AN AXIOMATIC FOUNDATION FOR MULTIDIMENSIONAL SPATIAL MODELS OF ELECTIONS WITH A VALENCE DIMENSION[†]

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January 12, 2010

Abstract. Recent works on political competition incorporate a valence dimension into the standard spatial model. The analysis of the game between candidates in these models is typically based on two assumptions about voters' preferences. One is that valence scores enter the utility function of a voter in an 'additively separable' way, so that the total utility can be decomposed into the 'ideological utility' from the implemented policy (based on the Euclidean distance) plus the valence of the winner. The second is that all the voters identically perceive the platforms of the candidates and agree about their valence score.

The goal of this paper is to axiomatize collections of preferences that satisfy these assumptions. Specifically, we consider the case where only the ideal point in the policy space and the ranking over candidates are known for each voter. We characterize the case where there are policies $x_1, \ldots, x_m$ for the m candidates and numbers $v_1, \ldots, v_m$ representing valence scores, such that a voter with an ideal policy y ranks the candidates according to $v_i - ||x_i - y||^2$.

Keywords: Elections, Spatial models, Valence, Euclidean preferences.

JEL Classification: D72

[†]This work grew out of a previous joint research with Ehud Lehrer (Azrieli and Lehrer [7]) to whom I'm grateful. I would also like to thank Jon Eguia, Itzhak Gilboa, Paul Healy, Jean–François Laslier, Jim Peck, David Schmeidler, Eran Shmaya, Roee Teper, participants of the Stony Brook (2009) conference on game theory and audience at Caltech's Theory Seminar for helpful comments and references.
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1. Introduction

Since the seminal works of Hotelling [24] and Downs [15], spatial models of elections have been widely used in the political economy literature. Typically, these models identify the policy space with a finite dimensional Euclidean space. Each potential voter in the electorate is assumed to have an ideal point in the policy space, and his utility is decreasing in the Euclidean distance between the implemented policy and his ideal point. Candidates then choose their platforms, and each voter votes for the candidate with the closest platform to her ideal policy. Usually, the emphasis is on equilibrium analysis of the resulting game between candidates.

More recently, researchers incorporated a "valence" dimension to the standard model. This additional dimension influences voters' preferences and was shown to have an important effect on the outcome of the political game, both in theory and in empirical studies. This additional dimension may represent any non–policy issue on which candidates differ in the "score" they get from voters. Examples include charisma, experience, past success, communication skills, etc. The difference between the valence dimension and other dimensions, which are part of the policy space, is that all voters prefer high valence scores to low. References to works that incorporate valence issues can be found in the related literature section below.

Let C denote the set of candidates competing in some elections, and let the d dimensional linear space $\mathbb{R}^d$ represent the policy space.[1] When valence issues are present, the preferences of voters are defined over $\mathbb{R}^{d+1}$. Indeed, the utility of a potential voter if a certain candidate $i \in C$ is elected depends both on the policy that i implements (a d-dimensional real vector) and on the valence score that this voter gives to i (a real number). Notice that we deal here with a collection of preference orders, one for each voter. The analysis of the game between candidates is typically based on two fundamental assumptions about this collection of preferences, which we now discuss.

The first assumption concerns the preferences of individual voters. Each voter is assumed to have an ideal point in the policy space. If candidate $i \in C$ is elected and implements the policy $x_i \in \mathbb{R}^d$, then the utility of a voter with an ideal point $y \in \mathbb{R}^d$ is given by $v_i - ||y - x_i||^2$, where v_i is the valence score of i (according to this voter) and $|| \cdot ||$ is the Euclidean norm in $\mathbb{R}^d$. This particular functional form can be interpreted

[1] The focus of this paper is on multidimensional policy spaces, i.e. $d \geq 2$. Most of our results do not hold in the case $d = 1$. See subsection 3.4 for a discussion of the one dimensional case.

as follows. Voters have an 'ideological' utility function based on the distance between their ideal policy and the platform of the candidate. In addition, they derive utility from having a leader with high valence. The total utility of a voter is 'additively separable' in the policy and valence dimensions, so that it can be decomposed into the ideological utility plus the valence index.[2] While this utility function is very natural, it is not clear why the Euclidean norm is the appropriate measure of ideological utility, and why valence scores enter in an additive way.

The second key assumption usually made is that all voters perceive the alternatives they face in the same way. First, the voters agree on the location of the candidates in the policy space. That is, the beliefs of all voters regarding the policy that a certain candidate is going to implement if elected coincide. Although this seems like a rather strong assumption, it can be justified by the claim that candidates commit to a certain policy prior to the elections, which is the policy that voters anticipate will be implemented if the candidate is elected. But voters are also supposed to agree about the valence of each candidate. This is harder to justify since it seems reasonable that voters with different ideological views will also have different views of the valence of candidates. Notice that, if one allows to each potential voter to perceive the platforms and/or valences of the candidates differently, then the model may become completely unfalsifiable.

Obviously, it is very hard (not to say impossible) to extract the entire preferences of each voter over $\mathbb{R}^{d+1}$. Therefore, it is not easy to check whether the aforementioned assumptions make sense in any particular political campaign. Thus, it is an important matter to identify conditions on more easily observable data that guarantee consistency with the spatial model assumptions. Introducing such necessary and sufficient conditions is the main result of this paper.

Specifically, we assume that, for each potential voter in the electorate, only his ideal policy and his ranking of the candidates can be observed. While this may also seem quite demanding, it is much more reasonable than observing the entire utility function of the voter. We characterize the case where this data is consistent with voters having utility functions as above. That is, we characterize the case where there are platforms $\{x_i\}_{i\in C} \subseteq \mathbb{R}^d$ and numbers $\{v_i\}_{i\in C} \subseteq \mathbb{R}$ representing valence scores, such that a voter with an ideal policy y ranks the candidates according to $v_i - ||x_i - y||^2$. We emphasize

[2]Notice that we take the square of the Euclidean norm (and not just the norm) as the ideological utility function. We discuss this point in subsection 3.2.

that the representation is for the collection of preference orders of all voters jointly, and not for the preferences of a single voter.

We use four axioms for the characterization. The first is that each voter preferences over candidates are rational (complete and transitive). The second, a continuity condition, is that the set of voters who strictly prefer one candidate over another is open. The third and perhaps most important condition is 'betweenness'. This condition is due to Grandmont [22]. Basically, this means that the set of voters preferring one candidate over another is convex. There is a close connection between convexity and the Euclidean norm, as other norms would typically induce non–convex sets. The last condition requires sufficient heterogeneity in the preferences. This is a more technical condition that is not necessary for the representation but is required for the sufficiency part of the proof.[3]

We think of our result as "good news" since it shows that if voters' preferences satisfy a set of rather natural axioms, then they are consistent with the standard spatial model with a valence dimension.[4] From a theoretical viewpoint, the result provides a possible justification for the assumptions (discussed above) that allow to study the game between candidates. From an empirical perspective, the axioms may help to check whether the spatial model makes sense in any particular campaign.

1.1. **Related literature.** A few recent papers study questions related to the implications of assuming Euclidean preferences in spatial models. Degan and Merlo [13] ask under what conditions the assumption that voters vote ideologically (i.e., according to Euclidean preferences) is falsifiable when data about the voting choices in several elections is available. Their answer is based on a relation between the dimension of the policy space and the number of elections.[5] Bogomolnaia and Laslier [9] find the exact number of dimensions required in order to be able to represent any preference profile of I voters over A alternatives. Knoblauch [27] provides a polynomial time algorithm to check whether a given finite preference profile has a one–dimensional Euclidean representation.

There are also several works that study similar questions for a more general class of preferences that includes Euclidean preferences as a special case. Eguia [16] axiomatizes

[3]Nevertheless, our main result (Theorem 1) is an equivalence theorem. See Section 2 for details.

[4]Our conditions are not sufficient if one doesn't allow for a valence dimension, and we do not know how to characterize data consistent with spatial models without this additive term. See subsection 3.2.

[5]Some of their results generalize to the case where candidates get different valence scores from voters. See Section 3.2 in [13].

preference relations over lotteries over multi–attribute objects that admit a representation by some l_p norm. He also studies the case of multiple voters and characterizes the case where their preferences can be jointly represented by such a norm. Kalandrakis [25] considers the case where a finite number of binary choices is observed, and characterizes the case where these choices can be rationalized by a concave utility function. He further studies the case where the rationalizing function has a bliss point.

An important difference between our paper and all of the above is that we assume that, for any point in the policy space, the preferences of a voter with this ideal point over candidates are observed. All of the above papers deal with either a single preference relation or with a finite number of relations. We note that many previous works assume a continuous distribution of voters' ideal points. We further discuss this point in subsection 3.1.

Papers using spatial models of elections with valence issues similar to the one studied here are numerous in recent years. Examples include Ansolabehere and Snyder [1], Aragones and Palfrey [2], Degan [12], Dix and Santore [14], Enelow and Hinich [17], Gersbach [18], Groseclose [23], Kim [26] and Schofield [31] among others. These papers study different aspects of the political competition and provide various interpretations for the additive constant in the utility functions of the voters. Krasa and Polborn [28] consider competition between political candidates when preferences of voters are more general than those studied here. In particular they allow for preferences which are not additively separable in the valence and policy dimensions.

From a technical point of view, our main result is closely related to Theorem 1 in Azrieli and Lehrer [7], who characterize categorization systems that are generated by proximity to a set of prototypical cases. There are several important differences however. First, the primitive in that paper consists of a collection of partitions of $\mathbb{R}^d$ indexed by subsets of the set of alternatives, while here the primitive is a collection of preference orders over alternatives indexed by points in $\mathbb{R}^d$. As a result, some of the axioms used for the characterization are different. Second, the representation in [7] is according to the Euclidean distance in $\mathbb{R}^{d+1}$ between the points $(y, 0)$ and (x_i, v_i). Thus, the valence dimension in the current paper works in the opposite direction than the 'extra dimension' in [7]. In addition, this paper contains several new results that did not appear in [7].

There is also a surprisingly close mathematical connection between the result of this paper and the characterization of a collection of preference orders that can be represented by linear functionals.[6] Such axiomatizations appear in works on scoring rules (Myerson [30], Smith [32], Young [33]), case–based decision theory (Gilboa and Schmeidler [19, 20]), expected utility in the context of games (Gilboa and Schmeidler [21]), relative utility (Ashkenazi and Lehrer [5]) and individual welfare functionals (Chambers and Hayashi [11]). While the axioms used in these works are similar to ours, the representation we obtain is significantly different.

Finally, the mathematical object we deal with here is known in the geometry literature as (generalized) Voronoi diagram or (generalized) Dirichlet tessellation.[7] The most relevant papers in this literature are Ash and Bolker [3], [4] and Aurenhammer [6]. The book by Boots et al. [10] surveys applications of Voronoi diagrams in many different fields.

1.2. **Organization.** The next section contains the model and the main result of the paper, as well as a result regarding the uniqueness of the representation. In Section 3 we discuss several issues related to the model. In particular, we study the case of a finite set of voters, discuss the importance of the valence dimension for the result, and consider the special cases of three candidates and of one–dimensional policy space.

2. Axioms and main result

2.1. **Setup.** Let $C = \{1, 2, \ldots, m\}$ be the set of candidates where $m \geq 2$. The policy space is taken to be $\mathbb{R}^d$ with $d \geq 2$. Each potential voter is identified with her ideal point in the policy space and we assume that for every $y \in \mathbb{R}^d$, there is a voter with y as her ideal policy. Thus, we can identify the electorate with $\mathbb{R}^d$. The letters i, j, k to denote candidates (elements of C) and x, y, z, w denote voters or policies (points in $\mathbb{R}^d$).

Our primitive is a collection of binary relations $\{\succeq_y\}_{y \in \mathbb{R}^d}$ over C, one for every voter $y \in \mathbb{R}^d$. The interpretation of $i \succeq_y j$ is that a voter with an ideal point y (weakly) prefers candidate i to candidate j. As usual, for any $i, j \in C$, we let $i \succ_y j$ if and only if both $i \succeq_y j$ and $j \not\succeq_y i$, and $i \sim_y j$ if and only if both $i \succeq_y j$ and $j \succeq_y i$.

2.2. **Axioms.** The following properties will be used for the characterization.

(A1) Weak order: For every $y \in \mathbb{R}^d$, $\succeq_y$ is complete and transitive.

(A2) Continuity: For every $i, j \in C$, the set $\{y \in \mathbb{R}^d \ : \ i \succ_y j\}$ is open.

[6]We thank Itzhak Gilboa for pointing out this connection.

[7]The word 'generalized' is added to indicate that there is an additive constant associated with each candidate. These objects are also called power diagrams in some places in the geometry literature.

(A3) Betweenness: For every $i, j \in C$ and $y, z \in \mathbb{R}^d$, if $i \succeq_y j$ ($i \succ_y j$) and $i \succeq_z j$ then $i \succeq_{\alpha y+(1-\alpha)z} j$ ($i \succ_{\alpha y+(1-\alpha)z} j$) for every $\alpha \in (0, 1)$.

(A4) Heterogeneity:

(i) For every (ordered) three distinct candidates (i, j, k) there is $y \in \mathbb{R}^d$ such that $i \succ_y j \succ_y k$; and

(ii) The sets $\{y \in \mathbb{R}^d \ : \ 1 \sim_y i \sim_y j\}$ and $\{y \in \mathbb{R}^d \ : \ 2 \sim_y i \sim_y j\}$ are not equal for every pair of candidates $\{i, j\}$ such that $\{1, 2\} \cap \{i, j\} = \emptyset$.

The first property is standard. The second implies that if a voter with ideal point y strictly prefers candidate i over j, then any voter with an ideal point sufficiently close to y also prefers i over j. *(A3)* states that if $w = \alpha y + (1-\alpha)z$ for some $0 < \alpha < 1$, then the preferences of a voter with an ideal point w are 'between' the preferences of voters with ideal points y and z. The definition of betweenness for binary relations that we use here is due to Grandmont [22]. This axiom implies that the set of voters who weakly prefer candidate i over j is convex for every i and j. Furthermore, if one voter z weakly prefers i over j and another voter y strictly prefers i over j, then every voter with an ideal point strictly between y and z *strictly* prefers i to j.

Finally, *(A4)* requires the population of voters to be sufficiently diverse in its preferences. Namely, for any (strict) ranking of every three candidates, there should be a voter who ranks these candidates according to that given order; and, for every $i \geq 3$ and $j \geq 3$ ($i \neq j$), there should be a voter who is indifferent between candidates 1, i and j but is not indifferent between 2, i and j. Similar conditions to *(A4)* appear already in the early social choice literature (see, e.g., Blau [8] page 309). Note that if $m = 2$, then *(A4)* is trivially satisfied, and if $m = 3$, then the second part of *(A4)* is trivially satisfied.

2.3. **Main result.** Before stating our main theorem we need one more definition.

Definition 1. *Fix two sets* $\{x_1, x_2, \ldots, x_m\} \subseteq \mathbb{R}^d$ *and* $\{v_1, v_2, \ldots, v_m\} \subseteq \mathbb{R}$. *The set* $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\} \subseteq \mathbb{R}^{d+1}$ *is in a* general position *if the following two conditions hold:*

(i) For every distinct $1 \leq i, j, k \leq m$, *the vectors* x_i, x_j, x_k *are affinely independent in* $\mathbb{R}^d$ *(equivalently,* $x_j - x_i$ *and* $x_k - x_i$ *are linearly independent in* $\mathbb{R}^d$*).*

(ii) For every $i \geq 3$ *and* $j \geq 3$ $(i \neq j)$ *the sets*

$$\{y \in \mathbb{R}^d : v_1 - ||x_1 - y||^2 = v_i - ||x_i - y||^2 = v_j - ||x_j - y||^2\}$$

and

$$\{y \in \mathbb{R}^d : v_2 - ||x_2 - y||^2 = v_i - ||x_i - y||^2 = v_j - ||x_j - y||^2\}$$

are not equal.

Informally speaking, if a set of points is *not* in a general position, then it has a 'degenerate structure'. We remark that if the points $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ are independently drawn from some continuous distribution over $\mathbb{R}^{d+1}$, then the resulting set will be in a general position with probability 1. The precise meaning of the term general position varies with the context in which it is used. A discussion of this term can be found in, e.g., Matoušek ([29], pp. 3-5).

Theorem 1. *The following two statements are equivalent:*
(i) The collection of binary relations $\{\succeq_y\}_{y\in\mathbb{R}^d}$ satisfies properties (A1) through (A4).
(ii) There are points $\{x_1, x_2, \ldots, x_m\} \subseteq \mathbb{R}^d$ and numbers $\{v_1, v_2, \ldots, v_m\} \subseteq \mathbb{R}$ such that $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ is in a general position and, for every $i, j \in C$ and every $y \in \mathbb{R}^d$, $i \succeq_y j$ if and only if $v_i - ||x_i - y||^2 \geq v_j - ||x_j - y||^2$.

Proof. ***(ii)* implies *(i)***:

A simple but useful observation is that for any $x_i \neq x_j \in \mathbb{R}^d$ and $v_i, v_j \in \mathbb{R}$, the set $\{y \in \mathbb{R}^d \ : \ v_i - ||x_i - y||^2 = v_j - ||x_j - y||^2\}$ is an affine subspace of dimension $d-1$ (a hyperplane), perpendicular to the direction $x_i - x_j$. Indeed, a simple computation shows that this set can be rewritten as[8] $\{y \in \mathbb{R}^d \ : \ y \cdot (x_i - x_j) = \frac{1}{2}(v_j - v_i + ||x_i||^2 - ||x_j||^2)\}$. Similarly, the set $\{y \in \mathbb{R}^d \ : \ v_i - ||x_i - y||^2 > v_j - ||x_j - y||^2\}$ is an open half space in $\mathbb{R}^d$ (given that $x_i \neq x_j$).

Fix the sets $\{x_1, x_2, \ldots, x_m\} \subseteq \mathbb{R}^d$ and $\{v_1, v_2, \ldots, v_m\} \subseteq \mathbb{R}$. Property *(A1)* is obviously satisfied. Denote $A_{ij} = \{y \in \mathbb{R}^d \ : \ v_i - ||x_i - y||^2 = v_j - ||x_j - y||^2\}$ and $B_{ij} = \{y \in \mathbb{R}^d \ : \ v_i - ||x_i - y||^2 > v_j - ||x_j - y||^2\}$. By property *(i)* of Definition 1, $x_i \neq x_j$ for every $i \neq j \in C$. Thus, each B_{ij} is open and convex and each A_{ij} is the boundary of the closed half space $B_{ij} \cup A_{ij}$. This shows that properties *(A2)* and *(A3)* are satisfied.

(A4) is satisfied because the set $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ is in a general position. To see that, take any distinct $i, j, k \in C$. We need to show that there is some y with $i \succ_y j \succ_y k$. If this was not true then it must be that B_{ij} and B_{jk} do not intersect. But this can only happen if $x_i - x_j$ and $x_j - x_k$ are linearly dependent, a contradiction to

[8]For two vectors $z, w \in \mathbb{R}^d$ we denote by $z \cdot w = \sum_{i=1}^{d} z_i w_i$ the standard inner product in $\mathbb{R}^d$.

the assumption of general position (property *(i)*). Finally, take any distinct $i \geq 3$ and $j \geq 3$. By the general position assumption (property *(ii)*), $B_{1i} \cap B_{ij}$ and $B_{2i} \cap B_{ij}$ are not equal. This proves that *(A4)* is satisfied. □

(i)* implies *(ii):

The proof is broken into several claims.

Claim 1. *For every ordered pair (i, j) of distinct candidates there is a non–zero vector $s_{ij} \in \mathbb{R}^d$ and a number $c_{ij} \in \mathbb{R}$ such that $\{y \in \mathbb{R}^d \ : \ i \succeq_y j\} = \{y \in \mathbb{R}^d \ : \ s_{ij} \cdot y \leq c_{ij}\}$. Moreover, these vectors and numbers can be chosen such that $s_{ji} = -s_{ij}$ and $c_{ji} = -c_{ij}$ for every (i, j).*

Proof. This follows from the betweenness axiom *(A3)* and the continuity axiom *(A2)* using a separation argument. See Grandmont [22] for the details. *(A4)* is used to guarantee that there is no total indifference between some pair of candidates, and that there is no domination of one candidate over another in the entire electorate. □

From now until the end of the proof we fix a collection $\{s_{ij}, c_{ij}\}_{i,j \in C}$ as in Claim 1.

Claim 2. *For every distinct $i, j, k \in C$, the vectors s_{ij} and s_{ik} are linearly independent.*

Proof. This follows from part (i) of *(A4)*. Indeed, if s_{ij} and s_{ik} are linearly dependent, then $s_{ij} = \alpha s_{ik}$ for some $\alpha \neq 0$ (recall that $s_{ij}, s_{ik} \neq 0$). Assume $\alpha > 0$. If $\alpha c_{ik} \geq c_{ij}$, then for any $y \in \mathbb{R}^d$ that satisfies $y \cdot s_{ik} > c_{ik}$ it holds that $y \cdot s_{ij} = \alpha y \cdot s_{ik} > \alpha c_{ik} \geq c_{ij}$. If $\alpha c_{ik} \leq c_{ij}$, then for any $y \in \mathbb{R}^d$ that satisfies $y \cdot s_{ik} < c_{ik}$ it holds that $y \cdot s_{ij} = \alpha y \cdot s_{ik} < \alpha c_{ik} \leq c_{ij}$. Either case contradicts *(A4)*. The case $\alpha < 0$ is similar. □

Claim 3. *For every $i, j, k \in C$, the vectors s_{ij}, s_{ik} and s_{jk} are* not *linearly independent.*

Proof. Consider the set $D = \{y \in \mathbb{R}^d \ : \ i \sim_y j\} \cap \{y \in \mathbb{R}^d \ : \ i \sim_y k\}$. By Claims 1 and 2 it is the intersection of two non–parallel hyperplanes, so it is an affine subspace of dimension $d - 2$. By transitivity it is contained in the hyperplane $\{y \in \mathbb{R}^d \ : \ j \sim_y k\}$. Thus, s_{jk} is in the orthogonal complement of D. Since s_{ij} and s_{ik} are also orthogonal to D, the claim is proved. □

Claim 4. *In the unique solution to the equation $s_{ik} = \alpha s_{ij} + \beta s_{jk}$ both α and β are positive.*

Proof. First, the previous claims imply that the above equation has a unique solution, and that both α and β are non–zero. Second, let y satisfy $i \sim_y j \sim_y k$ (existence of such y is guaranteed by the previous claims). Then $c_{ik} = y \cdot s_{ik} = \alpha y \cdot s_{ij} + \beta y \cdot s_{jk} = \alpha c_{ij} + \beta c_{jk}$.

Now, assume that $\alpha < 0$ and $\beta < 0$. Let $y \in \mathbb{R}^d$ be such that $i \succ_y j$ and $j \succ_y k$ (existence of such a voter is guaranteed by part (i) of *(A4)*). Then $y \cdot s_{ik} = \alpha y \cdot s_{ij} + \beta y \cdot s_{jk} > \alpha c_{ij} + \beta c_{jk} = c_{ik}$. Thus, $k \succ_y i$ which contradicts transitivity. Finally, assume $\alpha < 0$ and $\beta > 0$. Let $y \in \mathbb{R}^d$ be such that $i \succ_y k$ and $k \succ_y j$. Then $\alpha y \cdot s_{ij} = y \cdot s_{ik} - \beta y \cdot s_{jk} < c_{ik} - \beta c_{jk} = \alpha c_{ij}$, so $y \cdot s_{ij} > c_{ij}$. This implies that $j \succ_y i$ which contradicts transitivity. The case $\alpha > 0$ and $\beta < 0$ is similar. □

Fix $x_1 \in \mathbb{R}^d$ arbitrarily. Choose some positive number $\alpha_{12} > 0$ and let $x_2 = x_1 + \alpha_{12} s_{12}$. Let $\alpha_{21} = \alpha_{12}$.

Claim 5. *For every $3 \leq i \leq m$, there are unique positive numbers $\alpha_{1i}, \alpha_{2i} > 0$ such that $x_1 + \alpha_{1i} s_{1i} = x_2 + \alpha_{2i} s_{2i}$. That is, the rays from x_1 and x_2 in the directions s_{1i} and s_{2i} respectively intersect.*

Proof. Rearranging the equality in the claim we get $x_2 - x_1 = \alpha_{1i} s_{1i} - \alpha_{2i} s_{2i}$. We have $x_2 - x_1 = \alpha_{12} s_{12}$, and recall that $s_{2i} = -s_{i2}$. Thus, the claim is equivalent to the existence and uniqueness of $\alpha_{1i}, \alpha_{2i} > 0$ such that $s_{12} = \frac{\alpha_{1i}}{\alpha_{12}} s_{1i} + \frac{\alpha_{2i}}{\alpha_{12}} s_{i2}$. This is however an immediate consequence of the two previous claims. □

For every $3 \leq i \leq m$, define $x_i = x_1 + \alpha_{1i} s_{1i}$. By the previous claim, we also have $x_i = x_2 + \alpha_{2i} s_{2i}$. That is, x_i is placed at the intersection of the rays from x_1 in the direction s_{1i} and from x_2 in the direction s_{2i}. Denote $\alpha_{i1} = \alpha_{1i}$ and $\alpha_{i2} = \alpha_{2i}$ for every $3 \leq i \leq m$.

Claim 6. *For every distinct $1 \leq i, j \leq m$ there is $\alpha_{ij} > 0$ such that $x_j - x_i = \alpha_{ij} s_{ij}$.*

Proof. First, the claim is true by construction if either $i = 1$ or $i = 2$ or $j = 1$ or $j = 2$ (or two of the above). Assume therefore that both $i \geq 3$ and $j \geq 3$. We first show the existence of α_{ij} and then show that it must be positive.

Fix some $\bar{y}$ that satisfies $i \sim_{\bar{y}} 1 \sim_{\bar{y}} j$ (in particular, $i \sim_{\bar{y}} j$). To prove the existence of α_{ij} it is sufficient to show that $(x_j - x_i) \cdot (y - \bar{y}) = 0$ for every y that satisfies $i \sim_y j$. Indeed, this will show that $x_j - x_i$ is orthogonal to the hyperplane $\{y \ : \ s_{ij} \cdot y = c_{ij}\}$, which implies that s_{ij} and $x_j - x_i$ are linearly dependent.

If $i \sim_y 1 \sim_y j$ then we have

$$(x_j - x_i) \cdot (y - \bar{y}) = (x_j - x_1) \cdot (y - \bar{y}) + (x_1 - x_i) \cdot (y - \bar{y}) =$$
$$\alpha_{1j} s_{1j} \cdot (y - \bar{y}) - \alpha_{1i} s_{1i} \cdot (y - \bar{y}) = \alpha_{1j}(c_{1j} - c_{1j}) - \alpha_{1i}(c_{1i} - c_{1i}) = 0.$$

The second equality is by construction of the points x_i and x_j, and the third equality is due to the fact that both $i \sim_{\bar{y}} 1 \sim_{\bar{y}} j$ and $i \sim_y 1 \sim_y j$.

The set $\{y \;:\; i \sim_y 1 \sim_y j\}$ is an affine subspace of dimension $d-2$. Thus, if we can find a point y such that $i \sim_y j$, $i \not\sim_y 1$ and $(x_j - x_i) \cdot (y - \bar{y}) = 0$, we can conclude that $x_j - x_i$ is orthogonal to the ij indifference hyperplane. By part (ii) of *(A4)* there is y such that $i \sim_y 2 \sim_y j$ but $i \not\sim_y 1$. For this y we have

$$(x_j - x_i) \cdot (y - \bar{y}) = ((x_j - x_2) + (x_2 - x_1) + (x_1 - x_i)) \cdot (y - \bar{y}) =$$
$$(\alpha_{2j} s_{2j} + \alpha_{12} s_{12} - \alpha_{1i} s_{1i}) \cdot (y - \bar{y}) =$$
$$\alpha_{2j} c_{2j} + \alpha_{1i} c_{1i} + y \cdot (\alpha_{12} s_{12} - \alpha_{1i} s_{1i}) - \bar{y} \cdot (\alpha_{2j} s_{2j} + \alpha_{12} s_{12}) =$$
$$\alpha_{2j} c_{2j} + \alpha_{1i} c_{1i} - y \cdot \alpha_{2i} s_{2i} - \bar{y} \alpha_{1j} s_{1j} = \alpha_{2j} c_{2j} + \alpha_{1i} c_{1i} - \alpha_{2i} c_{2i} - \alpha_{1j} c_{1j} =$$
$$\alpha_{12} c_{12} - \alpha_{12} c_{12} = 0.$$

This proves existence of α_{ij}.

Finally, to show that $\alpha_{ij} > 0$, notice that

$$\alpha_{ij} s_{ij} = x_j - x_i = (x_1 - x_i) + (x_j - x_1) = \alpha_{i1} s_{i1} + \alpha_{1j} s_{1j}.$$

Divide by α_{ij} to obtain

$$s_{ij} = \frac{\alpha_{i1}}{\alpha_{ij}} s_{i1} + \frac{\alpha_{1j}}{\alpha_{ij}} s_{1j}.$$

By Claim 4, both $\frac{\alpha_{i1}}{\alpha_{ij}}$ and $\frac{\alpha_{1j}}{\alpha_{ij}}$ are positive. Since α_{i1} and α_{1j} are positive by construction it follows that $\alpha_{ij} > 0$. □

It remains to construct the valences $\{v_i\}_{i \in C}$. Choose v_1 arbitrarily and define for every $2 \leq i \leq m$

$$v_i = v_1 - ||x_1||^2 + ||x_i||^2 - 2\alpha_{1i} c_{1i}.$$

Claim 7. *The sets $\{x_1, x_2, \ldots, x_m\}$ and $\{v_1, v_2, \ldots, v_m\}$ constructed above represent the preferences as in Theorem 1.*

Proof.

$$
\begin{aligned}
i \succeq_y j &\iff s_{ij} \cdot y \le c_{ij} \iff (x_j - x_i) \cdot y \le \alpha_{ij} c_{ij} \iff (x_j - x_i) \cdot y \le \alpha_{1j} c_{1j} - \alpha_{1i} c_{1i} \\
&\iff (x_j - x_i) \cdot y \le \frac{1}{2}\left(v_1 - v_j + ||x_j||^2 - ||x_1||^2\right) - \frac{1}{2}\left(v_1 - v_i + ||x_i||^2 - ||x_1||^2\right) \\
&\iff (x_j - x_i) \cdot y \le \frac{1}{2}\left(v_i - v_j + ||x_j||^2 - ||x_i||^2\right) \iff v_i - ||x_i - y||^2 \ge v_j - ||x_j - y||^2.
\end{aligned}
$$

□

Claim 8. *The set $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ constructed above is in a general position.*

Proof. The vectors x_i, x_j, x_k are affinely independent since $x_j - x_i = \alpha_{ij} s_{ij}$ and $x_k - x_i = \alpha_{ik} s_{ik}$, and these are linearly independent vectors by Claim 2. Finally, the sets $\{y \in \mathbb{R}^d \ : \ 1 \sim_y i \sim_y j\}$ and $\{y \in \mathbb{R}^d \ : \ 2 \sim_y i \sim_y j\}$ are not equal by part (ii) of *(A4)*. □

This completes the proof of the theorem. □

2.4. **Uniqueness.** Examining the proof of Theorem 1, one can see that the platforms and valences derived from the properties *(A1)-(A4)* are not unique. However, we do have the following relation between any two representations of the voters' preferences.

Proposition 1. *Assume $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\} \subseteq \mathbb{R}^{d+1}$ represent the preferences $\{\succeq_y\}_{y \in \mathbb{R}^d}$ as in Theorem 1. Then $\{(x'_1, v'_1), (x'_2, v'_2), \ldots, (x'_m, v'_m)\} \subseteq \mathbb{R}^{d+1}$ also represent $\{\succeq_y\}_{y \in \mathbb{R}^d}$ if and only if there is a positive number $\alpha > 0$ and a vector $\beta \in \mathbb{R}^d$ such that $x'_i = \alpha x_i + \beta$ for every $1 \le i \le m$, and such that the equation*

$$
v'_i - \alpha v_i = v'_j - \alpha v_j + \alpha(1 - \alpha)(||x_j||^2 - ||x_i||^2) + 2\alpha\beta \cdot (x_i - x_j) \tag{1}
$$

holds for every $i, j \in C$. In particular, if $x_i = x'_i$ for $1 \le i \le m$ (i.e., $\alpha = 1$ and $\beta = 0$) then there is some $\gamma \in \mathbb{R}$ such that $v'_i = v_i + \gamma$ for $1 \le i \le m$.

This result can be interpreted as follows. We may rescale and change the origin of the policy space to get different sets of platforms that induce the same preferences. But once the unit of measurement and the origin are fixed, the platforms are uniquely determined by the preferences. Moreover, once platforms are fixed, the relative valences of the various candidates (the differences $v_i - v_j$) are also unique.

Proof. First, it is easy to check that if there are $\alpha > 0$ and $\beta \in \mathbb{R}^d$ such that $x'_i = \alpha x_i + \beta$ for $1 \le i \le m$ and in addition equation (1) is satisfied, then $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ and $\{(x'_1, v'_1), (x'_2, v'_2), \ldots, (x'_m, v'_m)\}$ represent the same preferences.

Now, assume that $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ and $\{(x'_1, v'_1), (x'_2, v'_2), \ldots, (x'_m, v'_m)\}$ represent the same preferences $\{\succeq_y\}_{y \in \mathbb{R}^d}$. It follows from the proof of Theorem 1 that for every $i, j \in C$, there is a positive number, say $t_{ij} > 0$, such that $x_j - x_i = t_{ij}(x'_j - x'_i)$ (with the convention $t_{ij} = -t_{ji}$). Fix some three candidates $i, j, k \in C$ (if there are only two candidates jump to the next paragraph). Sum up the equalities $x_j - x_i = t_{ij}(x'_j - x'_i)$, $x_i - x_k = t_{ki}(x'_i - x'_k)$, $x_k - x_j = t_{jk}(x'_k - x'_j)$ and rearrange the terms to obtain $(x'_i - x'_j)(t_{ki} - t_{ij}) + (x'_k - x'_j)(t_{jk} - t_{ki}) = 0$. But the vectors x'_i, x'_j, x'_k are affinely independent so $t_{ki} - t_{ij} = t_{jk} - t_{ki} = 0$. It follows that $t_{ij} = t_{ki} = t_{jk}$, so there is a number $\alpha > 0$ such that $x_j - x_i = \alpha(x'_j - x'_i)$ for every $i, j \in C$. Now, define $\beta = x_1 - \alpha x'_1$. For every $2 \le i \le m$ we have $x_1 - x_i = \alpha(x'_1 - x'_i)$ or $x_i - \alpha x'_i = x_1 - \alpha x'_1 = \beta$. That is, $x'_i = \alpha x_i + \beta$ for every $1 \le i \le m$.

Finally, we must have

$$\left\{ y \in \mathbb{R}^d \;:\; y \cdot (x_j - x_i) = \frac{1}{2}\left(v_i - v_j + ||x_j||^2 - ||x_i||^2\right) \right\} =$$
$$\left\{ y \in \mathbb{R}^d \;:\; y \cdot (x'_j - x'_i) = \frac{1}{2}\left(v'_i - v'_j + ||x'_j||^2 - ||x'_i||^2\right) \right\}$$

for every $i, j \in C$. Substituting $\alpha x_i + \beta$ for x'_i and $\alpha x_j + \beta$ for x'_j and rearranging we obtain equation (1). In particular, if $x'_i = x_i$ and $x'_j = x_j$, then $v'_i - v_i = v'_j - v_j$. Define $\gamma = v'_1 - v_1$. It follows that $v'_i = v_i + \gamma$ for every $1 \le i \le m$. □

3. Discussion and further results

3.1. **Finite set of voters.** From a practical point of view, it would be more interesting to find the testable implications of the spatial model assumptions for the preferences of a finite number of voters. The axiom *(A3)* implies that a finite sample of observations of voters' ideal points and rankings must have the property that the convex hulls of the ideal points of voters who prefer candidate i over j and of those who prefer j over i are disjoint in order for it to be consistent with the spatial model.[9] *(A1)* also gives an obvious necessary condition.

It is tempting to try to prove a similar representation result to that of Theorem 1 for the case of a finite sample of voters, where only *(A1)* and (the modified version of) *(A3)*

[9]Assume for simplicity that only strict preferences are allowed.

are assumed. For the case of two candidates, it is easy to see that these two axioms are sufficient for a representation.[10] However, if there are at least three candidates, this is no longer true. We demonstrate the problem with the following example. Let $d = 2$, $C = \{1, 2, 3\}$ and fix some $\epsilon > 0$. The set of voters, denoted Y, consists of six voters with the ideal points

$$Y = \{y_1 = (\epsilon, \epsilon),\ y_2 = (-\epsilon, -\epsilon),\ y_3 = (-\epsilon, -4),\ y_4 = (\epsilon, -4),\ y_5 = (4, \epsilon),\ y_6 = (4, -\epsilon)\}.$$

The preferences of these six voters are as follows. Voters $\{y_2, y_3\}$ prefer candidate 1 over candidate 2 (the rest of the voters prefer candidate 2 over candidate 1). Voters $\{y_1, y_2, y_3, y_5\}$ prefer candidate 1 over candidate 3, and voters $\{y_1, y_5\}$ prefer candidate 2 over candidate 3. Figure 1 illustrates the location of the voters' ideal points in the policy space and their rankings.

It is easy to check that the above condition of disjointness of the convex hulls is satisfied. However, we claim that these preferences are not consistent with the spatial model. Indeed, assume to the contrary that there are $\{(x_1, v_1), (x_2, v_2), (x_3, v_3)\}$ that represent these preferences as in Theorem 1. The locations of the points y_1, y_2, y_3, y_4 and the preferences of these voters imply that the line $\{y \in \mathbb{R}^2 \ : \ v_1 - ||y - x_1||^2 = v_2 - ||y - x_2||^2\}$ should be close to both points $(0, 0)$ and $(0, -4)$. Similarly, the line $\{y \in \mathbb{R}^2 \ : \ v_1 - ||y - x_1||^2 = v_3 - ||y - x_3||^2\}$ should be close to both points $(4, 0)$ and $(0, -4)$, and the line $\{y \in \mathbb{R}^2 \ : \ v_2 - ||y - x_2||^2 = v_3 - ||y - x_3||^2\}$ should be close to both points $(0, 0)$ and $(4, 0)$.

Now, for sufficiently small ϵ, it must be the case that the point $\bar{y} = (1, -1)$ is in the triangle generated by these three lines. It means that at this point we must have

$$v_1 - ||\bar{y} - x_1||^2 < v_2 - ||\bar{y} - x_2||^2 < v_3 - ||\bar{y} - x_3||^2 < v_1 - ||\bar{y} - x_1||^2,$$

a contradiction. If there was a voter with ideal point $\bar{y}$ and transitive preferences over candidates, this could not happen.

This example makes it clear that it is also necessary to impose some restrictions on the preferences of voters over triplets of candidates. Namely, it should be possible to choose the separating hyperplanes between voters with opposite preferences over pairs of candidates such that no cycles result. Denote by A_{ij} a possible hyperplane separating voters with opposite preferences over candidates i and j. Then the above no–cycles condition means that it is possible to choose the hyperplanes $\{A_{ij}\}_{i,j \in C}$ such that $A_{ij} \cap$

[10]Actually, the valence dimension is not needed in this case.

$A_{jk} \subseteq A_{ik}$ for every three candidates i, j, k. If this condition is satisfied, then it typically would be possible to represent the preferences by the spatial model (an additional minor condition similar to *(A4)* is required to guarantee a representation).

3.2. **The valence dimension.** The utility function of a voter with an ideal point y that we derive in Theorem 1 is of the form $v_i - ||x_i - y||^2$. Thus, we use the square of the Euclidean norm (and not just the norm) as the 'ideological' utility function. If instead voters' preferences are represented by the utility function $v_i - ||x_i - y||$, then the induced sets of supporters of candidates may not be convex. For instance, let $x_1 = (0,0)$, $x_2 = (1,0)$, $v_1 = 0$, and $v_2 = 1$. Then voters with ideal points $y = (0,1)$ and $y' = (0,-1)$ strictly prefer candidate 2 over candidate 1. However, a voter with an ideal point $y'' = \frac{y+y'}{2} = (0,0)$ is indifferent between the candidates. Thus, using the square of the norm is a consequence of *(A3)*.

Using the square of the norm is natural also for the following reason. We would like to think of the valence dimension as equally important to the policy dimensions. Recall that all voters agree that more is better on the valence domension. An alternative way to put this is to say that the ideal point of every voter is $+\infty$ along this dimension. For the sake of the argument, assume that we replace $+\infty$ by a large enough constant M. Then the utility of a voter if candidate i wins should be measured according to the distance between his ideal point (y, M) and the the point (x_i, v_i). This implies that we should add the valence score to the square of the norm of the difference in the policy space and not to the norm.

Theorem 1 is not true if we require all candidates to have the same score (zero, w.l.o.g.) on the valence dimension. To see this consider the case where $d = 2$ and $C = \{1, 2, 3, 4\}$. The preferences of voters are defined as follows:[11]

$$\begin{aligned}
\{y \ : \ 1 \succeq_y 2\} &= \left\{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ 2y_2 \leq -1 - y_1\right\} \\
\{y \ : \ 1 \succeq_y 3\} &= \left\{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ 2y_2 \geq 1 + y_1\right\} \\
\{y \ : \ 1 \succeq_y 4\} &= \left\{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ y_1 \leq 0\right\} \\
\{y \ : \ 2 \succeq_y 3\} &= \left\{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ y_2 \geq 0\right\} \\
\{y \ : \ 2 \succeq_y 4\} &= \left\{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ 2y_2 \geq -1 + y_1\right\} \\
\{y \ : \ 3 \succeq_y 4\} &= \left\{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ 2y_2 \leq 1 - y_1\right\}
\end{aligned}$$

[11]For simplicity we describe the preferences by the half–spaces of voters who prefer one candidate over another, for every pair of candidates. See Figure 2.

It is easy to check (see Figure 2) that the preferences defined by these half–spaces satisfy axioms *(A1)–(A4)*. Assume by way of contradiction that $\{(x_1, 0), \dots, (x_4, 0)\}$ represent these preferences. Then the point $(-1, 0)$ must be equidistant from x_1 and x_2, and the point $(1, 0)$ must be equidistant from x_2 and x_4. Furthermore, the second coordinates of x_1 and x_4 must be equal, and the first coordinate of x_1 must be equal to minus the first coordinate of x_4. Putting all these conditions together gives that the second coordinates of x_1 and x_4 must be zero. Similarly, it is not hard to check that the first coordinates of x_2 and x_3 must equal zero.

To summarize, if $\{(x_1, 0), \dots, (x_4, 0)\}$ represent the above preferences, then there must be $\alpha, \beta > 0$ such that $x_1 = (-\alpha, 0),\ x_2 = (0, \beta),\ x_3 = (0, -\beta)$ and $x_4 = (\alpha, 0)$. Since the point $(-1, 0)$ is equidistant from x_1 and x_2, we have $\alpha^2 - 2\alpha = \beta^2$. Similarly, since the point $(1, 0)$ is in equidistant from x_3 and x_4, we have $\alpha^2 = \beta^2 + \beta$. However, there is no positive solution to these two equations.

It follows that more restrictions must be imposed on preferences to allow a representation in the form $-||x_i - y||^2$. Finding natural additional axioms that distinguish this case from the more general one studied in this paper is an interesting direction for future research.

3.3. The cases $m = 2$ and $m = 3$.

In contrast to the claim of the previous subsection, if there are only two or three candidates, then it is possible to represent the voters' preferences without resorting to valences. The case $m = 2$ is trivial since one only needs to choose the platforms x_1 and x_2 equidistant from the hyperplane separating the voters that prefer candidate 1 from those preferring candidate 2. For $m = 3$ we state this fact as a proposition.

Proposition 2. *Assume $m = 3$. The preferences $\{\succeq_y\}_{y \in \mathbb{R}^d}$ satisfy properties (A1) through (A4) if and only if there are $x_1, x_2, x_3 \in \mathbb{R}^d$ in a general position such that $i \succeq_y j$ if and only if $||x_i - y||^2 \le ||x_j - y||^2$.*

Proof. The *if* part follows from Theorem 1, so we only need to prove the *only if* part. By Theorem 1, there are $(x_1, v_1), (x_2, v_2), (x_3, v_3)$ in a general position that represent the preferences. It follows that the vectors $x_1 - x_2$ and $x_1 - x_3$ are linearly independent. Therefore, there is $\beta \in \mathbb{R}^d$ that solves the two equations $\beta \cdot (x_1 - x_2) = \frac{v_2 - v_1}{2}$ and $\beta \cdot (x_1 - x_3) = \frac{v_3 - v_1}{2}$. Notice that the same vector β must satisfy also $\beta \cdot (x_2 - x_3) = \frac{v_3 - v_2}{2}$. Define $x'_i = x_i + \beta$ for $i = 1, 2, 3$.

By Proposition 1, the set $\{(x'_1, v'_1), (x'_2, v'_2), (x'_3, v'_3)\}$ represent the same preferences as $\{(x_1, v_1), (x_2, v_2), (x_3, v_3)\}$ if the equation $v'_i - v'_j = v_i - v_j + 2\beta \cdot (x_i - x_j)$ is satisfied for every $i, j \in C$. By construction, the vector β satisfies $\beta \cdot (x_i - x_j) = \frac{v_j - v_i}{2}$ for every i, j. It follows that $v'_1 = v'_2 = v'_3 = 0$ solve the above equations. That is, $\{(x'_1, 0), (x'_2, 0), (x'_3, 0)\}$ represent the preferences $\{\succeq_y\}_{y \in \mathbb{R}^d}$.

□

3.4. **One dimensional policy space.** Theorem 1 is no longer true if the policy space is one dimensional. To understand why, recall that in the proof of Theorem 1 we first constructed the platforms $\{x_1, \ldots, x_m\}$ such that $x_i - x_j$ is orthogonal to the hyperplane of indifferent voters between i and j. We then arbitrarily chose the valence v_1, and each v_i $(2 \leq i \leq m)$ was chosen such that the hyperplane corresponding to $(x_1, v_1), (x_i, v_i)$ is exactly the indifference hyperplane of candidates 1 and i. The final step of the proof (Claim 7) showed that, when the valences are chosen in this way, *for every pair of candidates i and j* the hyperplane corresponding to $(x_i, v_i), (x_j, v_j)$ is the indifference hyperplane of candidates i and j.

The reason why this construction works is that the set of voters who are indifferent between candidates 1 and i and also between 1 and j is not empty. But by transitivity these voters must also be indifferent between candidates i and j. Thus, if the pairs v_1, v_i and v_1, v_j are chosen appropriately, then it must be the case that v_i, v_j is also consistent with the given preferences. However, when $d = 1$ the set of voters who are indifferent between some three candidates is typically empty. Thus, there are too many independent equations that need to be satisfied in order to represent the preferences.

More formally, assume that we replace *(A4)* by the following weaker axiom:

(A4'): For every ordered pair of distinct candidates (i, j) there is $y \in \mathbb{R}^d$ such that $i \succ_y j$.

If the policy space has one dimension and the preferences of voters satisfy axioms *(A1)*–*(A3)* and *(A4')*, then for every $i \neq j$ there is a unique point $a_{ij} \in \mathbb{R}$ such that $i \sim_{a_{ij}} j$, voters to the left of a_{ij} strictly prefer one of these candidates and voters to the right of a_{ij} strictly prefer the other. Furthermore, it is easy to check that we can order the candidates such that $i < j$ implies that $i \succ_y j$ if and only if $y < a_{ij}$ and $j \succ_y i$ if and only if $y > a_{ij}$.

Now, assume that $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ represent the preferences as in Theorem 1. First it is obvious that we must have $x_1 < x_2 < \ldots < x_m$. Second, a simple computation shows that the equality

$$a_{ij} = \frac{v_i - v_j + x_j^2 - x_i^2}{2(x_j - x_i)}$$

must hold for every pair of candidates $i < j$. These equations imply (after some simple manipulations) that the platforms of every three candidates $i < j < k$ satisfy the linear equation

$$a_{ik}(x_k - x_i) - a_{ij}(x_j - x_i) - a_{jk}(x_k - x_j) = 0,$$

or alternatively

$$(a_{ij} - a_{ik})x_i + (a_{jk} - a_{ij})x_j + (a_{ik} - a_{jk})x_k = 0.$$

Consider the above $\binom{m}{3}$ linear equations corresponding to all triplets of candidates, and the corresponding matrix with m columns and $\binom{m}{3}$ rows. Notice that the sum of the columns is the zero vector, which means that $x_1 = x_2 = \ldots = x_m$ is always a solution to this system of equations. For $m \geq 4$, this matrix will typically have a rank of $m - 1$, which means that the solution space is one dimensional. Since constant vectors solve the system, there is typically no solution in which $x_1 < x_2 < \ldots < x_m$. This shows that typically the preferences cannot be represented as in Theorem 1 if there are at least four candidates. If there are two or three candidates, then it is easy to check that axioms *(A1)–(A3)* and *(A4')* are sufficient for a representation. We summarize this in the following proposition.

Proposition 3. *(i) Assume $m = 2$ or $m = 3$. The collection of voters' preferences satisfy axioms* (A1)–(A3) *and* (A4') *if and only if there are m distinct points $\{x_1, \ldots, x_m\} \subseteq \mathbb{R}$ and m numbers $\{v_1, \ldots, v_m\}$ that represent the preferences as in Theorem 1.*
(ii) Assume $m \geq 4$. If the preferences can be represented as in Theorem 1, then axioms (A1)–(A3) *are satisfied. However,* (A1)–(A3) *and* (A4') *are not enough to guarantee a representation.*

3.5. **The Heterogeneity axiom.** Although our main result is an equivalence theorem, the heterogeneity axiom *(A4)* is not always satisfied when preferences are as in the spatial model with a valence dimension. When the set $\{(x_1, v_1), (x_2, v_2), \ldots, (x_m, v_m)\}$ is not in general position, *(A4)* does not hold. A natural question is therefore whether we can dispense with this axiom and get a more general representation theorem. We now show that the answer is negative, and that both parts of *(A4)* are needed for the sufficiency result.

Consider first the following collection of preferences, where the policy space is two–dimensional ($d = 2$) and $C = \{1, 2, 3\}$:

$$\begin{aligned}
\{y \ : \ 1 \succeq_y 2 \succeq_y 3\} &= \{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ y_1 \leq 0, y_1 \leq y_2\} \\
\{y \ : \ 1 \succeq_y 3 \succeq_y 2\} &= \{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ y_1 \leq 0, y_1 \geq y_2\} \\
\{y \ : \ 2 \succeq_y 3 \succeq_y 1\} &= \{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ y_1 \geq 0, y_1 \leq y_2\} \\
\{y \ : \ 3 \succeq_y 2 \succeq_y 1\} &= \{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ y_1 \geq 0, y_1 \geq y_2\}
\end{aligned}$$

These preferences violate part (i) of *(A4)* since, for instance, there is no voter y such that $2 \succ_y 1 \succ_y 3$ (see Figure 3). It is easy to check that the rest of the axioms are satisfied. We claim that these preferences are not consistent with the spatial model. Indeed, assume that $\{(x_1, v_1), (x_2, v_2), (x_3, v_3)\}$ represent these preferences. Then it must be the case that $x_2 - x_1 = (\alpha, 0)$ and $x_3 - x_1 = (\beta, 0)$ for some $\alpha, \beta > 0$. Taking the difference of these two equalities we get $x_2 - x_3 = (\alpha - \beta, 0)$. However, the line separating voters with opposite preferences over candidates 2 and 3 is the main diagonal, and so $x_2 - x_3$ must be of the form $(-\gamma, \gamma)$ for some $\gamma > 0$.

To see that part (ii) of *(A4)* is necessary, consider the following preferences ($d = 2$ and $C = \{1, 2, 3, 4\}$):

$$\begin{aligned}
\{y \ : \ 1 \succeq_y 2 \succeq_y 3 \succeq_y 4\} &= \{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ y_1 \leq 0, -y_1 \leq y_2\} \\
\{y \ : \ 1 \succeq_y 3 \succeq_y 2 \succeq_y 4\} &= \{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ y_2 \geq 0, -y_1 \geq y_2\} \\
\{y \ : \ 3 \succeq_y 1 \succeq_y 4 \succeq_y 2\} &= \{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ y_2 \leq 0, y_1 \leq 2y_2\} \\
\{y \ : \ 3 \succeq_y 4 \succeq_y 1 \succeq_y 2\} &= \{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ y_1 \leq 0, y_1 \geq 2y_2\} \\
\{y \ : \ 4 \succeq_y 3 \succeq_y 2 \succeq_y 1\} &= \{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ y_1 \geq 0, -y_1 \geq y_2\} \\
\{y \ : \ 4 \succeq_y 2 \succeq_y 3 \succeq_y 1\} &= \{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ y_2 \leq 0, -y_1 \geq y_2\} \\
\{y \ : \ 2 \succeq_y 4 \succeq_y 1 \succeq_y 3\} &= \{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ y_2 \geq 0, y_1 \geq 2y_2\} \\
\{y \ : \ 2 \succeq_y 1 \succeq_y 4 \succeq_y 3\} &= \{y = (y_1, y_2) \in \mathbb{R}^2 \ : \ y_1 \geq 0, y_1 \leq 2y_2\}
\end{aligned}$$

The axioms *(A1)*–*(A3)* and part (i) of *(A4)* are satisfied, but part (ii) of *(A4)* is not satisfied since $\{y \ : \ 1 \sim_y 3 \sim_y 4\} = \{y \ : \ 2 \sim_y 3 \sim_y 4\} = \{(0, 0)\}$ (see Figure 4). We now show that these preferences are not consistent with the spatial model.

Assume to the contrary that $\{(x_1, v_1), \ldots, (x_4, v_4)\}$ represent the preferences. Consider first the triplet $\{1, 2, 3\}$. The preferences imply that there are $\alpha, \beta, \gamma > 0$ such that $x_2 - x_1 = (\alpha, 0), x_1 - x_3 = (0, \beta)$, and $x_2 - x_3 = (\gamma, \gamma)$. Summing up the first two equalities and subtracting the third we get $(0,0) = (\alpha - \gamma, \beta - \gamma)$. Thus, $\alpha = \beta = \gamma$. A similar argument for the triplet $\{2,3,4\}$ implies that there is $a > 0$ such that $x_2 - x_3 = (a,a)$, $x_4 - x_3 = (a,0)$ and $x_2 - x_4 = (0,a)$. Since the pair $\{2,3\}$ appears in both triplets, it must be the case that $a = \alpha$. Thus, $x_4 - x_1 = x_4 - x_3 + x_3 - x_1 = (\alpha, 0) + (0, -\alpha) = (\alpha, -\alpha)$. However, the line separating voters with opposite preferences over candidates 1 and 4 is $y_1 = 2y_2$, so $x_4 - x_1$ must be of the form $(\alpha, -2\alpha)$ for some $\alpha > 0$.

3.6. **Euclidean preferences.** Our model does not presume any specific kind of voter preferences over the policy space. The primitive only consists of a collection of preferences over candidates indexed by points in $\mathbb{R}^d$. The Euclidean preferences are derived from the axioms.[12]

Another approach would be to assume from the start that voters' preferences over policies are given by the Euclidean distance from their ideal point, and that valence scores are additively separable. In other words, one could test only the second assumption of the spatial model, that the subjective views of voters regarding the implemented policies and valences of the candidates are identical. In this case the model would consist of sets $\{x_i(y)\}_{i \in C} \subseteq \mathbb{R}^d$ and $\{v_i(y)\}_{i \in C} \subseteq \mathbb{R}$ for every $y \in \mathbb{R}^d$. It is easy to see that one can obtain a similar result to that of Theorem 1 in this case.

The Euclidean norm is intimately related to the betweenness axiom *(A3)*. Other norms, such as the 'sup–norm' or the 'city–block metric', typically induce non–convex sets. A thorough study of the relation between convexity and the Euclidean norm, as well as of the kind of preferences induced by other norms is beyond the scope of this paper.

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[12] This is not to say that representations using other metrics are impossible. However, we conjecture that the only kind of norms that induces convex sets (satisfy axiom *(A3)*) are those induced by some inner product.

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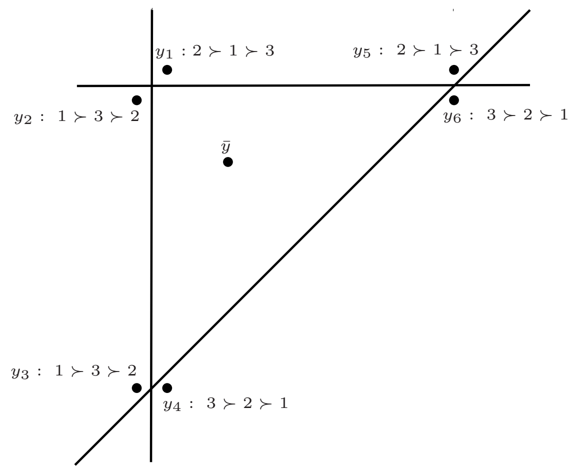

Figure 1: A finite set of voters

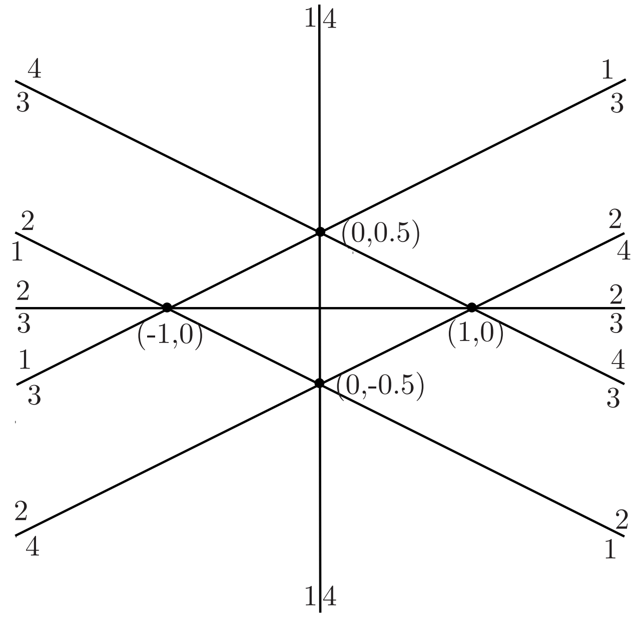

Figure 2: The valence dimension

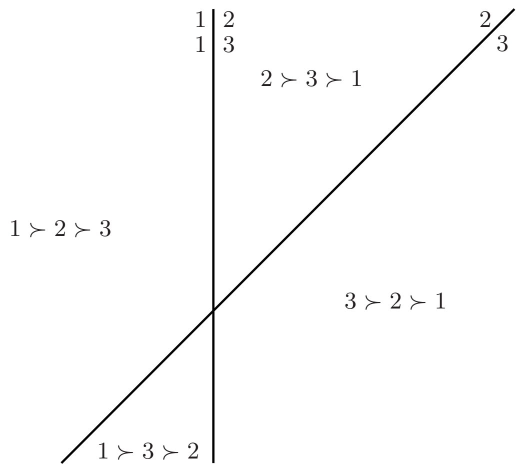

Figure 3: Violation of *(A4)* (i)

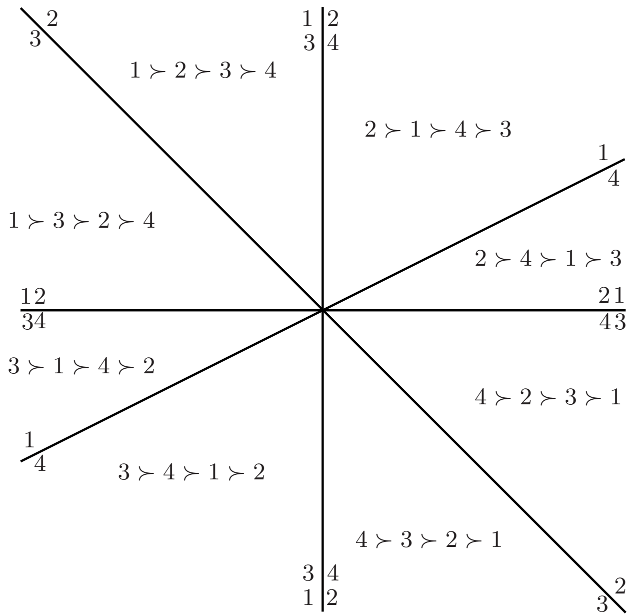

Figure 4: Violation of *(A4)* (ii)