DISCOUNTING WHEN INCOME IS STOCHASTIC AND DISCOUNTED UTILITY ANOMALIES

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Abstract. Several discounted utility anomalies are explained as rational choices of an agent with standard preferences and stochastic income. We define the term structure of absolute risk aversion and demonstrate that the gain-loss asymmetry is observed for small gains and losses and a general utility function if the term structure is non-decreasing. Agents, whose current income is less than the long-run average by a certain margin, exhibit hyperbolic discounting. The discount rate of agents, whose current income is above the central tendency, is increasing. Agents who are neither rich nor poor have hump-shaped discount rate curves.

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Behavioral economics emphasizes experimental findings that suggest inadequacies of standard economic theories, proceeds to provide psychological explanations of such inadequacies and advocates radical changes in the methods of economics. However, methods of the standard economics are much more flexible than it is assumed by radical behavioral economists. See Faruk Gul and Wolfgang Pesendorfer (2008) for the argument and examples demonstrating that small changes to standard choice-theoretic methods suffice to analyze variables that are often ignored in standard economic models. Even in hard sciences, new theoretical knowledge about the fine structure of reality does not usually supersede "outdated" theories. It is also extremely difficult, if at all possible, to derive the axioms of the old theory from the new: think of quantum mechanics and

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mechanics of continuous media as a prime example. Therefore, it is not clear if behavioral observations can be ever used as a basis of a consistent general economic theory. Note that Wolfgang Pesendorfer (2006) argues that it is difficult to use economic data to calibrate utility functions that depend on some variables observed in experiments on intertemporal choice.

Recently, a generalization of standard choice-theoretic welfare economics that encompasses a wide variety of non-standard behavioral models was proposed in B. Douglas Bernheim and Antonio Rangel (2008). Jawwad Noor (2009) uses a standard exponential discounting model to explain the hyperbolic (time dependent) discounting, which is regarded by behavioral economists as one of the most celebrated failures of the standard economics (the latter assumes that intertemporal choices do not depend on the decision date). His main argument is that "the most likely participants in experiments may be those with the most immediate need for money;" if the participants expect a small deterministic increase in the base consumption level, then their behavior in experiments agrees with hyperbolic discounting.

The idea that time dependent discounting can be explained with an urgent need for money agrees with existence of high priced credit products such as small personal loans, pawnbroker loans, payday loans, automobile title loans, and refund anticipation loans. Prices for these products are indeed high. Finance charges are large relative to loan amounts, and annual percentage rates often exceed 100 percent (see Gregory Elliehausen (2006) for details). Paige M. Skiba and Jeremy Tobacman (2008) demonstrate in an empirical model that consumers' behavior regarding payday loans is most consistent with partially naive quasi-hyperbolic discounting.

Noor (2009), however, does not explain other discounted utility (DU) anomalies. For example, how the immediate need for money can account for the fact that gains are discounted more than losses? Starting with the same standard preferences as in Noor (2009), we introduce uncertainty and derive general discount factors for gains and losses that imply several discounted utilities anomalies at once. To be more specific, we demonstrate that if, at date 0, the agent is asked to compare the dated rewards (or losses) (m', t) and $(m, t+T)$ $(t \geq 0, T > 0, m' > 0, m > 0)$, then the discount function is the marginal rate of substitution between the base consumption level at t and $t+T$ as perceived at time 0. Whether the discount function is an increasing function of time (that is hyperbolic discounting takes place) depends on the type of the stochastic process for income and agent's current base consumption level. Assuming that $u(\cdot)$ is the instantaneous utility function which is of the class C^2, increasing and concave, and b_t is the base consumption level[1] at date t, we introduce the ratio $E[-u''(b_t)]/E[u'(b_t)]$ (where E is the expectations operator) which we call *the term structure of absolute risk aversion*. We show that if the term structure of absolute risk aversion is a non-decreasing function of time (this condition is sufficient, but not necessary) then gains are discounted more than losses and the delay-speedup asymmetry follows. The gain-loss asymmetry is observed even when the discount function is exponential.

As examples, we consider two most popular forms of the instantaneous utility function: exponential utility (CARA utility) and constant relative risk aversion utility (CRRA utility). We show that relatively poor agents prefer to consume sooner than relatively rich agents, but relatively rich agents prefer to suffer a loss sooner than relatively poor agents. This fact, in particular, suggests an explanation of the unwillingness of poor countries to suffer costs of combat against the global warming now in order to save themselves from the losses in the future.

If the agent perceives her income as a process with i.i.d. increments, then the discounting is exponential, though it depends on the parameters of the underlying stochastic process used to model the base consumption stream and on the agent's risk aversion. In the case of the CRRA utility function and the stochastic base consumption stream following a geometric Brownian motion (GBM), the money discount factors for gains and losses depend on the agent's current base consumption level. Rich agents discount the future gains less than poor agents, which agrees with consumers' willingness to buy high priced credit products such as payday loans. On the other hand, poor agents discount future losses less than rich agents. For losses that may happen in the near future, negative discounting is observed; however, rich people do not exhibit negative discounting in this model.

In order to generate non-exponential discounting and preference reversal, we use two popular stochastic processes with mean reverting features to model the base consumption level stream (or its logarithm in the case of the CRRA utility function): the Cox-Ingersol-Ross (CIR) process and Ornstein-Uhlenbeck (OU) process. For both of these processes and both CARA and CRRA utlity functions, the following results obtain. If the agent is rich so that her current base consumption level is higher than the long run central tendency, then the effective discount rate increases in time (as the borrowing rate for a sound corporation), and no hyperbolic discounting is observed. This behavior of the effective discount rate corresponds to the case of so called normal yield curve in the bond markets (a pattern known as *contango* in the commodities futures markets).

[1] For the sake of brevity, we often call b_t the income; we understand the difference between income and base consumption level, though we believe that this difference can be ignored for poor people such as students who are the main participants in DU anomalies experiments.

If the agent is poor so that her current income is less than the long-run average by a certain non-zero margin (which depends of the risk attitude, type of uncertainty and the parameters of the income process), then the effective discount rate decreases with time, and the hyperbolic discounting is observed. This pattern is known as *backwardation* in the commodities futures markets or an inverted yield curve in the bond markets. Contango and backwardation patterns can be intuitively explained. Consider, for instance, oil spot and forward prices. If tanks are awash in oil, the market is less at risk of a supply shock and oil for prompt delivery is cheap relative to later-dated futures contracts (contango). Conversely, low inventory levels make immediate access to oil very valuable, giving promptly delivered oil a premium (backwardation). Similarly, the agent, whose current income is above the central tendency, values future consumption more than present consumption, because she expects her income to drop to the log-run average eventually. On the other hand, the agent, whose income is below the long-run average, expects her income to revert to the central tendency eventually, therefore she values immediate consumption more than distant consumption.

Finally, if the agent is neither too rich nor too poor, then there exists $t^* > 0$ such that the hyperbolic discounting is observed over the interval $[t^*, +\infty)$ but on $[0, t^*]$, the effective discount rate is increasing; i.e., the effective discount rate is hump-shaped. The poorer the agent becomes, the higher is the probability that the hyperbolic discounting will be observed in an experiment. Hump-shaped yield curves are also observed in real markets. The traditional crude oil futures curve, for example, is typically humped: it is normal in the short-term but gives way to an inverted market for longer maturities.

The rest of the paper is organized as follows. Section 1 gives an overview of the standard discounted utility, introduced by Paul Samuelson (1937) and departures from this theory observed in experiments. The time preference model is specified in Section 2. In Section 3, we consider the case of the CARA utility function and stochastic base consumption stream following the Brownian motion (BM) and CIR process. In Section 4, we derive formulas for discount factors for gains and losses for a general utility function and general uncertainty and introduce the notion of the term structure of absolute risk aversion. Section 5 deals with the case of the CRRA utility function and stochastic base consumption stream following the GBM and OU process. Section 6 concludes. Technical details are relegated to the Appendix.

1. Time preference

In 1937, Samuelson invented the DU theory, which compressed the influence of many factors affecting intertemporal choices into one number: the discount rate. In continuous time models, an individual with the time-separable utility u calculates the value of consumption of a stream b_t over time interval $[0, T]$

according to the formula

$$U = \int_0^T e^{-rt} u(b_t) dt, \tag{1}$$

where $r > 0$ is the discount rate. In discrete time models, the counterpart of equation (1) is

$$U = \sum_{t=0}^{T} \delta^t u(b_t), \tag{2}$$

where $\delta = e^{-r} \approx 1/(1+r)$. Due to the analytical simplicity, the exponential discounted utility model was almost instantly adopted as a standard tool in intertemporal models, although Samuelson (1937) suggested the DU model as a convenient tool only, and explicitly disavowed an idea that individuals really optimize an integral of the form (1). More than 20 years later, Tjalling C. Koopmans (1960) constructed an axiomatic theory of time preference which lead to the exponential discount factor in Samuelson's model. As a result, a general feeling emerged that the DU model was justified. However later, in many experimental studies, it was shown that the real behavior of individuals did not agree with the exponential discounting model. Shane Frederick, George Loewenstein, and Ted O'Donoghue (2002) present evidence that the instantaneous discount rate for gains *decreases* with time (*hyperbolic discounting*), gains are discounted more than losses (*sign effect, or gain-loss asymmetry*), greater discounting is demonstrated to avoid delay of a good than to expedite its receipt (*delay-speedup asymmetry*), an individual may prefer to expedite a payment (*negative discounting for losses*).

To account for DU anomalies, several alternative models have been developed. In the (β, δ)- model of quasi-hyperbolic discounting introduced first by E.S. Phelps and Robert Pollack (1968), equation (2) is replaced by

$$U = u(b_0) + \sum_{t=1}^{T} \beta\delta^t u(b_t), \tag{3}$$

where $\beta, \delta \in (0, 1)$. Equation (3) is analytically simple, and captures many qualitative features of hyperbolic discounting. Thus, as in Samuelson (1937), the discount factors are postulated. Another strand of literature initiated by Koopmans (1960) deals with the axiomatic systems for time preferences, which are consistent with DU anomalies - see Efe A. Ok and Yusufcan Masatlioglu (2008) and the bibliography therein. Drew Fudenberg and David K. Levine (2004) suggested a "dual-self" model as a unified explanation for several empirical regularities. Habit formation models, reference point models and a number of other models incorporate non-standard features into the utility function. See Frederick, Loewenstein and O'Donoghue (2002), Gul and Pesendorfer (2008) and the extensive bibliography there for the list of models that depart from the DU model.

Note that a natural model for hyperbolic discounting in discrete time would be

$$U = u(b_0) + \sum_{t=1}^{T} \prod_{t'=0}^{t-1} \beta_{t',t'+1} u(b_t), \tag{4}$$

where $\beta_{t,t+1}$, the discount factor between t and $t+1$, as viewed from time 0, is an *increasing* function in t. In the continuous time limit, we obtain that the hyperbolic discounting means that the instantaneous discount rate

$$r_t = -\lim_{T\to 0} \frac{\ln \beta_{t,t+T}}{T} = -\frac{\partial}{\partial T} \ln \beta_{t,t+T}|_{T=0} \tag{5}$$

is a *decreasing* function of t.

In finance, close analogs of the discount rates are zero-coupon bond yields. At time t, consider the bond maturing at $t+T$. Although at maturity, the payoff is deterministic (say, \$ 100), the bond price $B(t, t+T)$ is a random variable, and yield curves $t \mapsto -\ln B(t, t+T)/T$ are not flat; in fact, they can be of many shapes. The reason is that during the time period to maturity, many random events will happen in the world, and they will influence the value of the riskless zero-coupon bond. The example with the yield curves explains that however hard a researcher tries to exclude the uncertainty in an experiment on DU anomalies, the uncertainty will always remain in the background, and therefore, there is no reason to expect that the discount rate curve observed in experiments will be flat.

There is a substantial body of research in financial economics, where the behavior of bond prices and yields is derived endogenously in general equilibrium models from the exogenous stochastic dynamics of the production sector of the economy: see, e.g., John C. Cox, Jonathan E. Ingersoll, and Stephen A. Ross. (1985) for one of the most popular interest rate models and J. Darrell Duffie and Kenneth J. Singleton (2003) for the review of alternative models and further references. We conduct our analysis in the framework of a partial equilibrium model, and, to simplify the treatment of instantaneous payoffs, we consider the payoff streams in discrete time.

2. Model specification

We neither postulate the non-standard dependence of the discount factor on time as in the quasi-hyperbolic discounted utility models nor deduce it from time preference axioms. Instead, we derive general equations for the discount factors for gains and losses from several simple general assumptions.

As in Noor (2009), we define a preference relation $\succeq$ over the set of dated rewards $\mathcal{X} = \mathcal{M} \times \mathcal{T}$, where $\mathcal{M} = [0, M]$ (for some $M > 0$), and $\mathcal{T} = \mathbb{R}_+$. Let $\{b_t\}_{t\ge 0}$ be the consumer's base consumption stream (income), then if the stream is deterministic, the preference relation $\succeq$ on $\mathcal{X}$ is induced by a utility function (see Noor (2009))

$$U(m,t) = D(t)[u(b_t + m) - u(b_t)],$$

where $D(t)$ is the discount function. Using the present equivalent $\psi(m,t)$ of any dated reward (m,t), where the present equivalent is defined by $(\psi(m,t),0) \sim (m,t)$, Noor (2009) shows that for small rewards, the money discount function $\phi^m(t) = \psi(m,t)/m$ can be matched to a hyperbolic discount function by varying the standard discount factor δ and parameters of a concave utility function u (u is assumed to be a CARA utility function in Noor (2009)).

We depart from Noor (2009) by introducing uncertainty into the standard exponential DU model. Our starting point is that an individual perceives the future – hence the utility of consumption – as uncertain. To be more specific, in this paper, we assume that the base consumption stream, $\{b_t\}_{t\geq 0}$, is stochastic. In general, the uncertainty may be caused by changes both in the anticipated base consumption stream and/or utility function *p*er se: obviously, the satisfaction from possession of a certain widget may change (and typically, changes) in a not completely predictable fashion. Similar ideas are used in Faruk Gul and Wolfgang Pesendorfer (2005) ("changing tastes") and Paola Manzini and Marco Mariotti (2006) ("the perception of future events becomes increasingly "blurred" as the events are pushed further in time"), among the others. Partha Dasgupta and Eric Maskin (2005) show that if the "average" situation entails some uncertainty about the time when payoffs are realized, the corresponding preferences may well entail hyperbolic discounting. Peter D. Sozou (1998) derives the hyperbolic discounting from the Bayesian updating of the beliefs about the distribution of the random discount rate. Arthur J. Robson and Larry Samuelson (2009) demonstrate that aggregate uncertainty concerning survival rates can lead to non-exponential discount rates.

Consider, first, the case of gains (rewards). Suppose that, at time 0, the agent is asked to compare dated payoffs (m',t) and $(m,t+T)$, where $t \geq 0$ and $T > 0$. The agent evaluates consumption streams using the standard expected discounted utility model:

$$V(b_0, \ldots, b_t, \ldots, b_{t+T}) = E\left[\sum_{\tau=0}^{t+T} \delta^\tau u(b_\tau)\right],$$

where $\delta \in (0,1)$ is the discount factor, and E is the expectation operator. As in Noor (2009), we assume that both rewards m and m' are small, so that the agent does not consider spreading any of the rewards over time. Then $(m,t+T) \succeq (m',t)$ iff

$$V(b_0, \ldots, b_t, \ldots b_{t+T} + m) - V(b_0, \ldots, b_t + m', \ldots b_{t+T}) \geq 0,$$

equivalently, iff

$$E\left[\sum_{\tau=0}^{t-1}\delta^\tau u(b_\tau)\right]+\delta^t E\left[u(b_t)+\sum_{\tau=t+1}^{t+T-1}\delta^{\tau-t}u(b_\tau)+\delta^T u(b_{t+T}+m)\right]-$$
$$-E\left[\sum_{\tau=0}^{t-1}\delta^\tau u(b_\tau)\right]-\delta^t E\left[u(b_t+m')+\sum_{\tau=t+1}^{t+T-1}\delta^{\tau-t}u(b_\tau)+\delta^T u(b_{t+T})\right]\ge 0,$$

equivalently, iff

$$G(T,m;m';b_t):=\delta^T E[u(b_{t+T}+m)-u(b_{t+T})]-E[u(b_t+m')-u(b_t)]\ge 0. \tag{6}$$

Now, let the agent be asked to compare dated losses (m',t) and $(m,t+T)$, where $T>0$. Then $(m,t+T)\succeq(m',t)$ iff

$$V(b_0,\dots,b_t,\dots b_{t+T}-m)-V(b_0,\dots,b_t-m',\dots b_{t+T})\ge 0,$$

equivalently, iff

$$E\left[\sum_{\tau=0}^{t-1}\delta^\tau u(b_\tau)\right]+\delta^t E\left[u(b_t)+\sum_{\tau=t+1}^{t+T-1}\delta^{\tau-t}u(b_\tau)+\delta^T u(b_{t+T}-m)\right]-$$
$$-E\left[\sum_{\tau=0}^{t-1}\delta^\tau u(b_\tau)\right]-\delta^t E\left[u(b_t-m')+\sum_{\tau=t+1}^{t+T-1}\delta^{\tau-t}u(b_\tau)+\delta^T u(b_{t+T})\right]\ge 0,$$

equivalently, iff

$$L(T,m;m';b_t):=\delta^T E[u(b_{t+T}-m)-u(b_{t+T})]-E[u(b_t-m')-u(b_t)]\ge 0. \tag{7}$$

Both in case of gains and losses, we want to derive the relation between T,m,m' and b_t, which makes the agent indifferent between the dated payoffs (losses) (m',t) and $(m,t+T)$. We will start with the simplest case of preferences – the exponential utility function.

3. Exponential utility

Let $u(b)=(1-e^{-ab})/a$, where $a>0$.

3.1. Discount factors for gains. We can write (6) as

$$\begin{aligned}G(T,m;m';b_t) &= \delta^T E\left[\frac{1-e^{-a(b_{t+T}+m)}-1+e^{-ab_{t+T}}}{a}\right]\\ &\quad -E\left[\frac{1-e^{-a(b_t+m')}-1+e^{-ab_t}}{a}\right]\\ &= \delta^T E\left[e^{-ab_{t+T}}\right]\frac{1-e^{-am}}{a}-E\left[e^{-ab_t}\right]\frac{1-e^{-am'}}{a}\\ &= \delta^T E\left[e^{-ab_{t+T}}\right]u(m)-E\left[e^{-ab_t}\right]u(m').\end{aligned}$$

Letting $t = 0$, we derive that $G(T, m; m'; b_0) \geq 0$ iff

$$e^{ab_0} \geq \frac{1}{\delta^T E\left[e^{-ab_T}\right]} \cdot \frac{u(m')}{u(m)},$$

equivalently, iff $b_0 \geq K$, where

$$K = \frac{1}{a} \ln \left\{ \frac{1}{\delta^T E\left[e^{-ab_T}\right]} \cdot \frac{u(m')}{u(m)} \right\}.$$

We conclude that $(m', 0) \succeq (m, T)$ if and only if $b_0 \leq K$, i.e, relatively poor agents prefer immediate gratification.

To determine the present equivalent of $(m, t+T)$ at $t \geq 0$, as viewed from date 0, we need to find (m', t) such that $G(T, m; m'; b_t) = 0$. Set

$$P(t, t+T) = \delta^T \frac{E[u'(b_{t+T})]}{E[u'(b_t)]} = \delta^T \frac{E\left[e^{-ab_{t+T}}\right]}{E\left[e^{-ab_t}\right]}. \tag{8}$$

The function $P(t, t+T)$ is the marginal rate of substitution between consumption at t and $t+T$ as perceived at time $t = 0$. We will see it later, that in fact, $P(t, t+T)$ is the discount function if the size of the rewards is small. We have $G(T, m; m'; b_t) = 0$ iff

$$u(m') = P(t, t+T)u(m) \ \Leftrightarrow\ 1 - e^{-am'} = P(t, t+T)(1 - e^{-am}). \tag{9}$$

It is natural to assume that $m' < m$, and, since the utility function is increasing, we also have to assume that

$$P(t, t+T) < 1. \tag{10}$$

Let $m'_g = m'_g(t, T, m, b_t)$ be a (unique) solution to (9), then $(m'_g, t) \sim (m, t+T)$; set $M_g = e^{am'_g}$ and $M = e^{am}$. We find M_g from (9):

$$1 - M_g^{-1} = P(t, t+T)(1 - M^{-1}),$$

equivalently,

$$M_g = [1 - P(t, t+T)(1 - 1/M)]^{-1} \tag{11}$$

and

$$m'_g = -\frac{1}{a} \ln[1 - P(t, t+T)(1 - 1/M)]. \tag{12}$$

So, the money discount factor for gains is

$$\mathcal{D}_g(t, T; m) := \frac{m'_g}{m} = -\frac{1}{am} \ln[1 - P(t, t+T)(1 - e^{-am})]. \tag{13}$$

It remains to notice that the discount factor and the present equivalent above are well-defined due to (10).

3.2. **Discount factors for losses.** For the exponential utility function, we write (7) as

$$\begin{aligned} L(T,m;m';b_t) &= \delta^T E\left[\frac{1-e^{-a(b_{t+T}-m)}-1+e^{-ab_{t+T}}}{a}\right] \\ &\quad -E\left[\frac{1-e^{-a(b_t-m')}-1+e^{-ab_t}}{a}\right] \\ &= \delta^T E\left[e^{-ab_{t+T}}\right]\frac{1-e^{am}}{a} - E\left[e^{-ab_t}\right]\frac{1-e^{am'}}{a} \\ &= \delta^T E\left[e^{-ab_{t+T}}\right]u(-m) - E\left[e^{-ab_t}\right]u(-m'). \end{aligned}$$

Letting $t = 0$, we derive that $L(T, m; m'; b_0) \geq 0$ iff

$$e^{ab_0} \leq \frac{1}{\delta^T E\left[e^{-ab_T}\right]} \cdot \frac{u(-m')}{u(-m)},$$

equivalently, iff $b_0 \leq K_l$, where

$$K_l = \frac{1}{a}\ln\left\{\frac{1}{\delta^T E\left[e^{-ab_T}\right]} \cdot \frac{u(-m')}{u(-m)}\right\}.$$

We conclude that $(m', 0) \succeq (m, T)$ if and only if $b_0 \geq K_l$, i.e, relatively rich agents prefer to expedite the loss.

To determine the present equivalent of $(m, t+T)$ at $t \geq 0$, as viewed from date 0, we need to find (m', t) such that $L(T, m; m'; b_t) = 0$. Evidently, the agent is indifferent between (m', t) and $(m, t+T)$ iff

$$u(-m') = P(t,t+T)u(-m) \;\Leftrightarrow\; 1-e^{am'} = P(t,t+T)(1-e^{am}). \tag{14}$$

Let $m'_l = m'_l(t, T, m, b_t)$ be a (unique) solution to (14), then $(m'_l, t) \sim (m, t+T)$; set $M_l = e^{am'_l}$. We find M_l from (14):

$$M_l = 1 - P(t,t+T)(1-M) = 1 + P(t,t+T)(M-1), \tag{15}$$

and

$$m'_l = \frac{1}{a}\ln[1 + P(t,t+T)(M-1)]. \tag{16}$$

So, the money discount factor for losses is

$$\mathcal{D}_l(t,T;m) := \frac{m'_l}{m} = \frac{1}{am}\ln[1+P(t,t+T)(e^{am}-1)]. \tag{17}$$

3.3. **Gain-loss asymmetry.** It is easy to see that $\mathcal{D}_l(t,T;m) > \mathcal{D}_g(t,T;m)$ for some m if and only if $P = P(t,t+T) < 1$, and then $\mathcal{D}_l(t,T;m) > \mathcal{D}_g(t,T;m)$ for all m. Indeed,

$$\begin{aligned}
\mathcal{D}_l > \mathcal{D}_g \;\Leftrightarrow\; & \quad 1 + P(M-1) > \frac{1}{1 - P(1-1/M)} \\
\Leftrightarrow\; & \quad (1+P(M-1))(M - P(M-1)) > M \\
\Leftrightarrow\; & \quad M + P(M-1)M - P(M-1) - P^2(M-1)^2 > M \\
\Leftrightarrow\; & \quad (P - P^2)(M-1)^2 > 0 \;\Leftrightarrow\; P \in (0,1).
\end{aligned}$$

On the strength of condition (10), gains are discounted more than losses and the delay-speedup asymmetry follows immediately.

3.4. **Preference reversal.** Suppose that the agent is asked to compare two pairs of dated payoffs: $(m',0)$ vs. (m,T) and (m',t) vs. $(m,t+T)$. If the agent's preferences are $(m',0) \succ (m,T)$ and $(m,t+T) \succ (m',t)$, then we have the so-called preference reversal, or decreasing impatience. Let $(m_g^0, 0) \sim (m,T)$, and $(m_g^t, t) \sim (m, t+T)$. Then if the preference reversal is observed, we must have $m_g^0 < m' < m_g^t$. A sufficient condition is: m_g^t, the present equivalent at $t \geq 0$ as viewed from date 0, is an increasing function of t. This condition is necessary if we want to model the preference reversal between any two dates $0 \leq t' < t$, not only between 0 and t. In particular, it is clear from (12) that the preference reversal will be observed if $P(t,t+T)$ is an increasing function of t.

3.5. **Effective discount rates and hyperbolic discounting.** As in Noor (2009), we define the discount function as the limit of the money discount factor when the size of the reward vanishes. If $m > 0$ is small, we can study the shapes of the effective discount rate curves using linear approximations

$$\mathcal{D}_g(t,T;m) \sim -\frac{1}{am} \ln\left[1 - P(t,t+T)am\left(1 - \frac{am}{2}\right)\right], \tag{18}$$

$$\sim P(t,t+T)\left(1 - \frac{am}{2}(1 - P(t,t+T))\right), \tag{19}$$

$$\mathcal{D}_l(t,T;m) \sim \frac{1}{am} \ln\left[1 + P(t,t+T)am\left(1 + \frac{am}{2}\right)\right], \tag{20}$$

$$\sim P(t,t+T)\left(1 + \frac{am}{2}(1 - P(t,t+T))\right). \tag{21}$$

It follows from (19) and (21) that

$$\lim_{m\to 0} \mathcal{D}_g(t,T;m) = \lim_{m \to 0} \mathcal{D}_l(t,T;m) = P(t,t+T),$$

therefore $P(t,t+T)$ is the discount function for gains and losses. Considering the continuous time limit of the discrete time model, and assuming that $P(t,t+T)$

is differentiable at $T = 0$, we define the effective discount rate as:

$$\rho(t) = -\frac{\partial}{\partial T} \ln P(t, t+T)\Big|_{T=0} = -\frac{\partial}{\partial T} P(t, t+T)\Big|_{T=0}. \tag{22}$$

By definition, hyperbolic discounting means that $\rho(t)$ is a decreasing function of t. On the strength of definition (8) of $P(t, t+T)$ and (22) of $\rho(t)$, the behavior of the effective discount rate depends on the agent's current income, b_0, and specification of uncertainty.

In the standard models of uncertainty, it is convenient to work with the moment-generating function of the random variable b_t:

$$\mathrm{MGF}(b, t, \gamma) = E\left[e^{\gamma b_t} | b_0 = b\right].$$

Set $\Xi(b, t, \gamma) = \ln \mathrm{MGF}(b, t, \gamma)$ (the logarithm of the moment-generating function is called the cumulant-generating function) and $r = -\ln \delta$. Then

$$P(t, t+T) = \exp[-rT + \Xi(b_0, t+T, -a) - \Xi(b_0, t, -a)],$$

and

$$\rho(a, b_0; t) = r - \frac{\partial}{\partial T}\left(\Xi(b_0, t+T, -a) - \Xi(b_0, t, -a)\right)\Big|_{T=0}$$

Simplifying,

$$\rho(a, b_0; t) = r - \Xi_t(b_0, t, -a), \tag{23}$$

where $\Xi_t = \partial \Xi(\cdot, t, \cdot)/\partial t$. Hence, ρ is a decreasing function in t if Ξ_t is an increasing function in t, i.e., if Ξ is a (strictly) convex function in t. Notice that r represents the standard discount rate, and $-\Xi_t(b_0, t, -a)$ is the idiosyncratic discount rate that depends on the agent's current base consumption level, risk attitude and the underlying uncertainty. To avoid negative discounting, we need $\Xi_t(b_0, t, -a) < r$, which is satisfied automatically if Ξ is a decreasing function in t.

3.6. **The Brownian motion model.** First, we notice that if the agent perceives the evolution of her income as a process with i.i.d. increments (the leading example being the BM), then her effective discount rate is independent of time, because for such a process,

$$\Xi(b, t, \gamma) = \gamma b + t\Psi(\gamma),$$

where $\Psi(\gamma)$ is the so-called Lévy exponent. In particular, if $\{b_t\}$ is modeled as the BM with the drift μ and variance σ^2, then b_t is given by the following stochastic differential equation:

$$db_t = \mu dt + \sigma dW_t,$$

where dW_t is the increment of the standard BM with zero mean and unit variance; and $\Psi(\gamma) = \mu\gamma + \sigma^2\gamma^2/2$. In the case of a process with i.i.d. increments, we have

$$\rho(a, b_0; t) = r - \Psi(-a).$$

The effective discount rate is independent of t and the current consumption level, the discounting is exponential, but we observe that the effective discount rate depends on the agent's parameter of absolute risk aversion a. In particular, for the BM, $\rho(a) = r + \mu a - a^2\sigma^2/2$. If the drift $\mu \le 0$, then the effective discount rate decreases when the absolute risk aversion increases. If the drift $\mu > 0$, then ρ increases in a if $0 < a < \mu/\sigma^2$ and decreases in a if $\mu/\sigma^2 < a$. To avoid negative discounting, we must have $r + \mu a - a^2\sigma^2/2 > 0$. Since $a > 0$ the condition on a is $0 < a < \frac{\mu+\sqrt{\mu^2+2\sigma^2 r}}{\sigma^2}$.

In order to observe hyperbolic discounting and other related DU anomalies, the agent has to perceive her income as a mean-reverting process.

3.7. **Square root model.** Since the base consumption stream cannot be negative, we will use the square-root process on R_+ to model the evolution of b_t. This process is also known as the CIR process, because Cox, Ingersoll, and Ross (1985) used it to model the production process of the economy and derive the term structure of interest rates. To be more specific, b_t is given by the following stochastic differential equation:

$$db_t = \kappa(\theta - b_t)dt + \sigma\sqrt{b_t}dW_t, \tag{24}$$

where dW_t is the increment of the standard BM with zero mean and unit variance, and $\sigma, \theta, \kappa > 0$. Parameter θ represents the long-run (or central tendency) base consumption level of the agent, κ characterizes the rate of mean reversion, and σ is the volatility of the process. In the Appendix, we show that, in the case of the CIR process, the moment-generating function of b_t is an exponential of an affine function of b with the coefficients depending on t and γ (this fact is well-known, and similar formulas exist for numerous more complex situations but we were unable to find explicit formulas for this simple case in the literature):

$$\mathrm{MGF}(b, t, \gamma) = \exp \Xi(b, t, \gamma),$$

where

$$\Xi(b, t, \gamma) = A(t;\gamma)b + B(t;\gamma), \tag{25}$$

$$A(t;\gamma) = \frac{2\kappa}{\sigma^2(1 - C(\gamma)e^{\kappa t})} \tag{26}$$

$$B(t;\gamma) = \frac{2\kappa\theta}{\sigma^2} \ln\left|\frac{C(\gamma) - 1}{C(\gamma) - e^{-\kappa t}}\right|; \tag{27}$$

and

$$C(\gamma) = 1 - 2\kappa/(\gamma\sigma^2).$$

These formulas are valid for $\gamma < 0$, hence, for $\gamma = -a$. Now we can write the effective discount rate as

$$\begin{aligned}(28)\qquad \rho(a, b_0; t) &= r - \left.\frac{\partial A(t+T; -a)}{\partial T}\right|_{T=0} b_0 - \left.\frac{\partial B(t+T; -a)}{\partial T}\right|_{T=0} \\ &= r - A_t(t; -a) b_0 - B_t(t; -a),\end{aligned}$$

where $A_t = \partial A(t, \cdot)/\partial t$ and $B_t = \partial B(t, \cdot)/\partial t$. To satisfy (10), we must have

$$r - b_0 \frac{A(t+T, -a) - A(t, -a)}{T} - \frac{B(t+T, -a) - B(t, -a)}{T} > 0.$$

It follows from (26) and (27) that the last condition is satisfied for large T. For small T, (10) holds if $\rho(a, b_0; t) > 0$.

Theorem 1. *a) If $b_0 \le \frac{C(-a)-1}{C(-a)+1}\theta$, then $\rho(a, b_0; t)$ is a decreasing function of t on $[0, +\infty)$, and the hyperbolic discounting is observed for all t.*
b) If $\frac{C(-a)-1}{C(-a)+1}\theta < b_0 < \theta$, then $\rho(a, b_0; t)$ is an increasing function of t on $[0, t^*]$, where $t^* = \ln[(\theta + b_0)/((\theta - b_0)C(-a))]/\kappa$, and a decreasing function on $[t^*, +\infty)$. Hence, the hyperbolic discounting is observed on $[t^*, +\infty)$ only.
c) If $b_0 \ge \theta$, then $\rho(a, b_0; t)$ is an increasing function of $t \in [0, +\infty)$, and the hyperbolic discounting is never observed.

See the Appendix for the proof. Theorem 1 tells us, in particular, that agents, whose current base consumption level is sufficiently lower than their long run level, exhibit hyperbolic discounting. Since experiment participants are often college students, no wonder the hyperbolic discounting is observed. If the agent is sufficiently rich, then the effective discount rate is an increasing function in t, which reflects the rates for corporate borrowing (short term borrowing rates are smaller than long term ones). In the case of a moderately rich agent, the effective discount rate curve is hump-shaped. Notice that the normal yield curve in bond markets and forward curves in commodities markets are as in c); a downward sloping forward curve, as in an inverted yield curve, which corresponds to the hyperbolic discounting, and humped yield curves are also observed in real markets.

Similar result holds for the case when the base consumption level follows the Ornstein-Uhlenbeck process (see Section 5 for details), provided that the consumption level does not become negative.

Corollary 2. *The effective discount rate $\rho(a, b_0; t) > 0$ iff*

$$(29)\qquad r + \frac{2\kappa^2}{\sigma^2} \cdot \frac{1}{(C(-a)e^{\kappa t} - 1)^2} \left(C(-a)e^{\kappa t}(\theta - b_0) - \theta\right) > 0.$$

If $b_0 < (C(-a) - e^{-\kappa t})\theta/C(-a)$, then the idiosyncratic discount rate is positive, and (29) holds even if the standard discount rate $r = 0$. In particular, the idiosyncratic discount rate is positive if the discounting is hyperbolic for all t (case *a) in Theorem 1). If* $b_0 \geq (C(-a) - e^{-\kappa t})\theta / C(-a)$ *then the standard discount rate must be positive for* (29) *to hold.*

4. General utility and uncertainty

Let $u(\cdot) \in C^2$ be increasing and strictly concave.

4.1. **Discount factors for gains.** It is impossible to derive the exact relation between T, m, m' and b_t, which makes the agent indifferent between the dated payoffs (m', t) and $(m, t+T)$, using (6) in a simple algebraic form. Instead, we derive an approximation assuming that both m and m' are small relative to the current consumption level.

Using the Taylor expansion of order 2, we obtain the following approximation

$$G(T; m; m'; b_t) = \delta^T \left\{ mE[u'(b_{t+T})] + \frac{m^2}{2} E[u''(b_{t+T})] \right\} - m'E[u'(b_t)] - \frac{(m')^2}{2} E[u''(b_t)], \tag{30}$$

whence $G(T; m; m'; b_t) \geq 0$ iff

$$m' - (m')^2 \frac{E[-u''(b_t)]}{2E[u'(b_t)]} \leq m\delta^T \frac{E[u'(b_{t+T})]}{E[u'(b_t)]} - m^2 \delta^T \frac{E[-u''(b_{t+T})]}{2E[u'(b_t)]}.$$

As in Section 3, we set

$$P(t, t+T) = \delta^T \frac{E[u'(b_{t+T})]}{E[u'(b_t)]}$$

and assume that $P(t, t+T) < 1$ (i.e., condition (10) holds). We want to find m'_g such that $G(T; m; m'_g; b_t) = 0$. It follows from (30) that m'_g is a solution of the quadratic equation

$$\delta^T \left\{ mE[u'(b_{t+T})] - \frac{m^2}{2} E[-u''(b_{t+T})] \right\} - -m'_g E[u'(b_t)] + \frac{(m'_g)^2}{2} E[-u''(b_t)] = 0. \tag{31}$$

Since both m, m'_g are small, the linear approximation to $m'_g = m'_g(m)$ is

$$m'_g = m\delta^T \frac{E[u'(b_{t+T})]}{E[u'(b_t)]} = mP(t, t+T). \tag{32}$$

Taking square of (32) and substituting the result for the factor $(m'_g)^2$ in the last term of (31), we find the second order approximation to $m'_g = m'_g(m)$: modulo $o(m^2)$ term,

$$m'_g = mP(t, t+T) \left[1 + \frac{m}{2} \left(P(t, t+T) \frac{E[-u''(b_t)]}{E[u'(b_t)]} - \frac{E[-u''(b_{t+T})]}{E[u'(b_{t+T})]} \right) \right]. \tag{33}$$

Assuming that

$$\frac{E[-u''(b_{t+T})]}{E[u'(b_{t+T})]} \ge \frac{E[-u''(b_t)]}{E[u'(b_t)]}, \tag{34}$$

and taking into account the standing assumption $P(t, t+T) < 1$, we obtain that the coefficient at m inside the square brackets is negative.

It follows from (33) that, modulo $o(m^2)$ term, the money discount factor $\mathcal{D}_g(t, T; m) = m'_g/m$ for gains is

$$\mathcal{D}_g(t, T; m) = P(t, t+T)\left[1 + \frac{m}{2}\left(P(t, t+T)\frac{E[-u''(b_t)]}{E[u'(b_t)]} - \frac{E[-u''(b_{t+T})]}{E[u'(b_{t+T})]}\right)\right]. \tag{35}$$

hence, $P(t, t+T) = \lim_{m\to 0} \mathcal{D}_g(t, T; m)$ is the discount function.

4.2. **Discount factors for losses.** As before, we want to derive the relation between T, m, m' and b_t, which makes the agent indifferent between the dated losses (m', t) and $(m, t+T)$. Using the Taylor expansion of order 2, we obtain the following approximation

$$\begin{aligned} L(T; m; m'; b_t) \;&=\; \delta^T\left\{-mE[u'(b_{t+T})] + \frac{m^2}{2}E[u''(b_{t+T})]\right\} \\ &\quad + m'E[u'(b_t)] - \frac{(m')^2}{2}E[u''(b_t)], \end{aligned} \tag{36}$$

whence $L(T; m; m'; b_t) \ge 0$ iff

$$m' + (m')^2\frac{E[-u''(b_t)]}{2E[u'(b_t)]} \ge P(t, t+T)m + m^2\delta^T\frac{E[-u''(b_{t+T})]}{2E[u'(b_t)]}.$$

We want to find m'_l such that $L(T; m; m'_l; b_t) = 0$. It follows from (36) that m'_l is a solution of the quadratic equation

$$\begin{aligned} &-\delta^T\left\{mE[u'(b_{t+T})] + \frac{m^2}{2}E[-u''(b_{t+T})]\right\} + \\ &\quad + m'E[u'(b_t)] + \frac{(m')^2}{2}E[-u''(b_t)] = 0. \end{aligned} \tag{37}$$

The change of sign $m, m' \mapsto -m, -m'$ turns (31) into (37), therefore, (33) becomes

$$m'_l = mP(t, t+T)\left[1 + \frac{m}{2}\left(-P(t, t+T)\frac{E[-u''(b_t)]}{E[u'(b_t)]} + \frac{E[-u''(b_{t+T})]}{E[u'(b_{t+T})]}\right)\right]. \tag{38}$$

Assuming that (34) holds and $P(t, t+T) < 1$, we obtain that the coefficient at m inside the square brackets is positive.

It follows from (38) that, modulo $o(m^2)$ term, the money discount factor $\mathcal{D}_l(t,T;m) = m'_l/m$ for losses is
(39)
$$\mathcal{D}_l(t,T;m) = P(t,t+T)\left[1 - \frac{m}{2}\left(P(t,t+T)\frac{E[-u''(b_t)]}{E[u'(b_t)]} - \frac{E[-u''(b_{t+T})]}{E[u'(b_{t+T})]}\right)\right].$$

4.3. Normal term structure of absolute risk aversion, gain-loss asymmetry and delay-speedup asymmetry.

We call the mapping

$$t \mapsto E[-u''(b_t)]/E[u'(b_t)]$$

the term structure of absolute risk aversion. If (34) holds for all t, T, then the term structure of absolute risk aversion will be called normal (equivalently, the term structure is non-decreasing function in t). If the term structure is normal and $P(t,t+T) < 1$, then it follows from (33) and (38), that gains are discounted more than losses and the delay-speedup asymmetry follows.

5. CRRA utility function

Let $u(b) = \frac{b^{1-\alpha}-1}{1-\alpha}$, where $\alpha \in (0,\infty)$. In this section, we model the stochastic income as a geometric stochastic process: $b_t = e^{X_t}$. The moment-generating function of X_t is

$$\mathrm{MGF}(x,t,\gamma) = E\left[e^{\gamma X_t}\middle| X_0 = x = \ln b\right].$$

Set $\Xi(x,t,\gamma) = \ln \mathrm{MGF}(x,t,\gamma)$. Condition (34) can be written in terms of $\Xi(x,t,\gamma)$ as

(40) $$\Xi(x_0,t+T,-(\alpha+1)) - \Xi(x_0,t,-(\alpha+1)) \geq \Xi(x_0,t+T,-\alpha) - \Xi(x_0,t,-\alpha).$$

As it was shown in Section 4, the discount function is

$$P(t,t+T) = \delta^T \frac{E[u'(b_{t+T})]}{E[u'(b_t)]} = \delta^T \frac{E[b_{t+T}^{-\alpha}]}{E[b_t^{-\alpha}]} = \delta^T \frac{E[e^{-\alpha X_{t+T}}]}{E[e^{-\alpha X_t}]}.$$

Now we can use the definition (22) of the effective discount rate to find

(41) $$\rho(\alpha,x_0;t) = r - \Xi_t(x_0,t,-\alpha),$$

where $x_0 = \ln b_0$. As in Section 3, we see that the hyperbolic discounting is observed iff $\Xi_t(\cdot,t,\cdot)$ is an increasing function in t, i.e. iff $\Xi(\cdot,t,\cdot)$ is a convex function in t.

5.1. **The geometric Brownian motion model.** Let $\{b_t\}$ follow the GBM, i.e., b_t is given by the following stochastic differential equation:

$$db_t = \mu b_t dt + \sigma^2 b_t dW_t,$$

where where dW_t is the increment of the standard BM with zero mean and unit variance; and μ and σ^2 are, respectively, the drift and variance of the GBM. Then $X_t = \ln b_t$ is the BM with the drift $\mu - \sigma^2/2$ and variance σ^2. We have $\Xi(x,t,\gamma) = \gamma x + t\Psi(\gamma)$, where $\Psi(\gamma) = \gamma(\mu - \sigma^2/2) + \gamma^2\sigma^2/2$, therefore $\rho(\alpha, x_0, t) = r - \Psi(-\alpha)$ is independent of t and x_0. Thus, the discounting is exponential, however, we may observe the gain-loss asymmetry. By definition,

$$P(t, t+T) = \delta^T \frac{E[e^{-\alpha X_{t+T}}]}{E[e^{-\alpha X_t}]} = \delta^T \frac{e^{-\alpha x_0 + (t+T)\Psi(-\alpha)}}{e^{-\alpha x_0 + t\Psi(-\alpha)}} = \delta^T e^{T\Psi(-\alpha)} = e^{-T(r - \Psi(-\alpha))}.$$

Hence, condition (10) becomes $r - \Psi(-\alpha) > 0$. The term structure of absolute risk aversion is

$$\frac{E[-u''(b_t)]}{E[u'(b_t)]} = \alpha e^{t(\Psi(-\alpha-1) - \Psi(-\alpha))},$$

therefore, it is normal iff

$$\Psi(-\alpha - 1) \ge \Psi(-\alpha). \tag{42}$$

We conclude that the money discount factors for gains and losses are given by

$$\begin{aligned} \mathcal{D}_g(t, T :, m) &= \delta^T e^{T\Psi(-\alpha)} \left\{ 1 - \frac{m}{b_0}\frac{\alpha}{2} e^{(t+T)(\Psi(-\alpha-1) - \Psi(-\alpha))} \right. \\ &\quad \left. \times \left(1 - \delta^T e^{T(2\Psi(-\alpha) - \Psi(-\alpha-1))}\right) \right\}, \end{aligned} \tag{43}$$

$$\begin{aligned} \mathcal{D}_l(t, T :, m) &= \delta^T e^{T\Psi(-\alpha)} \left\{ 1 + \frac{m}{b_0}\frac{\alpha}{2} e^{(t+T)(\Psi(-\alpha-1) - \Psi(-\alpha))} \right. \\ &\quad \left. \times \left(1 - \delta^T e^{T(2\Psi(-\alpha) - \Psi(-\alpha-1))}\right) \right\}. \end{aligned} \tag{44}$$

Thus, we observe that the money discount factor for gains increases in the current consumption level b_0, and the money discount factor for losses decreases in b_0. Hence, rich people discount gains less and losses more than poor people. In particular, fairly rich agents do not exhibit the negative discounting in this model. Indeed, the negative discounting effect may be observed for small T, if $r - \Psi(-\alpha)$ is very close to 0 so that

$$-r + \Psi(-\alpha) + \frac{m}{b_0}\frac{\alpha}{2} e^{t(\Psi(-\alpha-1) - \Psi(-\alpha))}(r + \Psi(-\alpha - 1) - 2\Psi(-\alpha)) > 0. \tag{45}$$

For t in a finite interval $[0, \bar{t}]$, there exists b^* such that if $b_0 > b^*$, then the second term in (45) is less than $r - \Psi(-\alpha)$. Notice that for very large t, the probability that condition $m' << b_t$ will be violated is large, hence, for large t, the quadratic approximation we started with and resulting approximate formulas are too inaccurate and should not be applied.

It remains to analyze the normality condition (42). Assume that the agent perceives the utility of consumption as being the same on average, i.e. as a martingale, then for any t,

$$E[u(b_t)] = u(b_0) \;\Leftrightarrow\; E\left[e^{(1-\alpha)X_t}\right] = b_0^{1-\alpha} \;\Leftrightarrow\; b_0^{1-\alpha}e^{t\Psi(1-\alpha)} = b_0^{1-\alpha} \;\Leftrightarrow\; \Psi(1-\alpha) = 0.$$

Since $\Psi(0) = 0$, and $\Psi(\cdot)$ is a convex function, condition $\Psi(1-\alpha) = 0$ implies $\Psi(-\alpha-1) > \Psi(-\alpha)$. Notice, however, that (42) holds under much weaker conditions on the process. Straightforward calculations show that (42) is equivalent to $\mu \le \sigma^2(1+\alpha)$. One should expect that this condition holds. For instance, for stock and indices on stocks, typically, $\mu < \sigma^2$.

5.2. **Ornstein-Uhlenbeck model.** We assume that X_t is given by the stochastic differential equation

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t, \tag{46}$$

where $\kappa > 0$, $\theta > 0$, and dW_t is the increment of the standard BM with zero mean and unit variance. The procedures for the calculations of the expectation $E[e^{\gamma X_t}]$ and resulting formula are well-known (see, e.g., Andrei N. Borodin and Paavo Salminen (2002), p. 522–523)

$$E[e^{\gamma X_t} \mid X_0 = x] = \exp\left[\gamma e^{-\kappa t}x + \frac{\sigma^2\gamma^2}{4\kappa}(1 - e^{-2\kappa t}) + \theta\gamma(1 - e^{-\kappa t})\right], \tag{47}$$

hence

$$\Xi(x, t, \gamma) = \gamma e^{-\kappa t}x + \frac{\sigma^2\gamma^2}{4\kappa}(1 - e^{-2\kappa t}) + \theta\gamma(1 - e^{-\kappa t}). \tag{48}$$

To satisfy condition $P(t, t+T) < 1$, we must have

$$Tr - (\Xi(x, t+T, -\alpha) - \Xi(x, t, -\alpha)) > 0,$$

equivalently,

$$r + (\theta - x)\alpha e^{-\kappa t}\frac{1 - e^{-\kappa T}}{T} - \frac{\sigma^2\alpha^2}{4k}e^{-2\kappa t}\frac{1 - e^{-2\kappa T}}{T} > 0. \tag{49}$$

The LHS in (49) converges to $r > 0$ when $T \to \infty$, hence, (49) is satisfied for T sufficiently large. If T is small, then, to avoid negative discounting, the effective discount rate must be positive. Similarly to (28), the effective discount rate in the OU model is

$$\rho(\alpha, x_0; t) = r - A_t(t; -\alpha)x_0 - B_t(t; -\alpha), \tag{50}$$

where

$$\begin{aligned} A_t(t; -\alpha) &= \alpha\kappa e^{-\kappa t}, \\ B_t(t, -\alpha) &= \frac{\sigma^2\alpha^2}{2}e^{-2\kappa t} - \alpha\theta\kappa e^{-\kappa t} \end{aligned}$$

Substituting into (50), we obtain

$$\rho(\alpha, x_0; t) = r + (\theta - x_0)\alpha\kappa e^{-\kappa t} - \frac{\sigma^2\alpha^2}{2}e^{-2\kappa t}, \tag{51}$$

and, differentiating,

$$\rho_t(\alpha, x_0; t) = \alpha\kappa e^{-2\kappa t}(-(\theta - x_0)\kappa e^{\kappa t} + \sigma^2\alpha).$$

We can state the following

Theorem 3. *(i) if $x_0 \geq \theta$, then, for $t > 0$, $\rho_t(\alpha, x_0; t) > 0$, and the effective discount rate increases on $[0, +\infty)$;*

(ii) if $x_0 \leq \theta - \sigma^2\alpha/\kappa$, then, for $t > 0$, $\rho_t(\alpha, x_0; t) < 0$, and the effective discount rate decreases on $[0, +\infty)$: the hyperbolic discounting is observed;

(iii) if $\theta - \sigma^2\alpha/\kappa < x_0 < \theta$, then, for $0 < t < t^* = -(1/\kappa)\ln[(\theta - x_0)\kappa/(\sigma^2\alpha)]$, $\rho_t(\alpha, x_0; t) > 0$, hence, the effective discount rate increases on $[0, t^*]$; the hyperbolic discounting is observed on $[t^*, +\infty)$.

Corollary 4. *The effective discount rate $\rho(\alpha, x_0; t) > 0$ iff*

$$r + (\theta - x_0)\alpha\kappa e^{-\kappa t} - \frac{\sigma^2\alpha^2}{2}e^{-2\kappa t} > 0. \tag{52}$$

If $x_0 < \theta - \sigma^2\alpha e^{-kt}/(2\kappa)$, then the idiosyncratic discount rate is positive, and (52) holds even if the standard discount rate $r = 0$. In particular, the idiosyncratic discount rate is positive if the discounting is hyperbolic for all t (case (ii) in Theorem 3). If $x_0 \geq \theta - \sigma^2\alpha e^{-kt}/(2\kappa)$ then the standard discount rate must be positive for (52) to hold.

Recall that if (40) is satisfied, the gain-loss asymmetry is observed. Straightforward calculations show that for the OU process, (40) is equivalent to

$$\frac{\sigma^2(\alpha + 1)}{2\kappa}e^{-\kappa t}\left(1 - e^{-2\kappa T}\right) > (\theta - x_0)\left(1 - e^{-\kappa T}\right). \tag{53}$$

So, if $x_0 > \theta$, the gain-loss asymmetry is always observed. For small T, (53) becomes

$$\frac{\sigma^2(\alpha + 1)}{\kappa}e^{-\kappa t} > \theta - x_0.$$

In particular, poor agents who always exhibit hyperbolic discounting must not be too poor to discount gains more than losses, i.e., the following inequalities must hold:

$$\theta - \sigma^2(\alpha + 1)/\kappa < x_0 \leq \theta - \sigma^2\alpha/\kappa.$$

6. Conclusion

This paper shows that discounted utility anomalies can be explained within the standard discounted utility framework if there is uncertainty about, for example, the agent's base consumption level. We introduced the notion of the term structure of absolute risk aversion and demonstrated that, for a general utility function satisfying usual conditions, the gain-loss asymmetry and delay-speedup asymmetry for small gains and losses follow from a natural assumption that the term structure is normal, that is, non-decreasing. The gain-loss asymmetry can be observed even if the discounting is exponential. In order to observe non-exponential discounting, i.e., to have a time-dependent effective discount rate, the cumulant-generating function of the underlying stochastic variable must be a non-linear function of time. In particular, the hyperbolic discounting takes place if the cumulant-generating function is a convex function in time.

We used as model examples of stochastic base consumption level with non-linear cumulant-generating function the CIR and OU mean-reverting models. For these models, the shape of the effective discount rates depends on the current base consumption level. If the agent is rich (the current income b_0 is higher than the central tendency θ), then the effective discount rate increases in time (as the borrowing rate for a sound corporation), and no hyperbolic discounting is observed. If the agent is poor so that his current income is less than the long-run average by a certain non-zero margin (which depends on the risk attitude, type of uncertainty and the parameters of the income process): $b_0 \leq \theta_1 < \theta$, then the effective discount rate decreases with time, and the hyperbolic discounting is observed. Finally, if the agent is neither rich nor too poor: $\theta_1 < b_0 < \theta$, then there exists $t^* > 0$ such that the hyperbolic discounting is observed over the interval $[t^*, +\infty)$ but is not observed on $[0, t^*]$; as $b_0 \downarrow \theta_1$, $t^* \to +0$ (the poorer the agent becomes, the higher is the probability that the hyperbolic discounting will be observed in an experiment).

Apart from providing a robust explanation of DU anomalies without resorting to exotic time preference, our approach may potentially have other interesting applications such as, for example, contingent valuation of environmental goods. The contingent valuation method involves the use of sample surveys to elicit the willingness of respondents to pay for environmental programs or projects. For the history of the contingent valuation method and contingent valuation debate see Paul R. Portney (1994), and W. Michael Haneman (1994). According to Portney (1994), one of the most influential papers in natural resource and environmental economics was "Conservation Reconsidered" by John V. Krutilla (1967). That paper suggested that the difference between willingness-to-pay and willingness-to-accept compensation for "grand scenic wonders" may be large indeed. W. Michael Haneman (1991) presented a deterministic model that demonstrates that the differences in the willingness-to-pay and willingness-to-accept are due to the lack of substitutes for a public good. According to our results, compensation for losses requested by individuals is higher than the price the same individuals agree to pay for gains due to the presence of uncertainty. Thus, when facing a question of the sort "How much should the government pay for the damage to an endangered species", the same individual will name a greater price than when asked a question of the sort "How much should the government pay to preserve an endangered species." Long-lived environmental problems such as global warming, and native-exotic species protection are other potential applications of our results. In the context of the global warming, our model explains why it is natural that the poor countries do not want to commit themselves to costly emission reductions.

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Appendix A. Mathematical Appendix

Let b_t be given by (24). The moment-generating function $M(b, T; \gamma) = E[e^{\gamma b_T} \mid b_0 = b]$ is the time-0 value of the solution of the partial differential equation

$$(\partial_t + \kappa(\theta - b)\partial_b + \frac{\sigma^2}{2} b\partial_b^2)V(t, b; \gamma) = 0, t < T, \tag{54}$$

subject to the terminal condition $V(T, b; \gamma) = e^{\gamma b}$. We look for the solution in the form $V(t, b; \gamma) = \exp[A(\tau; \gamma)b + B(\tau; \gamma)]$, where $\tau = T - t$. Substitute into (54), perform the differentiations, multiply by $1/V(t, b; \gamma)$, and suppress for the moment the dependence on γ in the notation; the result is

$$-A'(\tau)b - B'(\tau) + \kappa(\theta - b)A(\tau) + \frac{\sigma^2}{2} bA(\tau)^2 = 0, \quad \tau > 0.$$

Considering separately terms with factors b and without these factors, we obtain the system of Riccati equations on the positive half-line

$$A'(\tau) = -\kappa A(\tau) + \frac{\sigma^2}{2} A(\tau)^2 \tag{55}$$

$$B'(\tau) = \kappa\theta A(\tau). \tag{56}$$

From the boundary condition $V(T, b; \gamma) = e^{\gamma b}$, we obtain the initial conditions for the system (55)-(56): $A(0) = \gamma, B(0) = 0$. We can find the general solution of the equation (55) by separation of variables, the result being $1 - 2\kappa/(\sigma^2 A) = Ce^{\kappa\tau}$, where $C \neq 0$ is a constant. Using $A(0; \gamma) = \gamma$, we find $C = C(\gamma) = 1 - 2\kappa/(\gamma\sigma^2)$, and, finally,

$$A(\tau; \gamma) = \frac{2\kappa}{\sigma^2(1 - C(\gamma)e^{\kappa\tau})}. \tag{57}$$

Substituting (57) into (56) and integrating, we find $B(\tau;\gamma)$:

$$B(\tau;\gamma) = \frac{2\kappa^2\theta}{\sigma^2}\int_0^\tau \frac{ds}{1-C(\gamma)e^{\kappa s}} = \frac{2\kappa\theta}{\sigma^2}\ln\left|\frac{C(\gamma)-1}{C(\gamma)-e^{-\kappa\tau}}\right|.$$

Finally, the moment-generating function

$$E[e^{\gamma b_T} \mid b_t = b] = \exp[A(\tau;\gamma)b + B(\tau;\gamma)]$$

is

$$E[e^{\gamma b_T} \mid b_t = b] = \exp\left[\frac{2\kappa}{\sigma^2}\left(\frac{b}{1-C(\gamma)e^{\kappa\tau}} + \theta\ln\left|\frac{C(\gamma)-1}{C(\gamma)-e^{-\kappa\tau}}\right|\right)\right]. \tag{58}$$

Proof of Theorem 1

We calculate

$$\begin{aligned}\frac{\partial A(t;-a)}{\partial t} &= \frac{2\kappa^2}{\sigma^2}\frac{C(-a)e^{\kappa t}}{(C(-a)e^{\kappa t}-1)^2},\\ \frac{\partial B(t;-a)}{\partial t} &= -\frac{2\kappa^2}{\sigma^2}\frac{\theta}{C(-a)e^{\kappa t}-1},\end{aligned}$$

and obtain

$$\begin{aligned}\rho(a,b_0;t) &= r - \frac{2\kappa^2}{\sigma^2}\left[\frac{C(-a)e^{\kappa t}b_0}{(C(-a)e^{\kappa t}-1)^2} - \frac{\theta}{C(-a)e^{\kappa t}-1}\right]\\ &= r + \frac{2\kappa^2}{\sigma^2}\left[-\frac{yb_0}{(y-1)^2} + \frac{\theta}{y-1}\right],\end{aligned}$$

where $y = y(t) = C(-a)e^{\kappa t}$. Clearly, $\rho'(t) < 0$ if and only if the derivative of

$$f(y) := -\frac{yb_0}{(y-1)^2} + \frac{\theta}{y-1}$$

is negative at $y = y(t)$:

$$f'(y) = -\frac{b_0(y-1-2y)}{(y-1)^3} - \frac{\theta}{(y-1)^2} = \frac{1}{(y-1)^3}(b_0(y+1) - \theta(y-1)) < 0.$$

Equivalently, $b_0(y+1) - \theta(y-1) < 0$, and finally, $b_0 < \frac{y-1}{y+1}\theta$. Similarly, one can prove the inequalities of the opposite sign. If $b_0 \geq \theta$, then $f'(y) > 0$ for all $y \geq C(-a) > 0$, and part c) follows. If $b_0 \leq \theta(C(-a)-1)/(C(-a)+1)$, then $f'(y) = 0$ at $y = C(-a)$ and positive for all $y > C(-a)$, which is equivalent to $t > 0$. This proves a). Statement b) follows straightforwardly from the fact that $f'(y) < 0$ iff $y > (b_0+\theta)/(\theta - b_0)$.