_2) \Big\{ \lambda \Big[v_1 I_{\{v_2 < p_1(v_2)\}} + p_1(v_2) I_{\{v_2 \geq p_1(v_2)\}} \Big] + (1-\lambda) v_1 \Big\} \tag{30}$$

$$+x_2(v_2) \Big\{ (1-\lambda)(v_1 - p_4(v_2)) I_{\{v_2 \leq p_4(v_2)\}} \Big\} \Big) dF(v_2) \tag{31}$$

Given that buyer 1 is obedient, buyer 2's payoff if he reports valuation $\tilde{v}_2$ and acts optimally in the resale market is given by:

$$U_2(v_2, \tilde{v}_2) \tag{32}$$

$$= x_1(\tilde{v}_2) \Big\{ \lambda \left[v_2 - p_1(\tilde{v}_2) \right] I_{\{v_2 \geq p_1(\tilde{v}_2)\}} + (1-\lambda)(v_2 - v_1) I_{\{v_2 \geq v_1\}} \Big\} \tag{33}$$

$$+x_2(\tilde{v}_2) \Big\{ \lambda \Big[v_2 I_{\{v_2 > v_1\}} + v_1 I_{\{v_2 \leq v_1\}} \Big] + (1-\lambda) \Big[p_4(\tilde{v}_2) I_{\{v_2 \leq p_4(\tilde{v}_2)\}} + v_2 I_{\{v_2 > p_4(\tilde{v}_2)\}} \Big] \Big\} \tag{34}$$

$$-t_2(\tilde{v}_2) \tag{35}$$

The formula follows directly from the outcomes in the four cases in the resale market. The incentive compatibility constraint and participation constraint imply that:

$$U_2(v_2, v_2) \geq U_2(v_2, \tilde{v}_2), \quad \forall\ v_2,\ \tilde{v}_2 \quad IC \tag{36}$$

$$U_2(v_2, v_2) \geq 0, \qquad \forall\ v_2 \qquad PC \tag{37}$$

As in the literature, we will first replace the IC with the first order condition, and then prove that the optimal mechanism satisfies the IC.

Lemma 1 *Buyer 2's incentive compatibility constrain is satisfied only if the following condition holds:*

$$
\begin{aligned}
& t_2(v_2) && (38)\\
= \quad & x_1(v_2)\left\{\lambda\left[v_2-p_1(v_2)\right]I_{\{v_2\geq p_1(v_2)\}}+(1-\lambda)(v_2-v_1)I_{\{v_2\geq v_1\}}\right\} && (39)\\
& +x_2(v_2)\left\{\lambda\left[v_2 I_{\{v_2>v_1\}}+v_1 I_{\{v_2\leq v_1\}}\right]+(1-\lambda)\left[p_4(v_2)I_{\{v_2\leq p_4(v_2)\}}+v_2 I_{\{v_2>p_4(v_2)\}}\right]\right\} && (40)\\
& -\int_a^{v_2}\left\{x_1(\xi)\left[\lambda I_{\{\xi\geq p_1(\xi)\}}+(1-\lambda)I_{\{\xi\geq v_1\}}\right]+x_2(\xi)\left[\lambda I_{\{\xi>v_1\}}+(1-\lambda)I_{\{\xi>p_4(\xi)\}}\right]\right\}d\xi && (41)\\
& -U_2(a,a). && (42)
\end{aligned}
$$

As shown the the proof of Lemma 1, the incentive compatibility constraints also imply that buyer 2's informational rent is increasing in his valuation. Therefore, buyer 2's PC can be replaced by $U_2(a,a)\geq 0$. The participation constraint for the lowest type must be binding, i.e., $U_2(a,a)=0$, otherwise the seller can enhance her revenue by decreasing the informational rent for the lowest valuation.

Now, the seller's revenue becomes:

$$
\begin{aligned}
& R && (43)\\
= \quad & \int_a^b t_1(v_2)dF(v_2)+\int_a^b t_2(v_2)dF(v_2) && (44)\\
= \quad & \int_a^b \Big(x_1(v_2)\left\{\lambda\left[v_1 I_{\{v_2<p_1(v_2)\}}+p_1(v_2)I_{\{v_2\geq p_1(v_2)\}}\right]+(1-\lambda)v_1\right\} && (45)\\
& +x_2(v_2)\left\{(1-\lambda)(v_1-p_4(v_2))I_{\{v_2\leq p_4(v_2)\}}\right\} && (46)\\
& +x_1(v_2)\left\{\lambda\left[v_2-p_1(v_2)\right]I_{\{v_2\geq p_1(v_2)\}}+(1-\lambda)(v_2-v_1)I_{\{v_2\geq v_1\}}\right\} && (47)\\
& +x_2(v_2)\left\{\lambda\left[v_2 I_{\{v_2>v_1\}}+v_1 I_{\{v_2\leq v_1\}}\right]+(1-\lambda)\left[p_4(v_2)I_{\{v_2\leq p_4(v_2)\}}+v_2 I_{\{v_2>p_4(v_2)\}}\right]\right\} && (48)\\
& -\frac{1-F(v_2)}{f(v_2)}\left\{x_1(v_2)\left[\lambda I_{\{v_2\geq p_1(v_2)\}}+(1-\lambda)I_{\{v_2\geq v_1\}}\right]\right\} && (49)\\
& -\frac{1-F(v_2)}{f(v_2)}\left\{x_2(v_2)\left[\lambda I_{\{v_2>v_1\}}+(1-\lambda)I_{\{v_2>p_4(v_2)\}}\right]\right\}\Big)dF(v_2) && (50)
\end{aligned}
$$

$$= \int_a^b x_1(v_2)\Big\{\lambda J_2(v_2)I_{\{v_2\geq p_1(v_2)\}} + \lambda v_1 I_{\{v_2<p_1(v_2)\}} \tag{51}$$

$$+ (1-\lambda)J_2(v_2)I_{\{v_2\geq v_1\}} + (1-\lambda)v_1 I_{\{v_2<v_1\}}\Big\}dF(v_2) \tag{52}$$

$$+\int_a^b x_2(v_2)\Big\{\lambda J_2(v_2)I_{\{v_2>v_1\}} + \lambda v_1 I_{\{v_2\leq v_1\}} \tag{53}$$

$$+ (1-\lambda)J_2(v_2)I_{\{v_2>p_4(v_2)\}} + (1-\lambda)v_1 I_{\{v_2\leq p_4(v_2)\}}\Big\}dF(v_2), \tag{54}$$

Define v_2^* such that $J_2(v_2^*) = v_1$.

Lemma 2 $\lambda J_2(v_2)I_{\{v_2\geq p_1(v_2)\}} + \lambda v_1 I_{\{v_2<p_1(v_2)\}} \leq \lambda J_2(v_2)I_{\{v_2\geq v_2^*\}} + \lambda v_1 I_{\{v_2<v_2^*\}}$

The lemma implies that, in Case 1, it is always the best to conceal all the information by making the full pooling recommendation $p_1(v_2) = v_2^*$.

Lemma 3 $(1-\lambda)J_2(v_2)I_{\{v_2>p_4(v_2)\}} + (1-\lambda)v_1 I_{\{v_2\leq p_4(v_2)\}} \leq (1-\lambda)J_2(v_2)I_{\{v_2>v_1\}} + (1-\lambda)v_1 I_{\{v_2\leq v_1\}}$

The lemma implies that, in Case 4, it is always the best to conceal all the information by making the fool pooling recommendation $p_4(v_2) = v_1$.

Thus, from the above two lemmas, we have

$$(54) \quad \leq \quad \int_a^b x_1(v_2)\Big\{\lambda J_2(v_2)I_{\{v_2\geq v_2^*\}} + \lambda v_1 I_{\{v_2<v_2^*\}} \tag{55}$$

$$+ (1-\lambda)J_2(v_2)I_{\{v_2\geq v_1\}} + (1-\lambda)v_1 I_{\{v_2<v_1\}}\Big\} \tag{56}$$

$$+\int_a^b x_2(v_2)\Big\{\lambda J_2(v_2)I_{\{v_2>v_1\}} + \lambda v_1 I_{\{v_2\leq v_1\}} \tag{57}$$

$$+ (1-\lambda)J_2(v_2)I_{\{v_2>v_1\}} + (1-\lambda)v_1 I_{\{v_2\leq v_1\}}\Big\}, \tag{58}$$

Lemma 4 $\lambda J_2(v_2)I_{\{v_2\geq v_2^*\}} + \lambda v_1 I_{\{v_2<v_2^*\}} \geq \lambda J_2(v_2)I_{\{v_2\geq v_1\}} + \lambda v_1 I_{\{v_2<v_1\}}$

Thus, from the above lemma, we have

$$(58) \quad \leq \quad \int_a^b [x_1(v_2) + x_2(v_2)] \tag{59}$$

$$\underbrace{\Big\{\Big[\lambda J_2(v_2)I_{\{v_2\geq v_2^*\}} + \lambda v_1 I_{\{v_2<v_2^*\}}\Big] + \Big[(1-\lambda)J_2(v_2)I_{\{v_2\geq v_1\}} + (1-\lambda)v_1 I_{\{v_2<v_1\}}\Big]\Big\}}_{H(v_2)} dF(v_2)$$

$$\leq \quad \int_a^b \max\{H(v_2), 0\} dF(v_2), \tag{60}$$

Therefore, the revenue of the seller is bounded by the right hand side of Inequality (60). If we can construct a mechanism generating this revenue, it will be the optimal mechanism. Since the upper bound revenue depends crucially on the function $H(v_2)$, it is necessary to examine this function in more detail. The following lemma characterizes the properties of $H(v_2)$.

Lemma 5 ***Scenario 1:*** *If $a \geq v_1$ and $\lambda v_1 + (1-\lambda)J_2(a) < 0$, then,*

$$H(v_2) \begin{cases} < 0 & if\ a \leq v_2 < \hat{v}_2 \\ \geq 0 & if\ \hat{v}_2 \leq v_2 \leq b \end{cases}, \tag{61}$$

where $\hat{v}_2$ is the unique solution to $\lambda v_1 + (1-\lambda)J_2(\hat{v}_2) = 0$.

Scenario 2: *If $a < v_1$ and $\lambda v_1 + (1-\lambda)J_2(v_1) < 0$, then,*

$$H(v_2) \begin{cases} \geq 0 & if\ a \leq v_2 \leq v_1\ and\ \hat{v}_2 \leq v_2 \leq b \\ < 0 & if\ v_1 < v_2 < \hat{v}_2 \end{cases}, \tag{62}$$

$\hat{v}_2$ is the unique solution to $\lambda v_1 + (1-\lambda)J_2(\hat{v}_2) = 0$.

Scenario 3: *In all other cases, $H(v_2) \geq 0$.*

Under Scenario 1, $H(v_2)$ is increasing and is equal to zero at $\hat{v}_2$. Under Scenario 2, there is a hole such that $H(v_2) < 0$. Under Scenario 3, $H(v_2)$ is always greater than zero. The following propositions confirms that a mechanism generating the revenue of the right hand side of Inequality (60) always exists.

Proposition 1 *Under **Scenario 1**, the following mechanism is optimal.*

$$x_1(v_2) = \begin{cases} 0 & if\ a \leq v_2 < \hat{v}_2 \\ 1 & if\ \hat{v}_2 \leq v_2 \leq b \end{cases} \tag{63}$$

and $x_2(v_2) = 0$. Recommendations $A_1(v_2;p_1)$,$A_2(v_2;p_2)$,$A_3(v_2;p_3)$,$A_4(v_2;p_4)$,$p_2(v_2)$,$p_3(v_2)$ are defined in (2), (3), (4), (5), (17), (19), respectively. Recommendations $p_1(v_2) = v_2^*$ and $p_4(v_2) = v_1$. The monetary transfers are:

$$\int_a^b t_1(v_2)dF(v_2) = v_1[1 - F(\hat{v}_2)] + (v_2^* - \lambda v_1)\left[1 - F(v_2^*)\right], \tag{64}$$

$$t_2(v_2) = \begin{cases} 0 & if\ a \leq v_2 < \hat{v}_2 \\ (1-\lambda)(\hat{v}_2 - v_1) & if\ \hat{v}_2 \leq v_2 \leq b \end{cases} \tag{65}$$

Proposition 2 *Under **Scenario 2**, the following mechanism is optimal.*

$$x_1(v_2) = \begin{cases} 0 & if\ v_1 < v_2 < \hat{v}_2 \\ 1 & if\ otherwise \end{cases} \tag{66}$$

and $x_2(v_2) = 0$. *Recommendations* $A_1(v_2; p_1)$,$A_2(v_2; p_2)$,$A_3(v_2; p_3)$,$A_4(v_2; p_4)$,$p_2(v_2)$,$p_3(v_2)$ *are defined in (2), (3), (4), (5), (17), (19), respectively. Recommendations* $p_1(v_2) = v_2^*$ *and* $p_4(v_2) = v_1$. *The monetary transfers are:*

$$\int_a^b t_1(v_2)dF(v_2) = v_1[1 - F(\hat{v}_2) + F(v_1) - F(a)] + (v_2^* - \lambda v_1)\left[1 - F(v_2^*)\right], \tag{67}$$

$$t_2(v_2) = \begin{cases} 0 & if \;\; a \leq v_2 < \hat{v}_2 \\ (1-\lambda)(\hat{v}_2 - v_1) & if \;\; \hat{v}_2 \leq v_2 \leq b \end{cases} \tag{68}$$

Proposition 3 *Under* ***Scenario 3****, the following mechanism is optimal.* $x_1(v_2) = 1, x_2(v_2) = 0$. *The recommendations* $A_1(v_2; p_1)$, $A_2(v_2; p_2)$, $A_3(v_2; p_3)$, $A_4(v_2; p_4)$, $p_2(v_2)$, $p_3(v_2)$ *are defined in (2), (3), (4), (5), (17), (19), respectively. The recommendations* $p_1(v_2) = v_2^*$ *and* $p_4(v_2) = v_1$. *The monetary transfers are:*

$$\int_a^b t_1(v_2)dF(v_2) = v_1 + (v_2^* - \lambda v_1)\left[1 - F(v_2^*)\right], \tag{69}$$

$$t_2(v_2) = \max\{(1-\lambda)(a - v_1), 0\}. \tag{70}$$

Basically, the seller always sells the right to the speculator and charges him all the revenue to be raised in the resale market. Although the seller does not allocate the object to the regular buyer directly, she also charges him the amount equal to the gain in the resale market of a regular buyer with the lowest valuation. Note that if we set buyer 1's monetary transfer as the expected payment, i.e.,

$$t_1(v_2) = \int_a^b t_1(v_2)dF(v_2), \tag{71}$$

then, buyer 1's monetary transfer no longer depends on buyer 2's valuation. As a result, the seller can collect the money right after the auction. The seller can collect money from buyer 2 right after the auction, since buyer 1 does not have any private information.

There are some features of the optimal mechanism, which are summarized in the follow corollaries.

Corollary 1 *Myerson's allocation can be achieved only when the winner has all the bargaining power, i.e.,* $\lambda = 1$.

When the winner has all the bargaining power, $H(v_2) \geq 0$ and revenue is $\int_a^b H(v_2)dF(v_2)$.

Corollary 2 *An efficient outcome is optimal if the loser has all the bargaining power in the resale market, i.e.,* $\lambda = 0$.

Corollary 3 *The seller should reveal no additional information to the resale market.*

Corollary 4 *Under **Scenario** 1, it is optimal for the seller to hold back the object if $a \leq v_2 < \hat{v}_2$. Under **Scenario** 2, it is optimal for the seller to hold back the object if $v_1 \leq v_2 < \hat{v}_2$.*

4 Conclusion

In this paper, we construct the optimal mechanism in an environment where there are one regular buyer and one speculator, and where resales between buyers cannot be forbidden. We model the resale market as a stochastic ultimatum bargaining game to allow for some bargaining power for both sides of the resale market. We find that the Myerson's optimal auction revenue can be achieved only when the selling side has all of the bargaining power in the resale market. Furthermore, the seller of the initial market retains the object more often than in the Myerson's optimal auction when the buying side also has some barganing power.

In this paper, we analyze a model with two buyers only. Generalizing it to more buyers and more general valuation distribution functions would be an interesting future research project

5 Appendix

Proof for Lemma 1
The envelope theorem implies that

$$\frac{dU_2(v_2,v_2)}{dv_2} = x_1(v_2)\left\{\lambda I_{\{v_2 \geq p_1(v_2)\}} + (1-\lambda) I_{\{v_2 \geq v_1\}}\right\} \tag{72}$$

$$+x_2(v_2)\left\{\lambda\left[I_{\{v_2 > v_1\}}\right] + (1-\lambda)\left[I_{\{v_2 > p_4(v_2)\}}\right]\right\} \tag{73}$$

Solving the above differential equation gives us:

$$U_2(v_2,v_2) \tag{74}$$

$$= \int_a^{v_2} \left\{x_1(\xi)\left[\lambda I_{\{\xi \geq p_1(\xi)\}} + (1-\lambda) I_{\{\xi \geq v_1\}}\right] + x_2(\xi)\left[\lambda I_{\{\xi > v_1\}} + (1-\lambda) I_{\{\xi > p_4(\xi)\}}\right]\right\} d\xi \tag{75}$$

$$+U_2(a,a). \tag{76}$$

Substituting Equation (76) into Equation (35) with $\tilde{v}_2 = v_2$ provides us the result. **Q.E.D**

Proof for Lemma 2
When $v_2 \geq v_2^*$, we have $J_2(v_2) \geq v_1$ since virtual valuation is increasing.

$$LHS \leq \lambda J_2(v_2) I_{\{v_2 \geq p_1(v_2)\}} + \lambda J_2(v_2) I_{\{v_2 < p_1(v_2)\}} \tag{77}$$

$$= \lambda J_2(v_2)[I_{\{v_2 \geq p_1(v_2)\}} + I_{\{v_2 < p_1(v_2)\}}] \quad (78)$$

$$= \lambda J_2(v_2) = RHS \quad (79)$$

When $v_2 < v_2^*$, we have $J_2(v_2) \leq v_1$ since virtual valuation is increasing.

$$LHS \leq \lambda v_1 I_{\{v_2 \geq p_1(v_2)\}} + \lambda v_1 I_{\{v_2 < p_1(v_2)\}} \quad (80)$$

$$= \lambda v_1 [I_{\{v_2 \geq p_1(v_2)\}} + I_{\{v_2 < p_1(v_2)\}}] \quad (81)$$

$$= \lambda v_1 = RHS \quad (82)$$

Q.E.D

Proof for Lemma 3
Note that when buyer 2 has the object and buyer 1 makes the offer, buyer 1 will not make an offer higher than his own valuation v_1, i.e.,$p_4(v_2) \leq v_1$.

When $v_2 \geq v_1$, we have $v_2 \geq p_4(v_2)$, and therefore,

$$LHS = (1 - \lambda)J_2(v_2) \quad (83)$$

$$RHS = (1 - \lambda)J_2(v_2) \quad (84)$$

When $v_2 < v_1$, we have $J_2(v_2) < v_1$, and therefore,

$$LHS \leq (1 - \lambda)v_1 I_{\{v_2 > p_4(v_2)\}} + (1 - \lambda)v_1 I_{\{v_2 \leq p_4(v_2)\}} = (1 - \lambda)v_1 \quad (85)$$

$$RHS = (1 - \lambda)v_1 \quad (86)$$

Q.E.D

Proof for Lemma 4
When $v_2 \geq v_2^*$, we have $J_2(v_2) \geq v_1$ since virtual valuation is increasing.

$$LHS = J_2(v_2) \quad (87)$$

and

$$RHS \leq \lambda J_2(v_2) I_{\{v_2 \geq v_1\}} + \lambda J_2(v_2) I_{\{v_2 < v_1\}} = \lambda J_2(v_2) = LHS \quad (88)$$

When $v_2 < v_2^*$, we have $J_2(v_2) \leq v_1$ since virtual valuation is increasing.

$$LHS = \lambda v_1 \quad (89)$$

$$RHS \leq \lambda v_1 I_{\{v_2 \geq v_1\}} + \lambda v_1 I_{\{v_2 < v_1\}} = \lambda v_1 = LHS \tag{90}$$

Q.E.D

Proof for Lemma 5
Note that $v_2^* \geq v_1$. When $v_2 \leq v_1$, $H(v_2) = \lambda v_1 + (1-\lambda)v_1 \geq 0$.

When $v_2 \geq v_2^*$, $H(v_2) = \lambda J_2(v_2) + (1-\lambda)J_2(v_2) \geq J_2(v_2^*) = v_1 \geq 0$.

When $v_1 < v_2 < v_2^*$, $H(v_2) = \lambda v_1 + (1-\lambda)J_2(v_2)$. In this case, $H(v_2)$ is increasing in v_2 since $J_2(v_2)$ is increasing. The upper bound of $H(v_2)$ is $H(v_2^*) = v_1 \geq 0$. Thus, if $a \geq v_1$ and $\lambda v_1 + (1-\lambda)J_2(a) < 0$, then there exist a unique $\hat{v}_2 \in (v_1, v_2^*)$, such that $H(\hat{v}_2) = 0$. If $a < v_1$ and $\lambda v_1 + (1-\lambda)J_2(v_1) < 0$, then there exist a unique $\hat{v}_2 \in (a, v_2^*)$, such that $H(\hat{v}_2) = 0$. Otherwise, i.e., $a \geq v_1$ and $\lambda v_1 + (1-\lambda)J_2(a) \geq 0$ or $a < v_1$ and $\lambda v_1 + (1-\lambda)J_2(v_1) \geq 0$, then, $H(v_2) \geq 0$. The lemma is a summary of all the cases. **Q.E.D**

Proof for Proposition 1
It is easy to see that the mechanism in this proposition generates the optimal revenue by simply substituting all of the functions into the seller's revenue function (54). Given the allocation rules and recommendations, the monetary transfers reduce to

$$\int_{\hat{v}_2}^{b} t_1(v_2)dF(v_2) = \int_{\hat{v}_2}^{b} \left\{\lambda \left[v_1 I_{\{v_2 < v_2^*\}} + v_2^* I_{\{v_2 \geq v_2^*\}}\right] + (1-\lambda)v_1\right\} dF(v_2) \tag{91}$$

$$= \lambda v_1[F(v_2^*) - F(\hat{v}_2)] + v_2^*[1 - F(v_2^*)] + (1-\lambda)v_1[1 - F(\hat{v}_2)] \tag{92}$$

$$= v_1[1 - F(\hat{v}_2)] + (v_2^* - \lambda v_1)[1 - F(v_2^*)]. \tag{93}$$

For $t_2(v_2)$, if $a \leq v_2 < \hat{v}_2$, then $t_2(v_2) = 0$. If $\hat{v}_2 \leq v_2$, then

$$t_2(v_2) = \lambda(v_2 - v_2^*)I_{\{v_2 \geq v_2^*\}} + (1-\lambda)(v_2 - v_1) \tag{94}$$

$$- \int_{\hat{v}_2}^{v_2} \left[\lambda I_{\{\xi \geq v_2^*\}} + (1-\lambda)\right] d\xi \tag{95}$$

$$= \lambda(v_2 - v_2^*)I_{\{v_2 \geq v_2^*\}} + (1-\lambda)(v_2 - v_1) \tag{96}$$

$$- \lambda(v_2 - v_2^*)I_{\{v_2 \geq v_2^*\}} - (1-\lambda)(v_2 - \hat{v}_2) \tag{97}$$

$$= (1-\lambda)(\hat{v}_2 - v_1). \tag{98}$$

which are consistent with those in the proposition. Thus, we only need to show that this mechanism satisfies all the incentive compatibility constraints and participation constraints.

For the resale market, since $A_1(v_2; p_1), A_2(v_2; p_2), A_3(v_2; p_3), A_4(v_2; p_4), p_2(v_2), p_3(v_2)$ are directly replaced by the incentive compatible ones, we only need to confirm that $p_1(v_2)$ and $p_4(v_2)$ are incentive compatible.

Let us first examine $p_1(v_2)$. The winner, if any, is always buyer 1 in the initial market and the recommendation is fully pooling. Thus, when buyer 1 chooses the price to offer in Case 1, he believes that buyer 2's valuation is always above $\hat{v}_2$, i.e., $G_1(v_2) = \frac{F(v_2)-F(\hat{v}_2)}{1-F(\hat{v}_2)}$. Thus, substituting $G_1(v_2) = F(v_2)$ into the FOC (15) determines the price offer, i.e., $v_1 = \tilde{p} - \frac{1-\frac{F(\tilde{p})-F(\hat{v}_2)}{1-F(\hat{v}_2)}}{\frac{f(\tilde{p})}{1-F(\hat{v}_2)}} = J_2(\tilde{p})$. Since we have assumed that $J_2(a) \le v_1 \le J_2(b)$ and $J_2(v_2)$ is increasing, there exists a unique solution $\tilde{p} = v_2^*$. As in the proof of Proposition 1, it is indeed optimal to be obedient and offer the price v_2^*.

Now, let us examine $p_4(v_2)$. Since, buyer 1 always wins in the initial market. Case 4 is actually on the out-of-equilibrium path.

The last issue to verify is buyer 2's incentive compatibility constraint in the initial market.

Substituting all the functions into Equation (35), we have

$$U_2(v_2, \tilde{v}_2) \tag{99}$$

$$= \begin{cases} 0 & if\ a \le v_2 < \hat{v}_2 \\ \lambda\,[v_2 - v_2^*]\, I_{\{v_2 \ge v_2^*\}} + (1-\lambda)(v_2 - v_1) - (1-\lambda)(\hat{v}_2 - v_1) & if\ \hat{v}_2 \le v_2 \le b \end{cases} \tag{100}$$

$$= \begin{cases} 0 & if\ a \le v_2 < \hat{v}_2 \\ \lambda\,(v_2 - v_2^*)\, I_{\{v_2 \ge v_2^*\}} + (1-\lambda)(v_2 - \hat{v}_2) & if\ \hat{v}_2 \le v_2 \le b \end{cases} \tag{101}$$

First consider $v_2 \le \hat{v}_2$. Note that now $v_2 \le v_2^*$ since $\hat{v}_2 \le v_2^*$. Truthfully reporting provides him $U_2(v_2, v_2) = 0$. If he deviates to any $\tilde{v}_2 \le \hat{v}_2$, it gives him the same payoff 0. If he deviates to $\tilde{v}_2 \ge \hat{v}_2$, then

$$U_2(v_2, \tilde{v}_2) = (1-\lambda)(v_2 - \hat{v}_2) \le 0 \tag{102}$$

Thus, he has no incentive to deviate.

Now consider $v_2 \ge \hat{v}_2$, then $v_2 \ge v_1$. Truthfully reporting provides him

$$U_2(v_2, v_2) \tag{103}$$

$$= \lambda\,(v_2 - v_2^*)\, I_{\{v_2 \ge v_2^*\}} + (1-\lambda)(v_2 - \hat{v}_2) \ge 0 \tag{104}$$

If he deviates to any $\tilde{v}_2 \ge \hat{v}_2$, then it give him the same payoff. If he deviates to $\tilde{v}_2 \le \hat{v}_2$, then $U_2(v_2, \tilde{v}_2) = 0$. Thus he has no incentive to deviate. **Q.E.D**

Proof for Proposition 2

It is easy to see that the above mechanism generate the optimal revenue by simply substituting all the functions into the seller's revenue function (54). Given the allocation rules and recommendations, the monetary transfers reduce to

$$\int_{\hat{v}_2}^{b} t_1(v_2)dF(v_2) = \int_{a}^{v_1} \left\{\lambda \left[v_1 I_{\{v_2<v_2^*\}} + v_2^* I_{\{v_2\geq v_2^*\}}\right] + (1-\lambda)v_1\right\} dF(v_2) \tag{105}$$

$$+\int_{\hat{v}_2}^{b} \left\{\lambda \left[v_1 I_{\{v_2<v_2^*\}} + v_2^* I_{\{v_2\geq v_2^*\}}\right] + (1-\lambda)v_1\right\} dF(v_2) \tag{106}$$

$$= \int_{a}^{v_1} \left\{\lambda v_1 + (1-\lambda)v_1\right\} dF(v_2) \tag{107}$$

$$+\int_{\hat{v}_2}^{b} \left\{\lambda \left[v_1 I_{\{v_2<v_2^*\}} + v_2^* I_{\{v_2\geq v_2^*\}}\right] + (1-\lambda)v_1\right\} dF(v_2) \tag{108}$$

$$= v_1[F(v_1) - F(a)] \tag{109}$$

$$+\lambda v_1[F(v_2^*) - F(\hat{v}_2)] + v_2^*\left[1 - F(v_2^*)\right] + (1-\lambda)v_1[1 - F(\hat{v}_2)] \tag{110}$$

$$= v_1[1 - F(\hat{v}_2) + F(v_1) - F(a)] + (v_2^* - \lambda v_1)\left[1 - F(v_2^*)\right]. \tag{111}$$

For $t_2(v_2)$, if $a \leq v_2 \leq v_1$,

$$t_2(v_2) = 0. \tag{112}$$

If $v_1 < v_2 < \hat{v}_2$,

$$t_2(v_2) = 0. \tag{113}$$

If $\hat{v}_2 \leq v_2 < b$,

$$t_2(v_2) = \lambda(v_2 - v_2^*)I_{\{v_2\geq v_2^*\}} + (1-\lambda)(v_2 - v_1) \tag{114}$$

$$-\int_{\hat{v}_2}^{v_2} \left[\lambda I_{\{\xi\geq v_2^*\}} + (1-\lambda)\right] d\xi \tag{115}$$

$$= \lambda(v_2 - v_2^*)I_{\{v_2\geq v_2^*\}} + (1-\lambda)(v_2 - v_1) \tag{116}$$

$$-\lambda(v_2 - v_2^*)I_{\{v_2\geq v_2^*\}} - (1-\lambda)(v_2 - \hat{v}_2) \tag{117}$$

$$= (1-\lambda)(\hat{v}_2 - v_1). \tag{118}$$

which are consistent with the ones in the proposition. Thus, we only need to show that this mechanism satisfies all the incentive compatibility constraints and participation constraints. For the resale market, since $A_1(v_2; p_1), A_2(v_2; p_2), A_3(v_2; p_3), A_4(v_2; p_4), p_2(v_2), p_3(v_2)$ are directly replaced by the incentive compatible ones, we only need to confirm that $p_1(v_2)$ and $p_4(v_2)$ are incentive compatible.

Let us first examine $p_1(v_2)$. The winner, if any, is always buyer 1 in the initial market and the recommendation is fully pooling. Thus, when buyer 1 chooses the price to offer in Case 1, he believes that buyer 2's valuation is always in $[a, v_1] \bigcup [\hat{v}_2, b]$, i.e.,

$$G_4(v_2) = \begin{cases} \frac{F(v_2)}{1-F(\hat{v}_2)+F(v_1)} & if\ a \leq v_2 \leq v_1 \\ \frac{F(v_1)}{1-F(\hat{v}_2)+F(v_1)} & if\ v_1 < v_2 < \hat{v}_2 \\ \frac{F(v_2)-F(\hat{v}_2)+F(v_1)}{1-F(\hat{v}_2)+F(v_1)} & if\ \hat{v}_2 \leq v_2 \leq b \end{cases}. \tag{119}$$

and

$$g_4(v_2) = \begin{cases} \frac{f(v_2)}{1-F(\hat{v}_2)+F(v_1)} & if\ a \leq v_2 \leq v_1 \\ 0 & if\ v_1 < v_2 < \hat{v}_2 \\ \frac{f(v_2)}{1-F(\hat{v}_2)+F(v_1)} & if\ \hat{v}_2 \leq v_2 \leq v_1 \end{cases}. \tag{120}$$

$$\frac{1-G_4(v_2)}{g_4(v_2)} = \begin{cases} \frac{1-F(\hat{v}_2)+F(v_1)-F(v_2)}{f(v_2)} & if\ a \leq v_2 \leq v_1 \\ +\infty & if\ v_1 < v_2 < \hat{v}_2 \\ \frac{1-F(v_2)}{f(v_2)} & if\ \hat{v}_2 \leq v_2 \leq b \end{cases}. \tag{121}$$

Thus, substituting into the FOC (15) determines the price offer, i.e.,

$$v_1 = \begin{cases} J_2(\tilde{p}) - \frac{F(\hat{v}_2)-F(v_1)}{f(\tilde{p})} & if\ a \leq v_2 \leq v_1 \\ -\infty & if\ v_1 < v_2 < \hat{v}_2 \\ J_2(\tilde{p}) & if\ \hat{v}_2 \leq v_2 \leq b \end{cases}. \tag{122}$$

We claim that this equation has a unique solution. Note that we have $J_2(a) \leq v_1 \leq J_2(b)$, $J_2(v_2)$ is increasing, $J_2(\hat{v}_2) = 0$ and $J_2(v_2^*) = v_1$. If $a \leq v_2 \leq v_1$, $RHS < J_2(\tilde{p}) < J_2(v_2^*) = v_1$. If $if\ v_1 < v_2 < \hat{v}_2$, it is obvious there is no solution. If $\hat{v}_2 \leq v_2 \leq b$, there is a unique solution $\tilde{p} = v_2^*$. Now, we need to show that $\tilde{p} = v_2^*$ is a global maximal.

$$\frac{d\Pi_1}{d\tilde{p}} = v_1 g_4(\tilde{p}) - \tilde{p} g(\tilde{p}) + [1 - G_1(\tilde{p})] \tag{123}$$

$$= g_4(\tilde{p}) \left[v_1 - \tilde{p} + \frac{1-G_4(\tilde{p})}{g_4(\tilde{p})} \right] \tag{124}$$

$$= \begin{cases} g_4(\tilde{p})\left[v_1 - J_2(\tilde{p}) - \frac{F(\hat{v}_2)-F(v_1)}{f(\tilde{p})}\right] & if\ a \le v_2 \le v_1 \\ 1 - G_1(\tilde{p}) & if\ v_1 < v_2 < \hat{v}_2 \\ v_1 - J_2(\tilde{p}) & if\ \hat{v}_2 \le v_2 \le b \end{cases} . \tag{125}$$

Since $v_1 = J_2(v_2^*)$ and $J_2(\cdot)$ is increasing, $\frac{d\Pi_1}{d\tilde{p}} \ge 0$ if $\tilde{p} \le v_2^*$ and $\frac{d\Pi_1}{d\tilde{p}} \le 0$ if $\tilde{p} \ge v_2^*$. Thus, it is indeed optimal to be obedient and offer the price v_2^*.

Now, let us examine $p_4(v_2)$. Since, buyer 1 always wins in the initial market. Case 4 is actually on the out-of-equilibrium path.

The last issue to verify is buyer 2's incentive compatibility constraint in the initial market.

Substituting all the functions into Equation (35), we have

$$U_2(v_2, \tilde{v}_2) \tag{126}$$

$$= \begin{cases} 0 & if\ a \le v_2 \le v_1 \\ 0 & if\ v_1 \le v_2 < \hat{v}_2 \\ \lambda\left[v_2 - v_2^*\right] I_{\{v_2 \ge v_2^*\}} + (1-\lambda)(v_2 - v_1) - (1-\lambda)(\hat{v}_2 - v_1) & if\ \hat{v}_2 \le v_2 \le b \end{cases} \tag{127}$$

$$= \begin{cases} 0 & if\ a \le v_2 < \hat{v}_2 \\ \lambda\left(v_2 - v_2^*\right) I_{\{v_2 \ge v_2^*\}} + (1-\lambda)(v_2 - \hat{v}_2) & if\ \hat{v}_2 \le v_2 \le b \end{cases} \tag{128}$$

First consider $v_2 \le \hat{v}_2$. Note that now $v_2 \le v_2^*$ since $\hat{v}_2 \le v_2^*$. Truthfully reporting provides him $U_2(v_2, v_2) = 0$. If he deviates to any $\tilde{v}_2 \le \hat{v}_2$, it gives him the same payoff 0. If he deviate to $\tilde{v}_2 \ge \hat{v}_2$, then

$$U_2(v_2, \tilde{v}_2) = (1-\lambda)(v_2 - \hat{v}_2) \le 0 \tag{129}$$

Thus, he has no incentive to deviate.

Now consider $v_2 \ge \hat{v}_2$, then $v_2 \ge v_1$. Truthfully reporting provides him

$$U_2(v_2, v_2) \tag{130}$$

$$= \lambda\left(v_2 - v_2^*\right) I_{\{v_2 \ge v_2^*\}} + (1-\lambda)(v_2 - \hat{v}_2) \ge 0 \tag{131}$$

If he deviates to any $\tilde{v}_2 \ge \hat{v}_2$, then it give him the same payoff. If he deviates to $\tilde{v}_2 \le \hat{v}_2$, then $U_2(v_2, \tilde{v}_2) = 0$. Thus he has no incentive to deviate. **Q.E.D**

Proof for Proposition 3

It is easy to check that the above mechanism generates the right hand side revenue of Inequality (60) by simply substituting all the functions into the seller's revenue function (54). Given the allocation rules and recommendations, the monetary transfers reduce to

$$\int_a^b t_1(v_2)dF(v_2) = \int_a^b \left\{\lambda\left[v_1 I_{\{v_2<v_2^*\}} + v_2^* I_{\{v_2\geq v_2^*\}}\right] + (1-\lambda)v_1\right\} dF(v_2) \tag{132}$$

$$= \lambda v_1 F(v_2^*) + v_2^*\left[1-F(v_2^*)\right] + (1-\lambda)v_1 \tag{133}$$

$$= v_1 + (v_2^* - \lambda v_1)\left[1-F(v_2^*)\right], \tag{134}$$

and

$$t_2(v_2) = \lambda(v_2 - v_2^*)I_{\{v_2\geq v_2^*\}} + (1-\lambda)(v_2-v_1)I_{\{v_2\geq v_1\}} \tag{135}$$

$$-\int_a^{v_2}\left[\lambda I_{\{\xi\geq v_2^*\}} + (1-\lambda)I_{\{v_2\geq v_1\}}\right]d\xi \tag{136}$$

$$= \lambda(v_2 - v_2^*)I_{\{v_2\geq v_2^*\}} + (1-\lambda)(v_2-v_1)I_{\{v_2\geq v_1\}} \tag{137}$$

$$-\lambda(v_2 - v_2^*)I_{\{v_2\geq v_2^*\}} - (1-\lambda)(v_2-\max\{a,v_1\})I_{\{v_2\geq v_1\}} \tag{138}$$

$$= (1-\lambda)(v_2-v_1)I_{\{v_2\geq v_1\}} - (1-\lambda)(v_2-\max\{a,v_1\})I_{\{v_2\geq v_1\}} \tag{139}$$

$$= (1-\lambda)(\max\{a,v_1\}-v_1)I_{\{v_2\geq v_1\}} \tag{140}$$

$$= \max\{(1-\lambda)(a-v_1),0\}. \tag{141}$$

Those monetary transfers are consistent with the ones in the proposition. Thus, we only need to show that this mechanism satisfies all the incentive compatibility constraints and participation constraints.

For the resale market, since $A_1(v_2;p_1)$, $A_2(v_2;p_2)$, $A_3(v_2;p_3)$, $A_4(v_2;p_4)$, $p_2(v_2)$, $p_3(v_2)$ are directly replaced by the incentive compatible ones, we only need to confirm that $p_1(v_2)$ and $p_4(v_2)$ are incentive compatible.

Let us first examine $p_1(v_2)$. Since buyer 1 is always the winner in the initial market and the recommendation is fully pooling, buyer 1's belief about buyer 2's valuation would not change when he chooses the price to offer in Case 1, i.e., $G_1(v_2) = F(v_2)$. Thus, substituting $G_1(v_2) = F(v_2)$ into the FOC (15) determines the price offer, i.e., $v_1 = \tilde{p} - \frac{1-F(\tilde{p})}{f(\tilde{p})} = J_2(\tilde{p})$. Since we have assumed that $J_2(a) \leq v_1 \leq J_2(b)$ and $J_2(v_2)$ is increasing, there exists a unique solution $\tilde{p} = v_2^*$. Now, we need to show that $\tilde{p} = v_2^*$ is a global maximal.

$$\frac{d\Pi_1}{d\tilde{p}} = v_1 f(\tilde{p}) - \tilde{p}f(\tilde{p}) + \left[1-F(\tilde{p})\right] \tag{142}$$

$$= f(\tilde{p})\left[v_1 - \tilde{p} + \frac{1-F(\tilde{p})}{f(\tilde{p})}\right] \tag{143}$$

$$= f(\tilde{p})\left[v_1 - J_2(\tilde{p})\right]. \tag{144}$$

Since $v_1 = J_2(v_2^*)$ and $J_2(\cdot)$ is increasing, $\frac{d\Pi_1}{d\tilde{p}} \geq 0$ if $\tilde{p} \leq v_2^*$ and $\frac{d\Pi_1}{d\tilde{p}} \leq 0$ if $\tilde{p} \geq v_2^*$. Thus, it is indeed optimal to be obedient and offer the price v_2^*.

Now, let us examine $p_4(v_2)$. Since, buyer 1 always wins in the initial market. Case 4 is actually on the out-of-equilibrium path. The price offer $p_4(v_2)$ can be supported by the belief $v_2 = b$.

The last issue to verify is buyer 2's incentive compatibility constraint in the initial market.

Substituting all the functions into Equation (35), we have

$$U_2(v_2, \tilde{v}_2) \tag{145}$$

$$= \Big\{\lambda\,[v_2 - v_2^*]\, I_{\{v_2 \geq v_2^*\}} + (1-\lambda)(v_2 - v_1) I_{\{v_2 \geq v_1\}}\Big\} - \max\{(1-\lambda)(a - v_1), 0\} \tag{146}$$

This payoff does not depend on $\tilde{v}_2$, and therefore, buyer 2 has no incentive to lie about his valuation. **Q.E.D**

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