Rationalizability in general settings*

Yi-Chun Chen[a], Xiao Luo[a,†]

[a]*Department of Economics, National University of Singapore, Singapore 117570*

This version: December 2009

Abstract

We show that rationalizability in a variety of general preference models yields the unique outcome of iterated strict dominance. The result implies that rationalizable strategic behavior in these preference models is observationally indistinguishable from that in the subjective expected utility model. Our indistinguishability result can be applied not only to mixed extensions of finite games, but also to other important applications in economics, for example, the Cournot-oligopoly model. *JEL Classification:* C70, D81.

Keywords: Rationalizability; iterated dominance; general preferences

*This paper is based on part of our earlier manuscript entitled "A Unified Approach to Information, Knowledge, and Stability." Acknowledgements to be added.

†Corresponding author. Fax:+65 6775 2646. *E-mail:* johnchen@northwestern.edu (Y.-C. Chen), ecslx@nus.edu.sg (X. Luo).

1 Introduction

The subjective expected utility (SEU) model axiomatized by Savage [24] is the most popular model of preference under uncertainty used in game theory and economic theory. In the decision-theoretic approach to game theory, each player's problem of strategy choice can be cast as a single agent decision problem under uncertainty. Accordingly, a player's beliefs about the strategies played by opponents are represented by a probability measure and the player is (Bayesian) rational if the player chooses a strategy that maximizes the subjective expected utility whereby expected values are calculated by using the probability measure. Bernheim [3] and Pearce [22] proposed the game-theoretic solution concept of rationalizability as the logical implication of common knowledge of rationality in the standard expected utility model (see also Tan and Werlang [27]). In particular, the set of (correlated) rationalizable strategies can be derived from iterative deletion of never best response strategies and it is equivalent to iterated strict dominance (see Osborne and Rubinstein [21, Chapter 4]).

The Ellsberg Paradox and related experimental evidence demonstrate that a decision maker usually displays an aversion to uncertainty or ambiguity and, thereby, motivates generalizations of the subjective expected utility model. The notion of "rationality" can thus be defined as the maximization with respect to a preference ordering in an alternative model of preference, such as ordinal expected utility, probabilistically sophisticated preferences, Choquet expected utility theory or the multi-priors model. Epstein [10, Theorems 3.2 and 6.3] extended the concept of rationalizability to a variety of general preference models by characterizing rationalizability and survival of iterated deletion of never best response strategies as the (equivalent) implications of rationality and common knowledge of rationality.[1]

[1] See, e.g., Klibanoff [15], Dow and Werlang [8], Lo [16], and Marinacci [19] for generalizations

As an outside observer, one only observes the actual strategy choice but not the preferences of each player. The outstanding question is whether or not for an outside observer to distinguish Bayesian players from other types of players with general preferences. The purpose of this paper is to study the observational indistinguishability about rationalizable strategic behavior in different preference models. In the context of single-person decision making, the Ellsberg Paradox demonstrates that there do exist some observable implications of different preferences. Somewhat surprisingly perhaps, we show that rationalizability in a class of very general preference models yields the same outcome of iterated strict dominance (see Theorem 1). Since the notion of (payoff) dominance is "preference-free," this result implies that rationalizable strategic behavior in these preference models is observationally indistinguishable from that in the standard expected utility model. This paper thus provides some insight into the notion of rationalizability under general preferences in the context of strategic interaction.

To illustrate our main result in this paper, consider the following one-person game with nature (where the entries are utility payoffs and with $0 < \varepsilon < 1/2$); cf. Ghirardato and Le Breton [14, Sec. 2].

	ω_1	ω_2
f	1	0
g	0	1
h	ε	ε

Observe that in this example, if the player chooses an act which maximizes expected utility with respect to additive beliefs about states, then the player will not choose h. If, instead, the player maximizes the Choquet expected utility (CEU) or multi-prior expected utility (MPEU) of Gilboa and Schmeidler [13], then h can be optimal. Thus, there are some observable implications of different

of Nash equilibrium with general preferences.

preferences in this case; see Sec. 4 for more discussion. However, this discrepancy between Bayesian rationality and other general rationality disappears once we consider the mixed extension of this game. Since h is dominated by the mixed action $\frac{1}{2}f + \frac{1}{2}g$ (under risk neutrality), h is no longer CEU or MPEU rational in the mixed extension of the game. Intuitively, the mixed action $\frac{1}{2}f + \frac{1}{2}g$ generates strictly higher payoffs in both states and the CEU and MPEU functionals are strongly monotone. Subsequently, Bayesian rationality is observably indistinguishable from other general notions of rationality, e.g., CEU and MPEU rationality. In this paper, this kind of indistinguishability result is shown to be true in a class of games with a variety of general preference models, when payoff functions are concave-like (see Sec. 2) and preferences are strongly monotonic and compatible with the SEU model. Our indistinguishability result can be applied not only to mixed extensions of finite games (see Corollary 1), but also to many other important applications in economics such as "nice games" defined in Moulin [20], for example, the Cournot-oligopoly model (see Corollary 2).

As we emphasized above, this paper focuses on observable implications for rationalizability in strategic games under various models of preference. Our paper is therefore related to the work of Bergemann and Morris [2] regarding "strategic distinguishability and robust virtual implementation." Bergemann and Morris [2] studied the related question of "strategic revealed preference" in a different framework, and defined the notion of (rationalizable) strategic indistinguishability over different (interdependent) payoff-relevant types in an SEU environment where payoff-relevant types may not be observable. In addition, Bergemann and Morris [2] studied robust virtual implementation by the notion of strategic indistinguishability in their framework.

To conclude this introduction, we provide a perspective on the basic idea behind the observational indistinguishability result of this paper. The argument for the indistinguishability result is as follows: Under the "strong monotonicity" property of preferences (**C2**), rationalizable strategies in a preference model must survive iterated deletion of strict dominated strategies. As the subjective expected utility preferences are "admissible" in a preference model (**C1**), rationalizable strategies in the SEU model cannot be excluded by any preference model. The equivalence result between the notion of "payoff dominance" and the notion of "never-best response" in the SEU model subsequently yields the indistinguishability result. The argument for the equivalence result in our framework rests mainly on K. Fan's [12] minimax theorem (see the proof of Lemma 1 in the Appendix).

The rest of this paper is organized as follows. Section 2 lays out the set-up. Section 3 presents the main result. Section 4 offers some concluding remarks. To facilitate reading, the proofs of lemmas are relegated to the Appendix.

2 The set-up

We consider an *n-person strategic game*

$$\mathcal{G} \equiv \left(N, \{X_i\}_{i\in N}, \{\zeta_i\}_{i\in N}\right),$$

where X_i, for each player $i \in N$, is a nonempty compact Hausdorff space, and $\zeta_i : X \to [0,1]$ (where $X \equiv \times_{i\in N} X_i$ is endowed with the product topology) is a continuous payoff function that assigns each strategy profile $x \in X$ to a number in $[0,1]$ and is *concavelike* in X_i, i.e., for every x_i, $x'_i \in X_i$ and $a \in [0,1]$, there is some $x''_i \in X_i$ such that $\zeta_i(x''_i, x_{-i}) \geq a\zeta_i(x_i, x_{-i}) + (1-a)\zeta_i(x'_i, x_{-i})$ for all $x_{-i} \in X_{-i}$ (cf. Sion [26]).[2]

[2] The payoff function ζ_i is *concave* on a convex set X_i if, for every x_i, $x'_i \in X_i$ and $a \in [0,1]$, $\zeta_i(ax_i + (1-a)x'_i, x_{-i}) \geq a\zeta_i(x_i, x_{-i}) + (1-a)\zeta_i(x'_i, x_{-i})$ for all $x_{-i} \in X_{-i}$. Clearly, if ζ_i is concave on X_i, then ζ_i is concavelike in X_i.

For the purpose of this paper, we consider a "model of preference" $\mathcal{P}(\cdot) \equiv \{\mathcal{P}_i(\cdot)\}_{i \in N}$, where $\mathcal{P}_i(\cdot)$ is defined for any compact subset $Y \subseteq X$ satisfying $Y = \times_{i \in N} Y_i$, and $\mathcal{P}_i(Y)$ is interpreted as player i's admissible preferences over X_i when i faces the opponents' strategy uncertainty in Y_{-i}.[3] Let $\mathcal{P}_i(X|Y)$ denote the set of player i's conditional preferences for which the complement of Y is null in the sense of Savage [24] – i.e. any two strategies of i that yield the same payoff outcome for each profile in Y_{-i} are ranked as being indifferent. We require that there is a one-to-one correspondence between $\mathcal{P}_i(X|Y)$ and $\mathcal{P}_i(Y)$ such that any preference ordering in $\mathcal{P}_i(X|Y)$ can be regarded as a preference ordering in $\mathcal{P}_i(Y)$. We first introduce the following definition.

Definition 1. Let $Y = \times_{i \in N} Y_i$ be a nonempty subset of X.
(1.1) A strategy $y_i \in X_i$ is *strictly dominated given* Y if there exists $x_i \in X_i$ such that $\zeta_i(x_i, y_{-i}) > \zeta_i(y_i, y_{-i})$ $\forall y_{-i} \in Y_{-i}$, which is denoted by $x_i >^Y y_i$.
(1.2) A strategy $x_i \in X_i$ is i's *best* $\mathcal{P}$*-response given* Y if there exist some compact subset $\overline{Y} \subseteq Y$ and preference ordering $\succeq$ in $\mathcal{P}_i(\overline{Y})$ such that $x_i \succeq y_i$ $\forall y_i \in X_i$.

Definition 1(1.1) is a notion of "pure-strategy" dominance in the sense that a strategy can be strictly dominated only by using a pure strategy, excluding using a mixed strategy as a dominator. Throughout this paper, we impose the following two conditions on a model of preference: $\forall i \in N$ and compact subset $Y \subseteq X$ satisfying $Y = \times_{i \in N} Y_i$,

[3] In this paper we are mainly concerned with the aspect of strategic implications. For simplicity, we adopt a simple version of "model of preference;" see Epstein [10, pp. 5-7] for the more complete description of "model of preference." Note that here we do not assume that preferences have utility function representations.

C1 [**SEU Admissibility**] $\mathcal{P}_i^{SEU}(Y) \subseteq \mathcal{P}_i(Y)$, where[4]

$$\mathcal{P}_i^{SEU}(Y) \equiv \{u^{\mu} | \ \exists \mu \in \Delta(Y_{-i}), \ u^{\mu}(x_i) = \int_{Y_{-i}} \zeta_i(x_i, y_{-i})\, d\mu(y_{-i}), \forall x_i \in X_i\}.$$

C2 [**Strong Monotonicity**] For preference ordering $\succeq$ in $\mathcal{P}_i(Y)$, $x_i \succ y_i$ if $x_i >^Y y_i$, $\forall x_i, y_i \in X_i$.

The SEU Admissibility condition requires all subjective expected utility preferences to be admissible preferences. The Strong Monotonicity condition requires that a strategy is strictly preferred to another strategy if the former strategy strictly dominates the latter one. The two conditions seem to be very natural, and are satisfied by many preference models discussed in the literature, e.g., the subjective expected utility model [24], the ordinal expected utility model [5], the probabilistic sophistication model [18], the multi-priors model [13], the Choquet expected utility model [25], the lexicographic preference model [4], and so on.

We define the notion of rationalizability in a preference model $\mathcal{P}(\cdot)$ as follows:

Definition 2. A strategy profile $x^* \in X$ is *$\mathcal{P}$-rationalizable* if for each $i \in N$ there is a compact subset $Y_i \subseteq X_i$ such that

- $x_i^* \in Y_i$, and
- $\forall y_i \in Y_i$ is a best $\mathcal{P}$-response given Y, i.e., there exists a preference ordering $\succeq$ in $\mathcal{P}_i(Y)$ such that $y_i \succeq x_i \ \forall x_i \in X_i$.

Denote by $R[\mathcal{P}(\cdot)]$ the set of $\mathcal{P}$-rationalizable strategy profiles.

We also need the following definition.

[4] $\Delta(Y)$ is the set of all regular Borel probability measures endowed with the weak* topology.

Definition 3. (3.1) The *set of strategies that survive iterated deletion of strictly dominated strategies* is defined as:[5] $D^{\infty} \equiv \cap_{k=0}^{\infty} D^k$, where $D^0 = X$ and for $k \geq 1$

$$D^k = \left\{ x \in D^{k-1} |\ \forall i \in N,\ x_i \text{ is not strictly dominated given } D^{k-1} \right\}.$$

(3.2) The *set of strategies that survive iterated deletion of never-best $\mathcal{P}$-responses* is defined as: $R[\mathcal{P}(\cdot)]^{\infty} \equiv \cap_{k=0}^{\infty} R[\mathcal{P}(\cdot)]^k$, where $R[\mathcal{P}(\cdot)]^0 = X$ and for $k \geq 1$

$$R[\mathcal{P}(\cdot)]^k = \left\{ x \in R[\mathcal{P}(\cdot)]^{k-1} \mid \forall i \in N,\ x_i \text{ is a best } \mathcal{P}\text{-response given } R[\mathcal{P}(\cdot)]^{k-1} \right\}.$$

Epstein [10] characterized $\mathcal{P}$-rationalizability as the implications of rationality and common knowledge of rationality by using Epstein and Wang's [11] construction of universal type space. In particular, Epstein showed that the set of $\mathcal{P}$-rationalizable strategy profiles can be derived by iterated deletion of never-best $\mathcal{P}$-response strategies.

3 Result

The central result of this paper shows that, for any arbitrary model of preference which satisfies **C1** and **C2**, the set of $\mathcal{P}$-rationalizable strategies coincides with the "preference-free" set of strategies that survive iterated deletion of strictly dominated strategies. This indistinguishability result on rationalizability is valid for many preference models discussed in the literature, including the subjective expected utility model [24], the ordinal expected utility model [5], the probabilistic sophistication model [18], the multi-priors model [13], the Choquet expected utility model [25], and the lexicographic preference model [4]. Formally, we have

[5] See Dufwenberg and Stegeman [9] and Chen et al. [6] for extensive discussions on the iterated strict dominance in general games.

Theorem 1. *Under C1 and C2,* $R[\mathcal{P}(\cdot)] = D^{\infty}$.

To prove Theorem 1, we need the following Lemmas 1-3 (see Appendix for proofs).

Lemma 1 *Suppose that* $Y = \times_{i \in N} Y_i$ *is a nonempty compact subset of* X. *Then, any never-best* $\mathcal{P}^{SEU}$*-response given* Y *is a strictly dominated strategy given* Y.

Lemma 2 $\forall k \geq 0$, D^k *is nonempty and compact.*

Lemma 3 $D^{\infty} \subseteq R[\mathcal{P}(\cdot)] \subseteq R[\mathcal{P}(\cdot)]^{\infty}$.

Proof of Theorem 1: By Lemma 3, it suffices to show that $\cap_{k=0}^{\infty} R[\mathcal{P}(\cdot)]^k \subseteq \cap_{k=0}^{\infty} D^k$. We proceed to show $R[\mathcal{P}(\cdot)]^k \subseteq D^k$ by induction on k. Clearly, $R[\mathcal{P}(\cdot)]^0 = X = D^0$. Assume $R[\mathcal{P}(\cdot)]^{k-1} \subseteq D^{k-1}$ and prove $R[\mathcal{P}(\cdot)]^k \subseteq D^k$.

By Lemma 2, D^{k-1} is nonempty and compact. By **C2**, no strictly dominated strategy given D^{k-1} is a best $\mathcal{P}$-response strategy given $R[\mathcal{P}(\cdot)]^{k-1}$. Thus, $R[\mathcal{P}(\cdot)]^k \subseteq D^k$.■

In the framework of this paper, the notion of "dominance" in Definition 1(1.1) is defined by taking only a pure strategy as a dominator. Theorem 1 is, however, easy to be applied to the case of "dominance" by using a mixed strategy. Consider the "mixed extension" of a finite game, in which the set of strategies of each player i is the set of i's mixed strategies.[6] As the set of i's mixed strategies can be viewed as a simplex in a finite-dimensional Euclidian space, the following Corollary 1 is

[6] With non-expected utility preferences, players may strictly prefer to randomize (see, e.g., Lo [16]).

an immediate implication of Theorem 1. This corollary generalizes the elementary equivalence result between iterated strict dominance and rationalizability in the standard SEU model (see Pearce [22, Lemma 3]), as well as the equivalence result about the "uncertainty aversion" rationalizability (see Klibanoff [15, Theorem 4]).[7]

Corollary 1. *In the mixed extension of a finite game, if the preference model $\mathcal{P}(\cdot)$ satisfies C1 and C2, then the set of $\mathcal{P}$-rationalizable mixed strategy profiles coincides with the set of strategies that survive iterated deletion of strictly dominated mixed strategies.*

Our indistinguishability result in Theorem 1 can also be applied to a class of "nice games," which is used frequently in economic models. In these games, each player's strategy set is (one-dimensional) compact convex set and each player's payoff function is continuous and (strictly) concave with respect to the player's own strategy, as in Cournot competition and differentiated Bertrand competition.[8] Since every concave function is clearly "concave-like," we have the following immediate corollary of Theorem 1.

Corollary 2. *In the "nice games" where each player's strategy set is compact convex set and each player's payoff function is continuous and concave in the player's own strategy, if the preference model $\mathcal{P}(\cdot)$ satisfies C1 and C2, then the set of $\mathcal{P}$-rationalizable strategy profiles coincides with the set of strategies that survive iterated deletion of strictly dominated strategies.*

[7] We note that Corollary 1 is true for games with compact Hausdorff strategy spaces and continuous payoff functions. Moreover, Lemma 1 can be used to show that a strategy is a never-best $\mathcal{P}^{SEU}$-response iff it is dominated by a mixed strategy with finite support. In the SEU model Daniëls [7] recently presented a general equivalence result between pure-strategy dominance and never-best response for games in which payoff functions are continuous and own-quasiconcave, and strategy spaces are compact, metrizable, and convex. In this paper we do not require that strategy spaces are metrizable and convex.

[8] Moulin [20] firstly introduced the notion of "nice games." We here adopt a slight variant of "nice games" (cf. Weinstein and Yildiz [28]).

Remark. We note that, in the "strongly monotonic" preference model,[9] a strategy is a best response for some preference ordering in the "strongly monotonic" preference model if, and only if, it is not strictly dominated in the pure strategy sense. Let us say that a preference model $\mathcal{P}(\cdot)$ satisfies **C1*** if, for any $i \in N$, any strictly undominated strategy x_i^*, and any compact subset $Y \subseteq X$ satisfying $Y = \times_{i \in N} Y_i$,

$$\mathcal{P}_i(Y) \supseteq \left\{ u \middle| \; u(x_i) \equiv \min_{y_{-i} \in Y_{-i}} \left[\zeta_i(x_i, y_{-i}) - \zeta_i(x_i^*, y_{-i}) \right] \right\}.$$

We can obtain an indistinguishability result with imposing no "concave-like" condition on payoff functions: for any preference model $\mathcal{P}(\cdot)$ that satisfies **C1*** and **C2**, $\mathcal{P}$-rationality is observationally indistinguishable in the sense that a strategy is $\mathcal{P}$-rational iff it is strictly undominated. However, most of the models studied in the literature do not satisfy this condition, **C1***.

4 Concluding remarks

In this paper, we have presented an observational indistinguishability result (Theorem 1) about the rationalizable strategic behavior under general preferences, i.e., the subjective expected utility model is observationally indistinguishable from all models of preference that satisfy the conditions of SEU Admissibility (**C1**) and Strong Monotonicity (**C2**). This result is suitable for many preference models discussed in the literature, e.g., the probabilistic sophistication model, the multi-priors model, the Choquet expected utility model, the ordinal expected utility model, and the lexicographic preference model.

[9] The "strongly monotonic" preference model $\mathcal{P}^{mon}(\cdot)$ is defined as: for compact subset $Y \subseteq X$ satisfying $Y = \times_{i \in N} Y_i$, $\mathcal{P}_i^{mon}(Y) = \left\{ \succeq \in \mathcal{P}_i(Y) \,|\, x_i \succ y_i \text{ if } x_i >^Y y_i, \forall x_i, y_i \in X_i \right\}$. That is, $\mathcal{P}_i^{mon}(\cdot)$ contains *all* strongly monotonic preferences for player i; see Epstein [10, p. 16].

It is worthwhile to point out that Theorem 1 holds to be true only for games in which each player's utility function is concavelike on the player's own strategy set. In particular, the indistinguishability result is also valid in the mixed extension of a finite game (Corollary 1). However, the result need not hold in case of the finite game where only pure strategies are considered. In a finite game, Epstein [10, Sec. 4] found that the notion of "pure-strategy" dominance associated with the "rationality" is, in general, distinct and depends upon specific preference models. From the perspective of this paper, the "concave-like" assumption on payoff functions do not necessarily hold true in finite games, e.g., the payoff function ζ_i of the game in the Introduction is not concave-like in X_i (since no pure action for the player guarantees the payoff $1/2 = \frac{1}{2}\zeta_i(f,\omega) + \frac{1}{2}\zeta_i(g,\omega)$ for $\omega = \omega_1, \omega_2$, for example). Epstein observed that the rationalizable strategic behavior in the probabilistic sophistication model is indistinguishable from that in the ordinal expected utility model. In the setting of finite games, Lo [17] extended this observational indistinguishability result to all models of preference that satisfy Savage's axiom P3 of "Eventwise Monotonicity" by using Borgers's [5] notion of "pure-strategy" dominance.[10]

Finally, it is interesting to note that Bergemann and Morris [2] studied the related "(rationalizable) strategic distinguishability" in the SEU environment where payoff-relevant types may not be observable. Bergemann and Morris provided a full characterization of "strategic distinguishability" and applied it to robust virtual implementation. In this respect, this paper has presented a "(rationalizable) strategic distinguishability" result for very general preferences.

[10] In a finite game, it is easy to see that the notion of (pure-strategy) strict dominance used in this paper implies Borgers's [5] notion of "pure-strategy" dominance which implies the commonly used notion of "mixed-strategy" strict dominance. In the mixed extension of a finite game, these dominance notions become identical and, thus, the indistinguishability result presented in this paper is consistent with Lo's [17] one in this case.

Appendix

Lemma 1. *Suppose that $Y = \times_{i \in N} Y_i$ is a nonempty compact subset of X. Then, any never-best $\mathcal{P}^{SEU}$-response given Y is a strictly dominated strategy given Y.*

Proof. Let y_i be a never-best $\mathcal{P}^{SEU}$-response given Y. For all $x_i \in X_i$ and $\mu \in \Delta(Y_{-i})$, define

$$f(x_i, \mu) \equiv \int_{Y_{-i}} \left[\zeta_i(x_i, y_{-i}) - \zeta_i(y_i, y_{-i})\right] d\mu(y_{-i}).$$

By Riesz Representation Theorem and Alaoglu's Theorem, $\Delta(Y_{-i})$ is a compact Hausdorff space. Since ζ_i is concavelike on X_i, f is concavelike on X_i. Similarly, f is linear on $\Delta(Y_{-i})$, and is therefore convex on $\Delta(Y_{-i})$. Since ζ_i (jointly) continuous, $f(x_i, \mu)$ is lower semi-continuous in μ by the proof of Reny's [23] Proposition 5.1. By Fan's [12] Theorem 2, we have

$$\sup_{x_i \in X_i} \min_{\mu \in \Delta(Y_{-i})} f(x_i, \mu) = \min_{\mu \in \Delta(Y_{-i})} \sup_{x_i \in X_i} f(x_i, \mu). \tag{1}$$

Since y_i is a never-best $\mathcal{P}^{SEU}$-response given Y, $\sup_{x_i \in X_i} f(x_i, \mu) > 0$ for each μ and, hence, $\min_{\mu \in \Delta(Y_{-i})} \sup_{x_i \in X_i} f(x_i, \mu) > 0$. Therefore, by (1) there is some $x_i^* \in X_i$ such that $\min_{\mu \in \Delta(Y_{-i})} f(x_i^*, \mu) > 0$. That is,

$$\int_{Y_{-i}} \left[\zeta_i(x_i^*, y_{-i}) - \zeta_i(y_i, y_{-i})\right] d\mu(y_{-i}) > 0 \text{ for all } \mu \in \Delta(Y_{-i}).$$

Taking μ be the Dirac measure on y_{-i}, we get

$$\zeta_i(x_i^*, y_{-i}) - \zeta_i(y_i, y_{-i}) > 0 \text{ for all } y_{-i} \in Y_{-i}.$$

Hence, x_i^* strictly dominates y_i given Y.■

Lemma 2. $\forall k \geq 0$, *D^k is nonempty and compact.*

Proof. See the proof of Dufwenberg and Stegeman's [9, p.2013] Lemma.■

Lemma 3. $D^\infty \subseteq R[\mathcal{P}(\cdot)] \subseteq R[\mathcal{P}(\cdot)]^\infty$.

Proof. Suppose $y \in D^{\infty}$. By Dufwenberg and Stegeman's [9, p.2013] Theorem 1, we know y_i is not strictly dominated given D^{∞} for every i. By Lemma 2, D^{∞} is nonempty and compact. By Lemma 1, y_i is a best $\mathcal{P}^{SEU}$-response given D^{∞}. By **C1**, y_i is a best $\mathcal{P}$-response given D^{∞}. Thus, $y \in R[\mathcal{P}(\cdot)]$.

Now, suppose $y \in R[\mathcal{P}(\cdot)]$. Then, for each player i, there are compact set $Y_i \subseteq X_i$ and preference ordering $\succeq$ in $\mathcal{P}_i(Y)$ such that $y_i \in Y_i$ and, for every $y_i' \in Y_i$, $y_i' \succeq x_i$ $\forall x_i \in X_i$. Since $Y \subseteq X$ is compact, by Definition 1(1.2), we obtain that $y \in R[\mathcal{P}(\cdot)]^1$ and $Y \subseteq R[\mathcal{P}(\cdot)]^1$ and, moreover, $y \in Y \subseteq R[\mathcal{P}(\cdot)]^k$ for each $k \geq 2$. Thus, $y \in R[\mathcal{P}(\cdot)]^{\infty}$.■

References

[1] C. Aliprantis, K. Border, Infinite Dimensional Analysis, Springer-Verlag, Berlin, 1999.

[2] D. Bergemann, S. Morris, Strategic distinguishability with an application to robust virtual implementation, Yale University, Mimeo (2007).

[3] B.D. Bernheim, Rationalizable strategic behavior, Econometrica 52 (1984), 1007-1028.

[4] L. Blume, A. Brandenburger, E. Dekel, Lexicographic probabilities and choice under uncertainty, Econometrica 59 (1991), 61-79.

[5] T. Borgers, Pure strategy dominance, Econometrica 61 (1993), 423-430.

[6] Y.C. Chen, N.V. Long, X. Luo, Iterated strict dominance in general games, Games Econ. Behav. 61 (2007), 299-315.

[7] T. Daniëls, Pure strategy dominance with quasiconcave utility functions, Econ. Bulletin 3 (2008), 1-8.

[8] J. Dow and S. Werlang, Nash equilibrium under Knightian uncertainty: Breaking down backward induction, J. Econ. Theory 64 (1994), 305-324.

[9] M. Dufwenberg, M. Stegeman, Existence and uniqueness of maximal reductions under iterated strict dominance, Econometrica 70 (2002), 2007-2023.

[10] L. Epstein, Preference, rationalizability and equilibrium, J. Econ. Theory 73 (1997), 1-29.

[11] L. Epstein, T. Wang, 'Belief about belief' without probabilities, Econometrica 64 (1996), 1343-1373.

[12] K. Fan, Minimax theorems, Proc. Nat. Acad. Sci. USA 39 (1953), 42-47.

[13] I. Gilboa, D. Schmeidler, Maxmin expected utility with non-unique prior. J. Math. Econ. 18 (1989), 141-153.

[14] P. Ghirardato, M. Le Breton, Choquet rationality, J. Econ. Theory 90 (2000), 277-285.

[15] K. Klibanoff, Uncertainty, decision, and normal-form games, Northwestern University, Mimeo (1996).

[16] K.C. Lo, Equilibrium in beliefs under uncertainty, J. Econ. Theory 71 (1996), 443-484.

[17] K.C. Lo, Rationalizability and the savage axioms, Econ. Theory 15 (2000), 727-733.

[18] M. Machina, D. Schmeidler, A more robust definition of subjective probability, Econometrica 60 (1992), 745-780.

[19] M. Marinacci, Ambiguous games, Games Econ. Behav. 31 (2000), 191-219.

[20] H. Moulin, Dominance solvability and Cournot stability, Math. Soc. Sci. 7 (1984), 83-102.

[21] M.J. Osborne, A. Rubinstein, A Course in Game Theory, The MIT Press, MA, 1994.

[22] D. Pearce, Rationalizable strategic behavior and the problem of perfection, Econometrica 52 (1984), 1029-1051.

[23] P. Reny, On the existence of pure and mixed strategy Nash equilibrium in discontinuous games, Econometrica 67 (1999), 1029-1056.

[24] L. Savage, The Foundations of Statistics, Wiley, NY, 1954.

[25] D. Schmeidler, Subjective probability and expected utility without additivity, Econometrica 57 (1989), 571-587.

[26] M. Sion, On general minimax theorems, Pacific J. Math. 8 (1958), 171-176.

[27] T. Tan, S. Werlang, The Bayesian foundations of solution concepts of games, J. Econ. Theory 45 (1988), 370-391.

[28] J. Weinstein, M. Yildiz, Sensitivity of equilibrium behavior to higher-order beliefs in nice games, MIT, Mimeo (2008).