Linking the Kar and Folk Solutions Through a Problem Separation Property

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Abstract

Minimum cost spanning tree problems try to connect agents efficiently to a source when agents are located at different points in space and the cost of using an edge is fixed. We examine solution concepts to divide the common cost of connection among users and revisit the dispute between two of the most familiar solutions: the Kar and folk solutions, both based on the familiar Shapley value. We characterize the family of solutions corresponding to the affine combination of the Kar and folk solutions. The weights being put on the Kar and folk solutions are related to how we share cost in a simple two-agent problem. This family is characterized using a new property called Weak Problem Separation that allows, under conditions, to divide the problem in two: the connection of an agent to the source and the connection of agents to each other. New characterizations of the Kar and folk solutions are then offered. In addition, a new rule is proposed and characterized.

1 Introduction

Minimum cost spanning tree (mcst) problems study situations where a group of agents, located at different points in space, need to be connected to a source. Agents can be connected directly to this source or indirectly through other agents already connected. Connection costs on an edge between two agents or between an agent and the source is a fixed cost, invariant with the number of users connecting through it. Examples of economic situations that can be modeled as mcst problems include electricity distribution networks as well as communication networks such as Internet, cable TV or telephone.

Finding the optimal configuration of the network, a minimum cost spanning tree, was the focus of the early operations research literature, and it provided efficient algorithms (Kruskal (1956), Prim (1957)).

The next task was to define methods to share the common costs of these networks. Bird (1976) was the first to study the problem with cooperative game theory tools. He provided a method based on Prim's algorithm. Dutta and Kar (2004) introduced a distinct method based on the algorithm. Their method also satisfies a cost monotonicity property; if the cost of the edge between i and j increases, these agents cannot benefit from this, a property that Bird's solution does not always satisfy. The two methods share the unappealing characteristic that they depend heavily on the minimum cost spanning tree and do not behave smoothly when there are more than one mcst, a clear possibility.

Two other cost sharing solutions are built around the familiar Shapley value. Kar (2002) applies the Shapley value to the stand-alone game associated with the mcst problem, where a coalition can only use edges between its agents to build an optimal network. This solution is known as the Kar solution. The second method was discovered independently by Feltkamp et al. (1994) and Bergantinos and Vidal-Puga (2007a), as well as being the average of the family of solutions proposed by Norde et al. (2004). While there are many ways to interpret the solution, one of them consists in defining the irreducible cost matrix, which is such that the cost of the edges on the mcst remains unchanged, while the cost on other edges is reduced up to the point where further reductions would change the total cost to connect everybody to the source. The so-called folk solution is the Shapley value of the stand-alone game associated with the irreducible cost matrix. Remarkably, the two solutions have not been characterized together. However, Bergantinos and Vidal-Puga (2010) show that they can be implemented using similar non-cooperative mechanisms.

The Kar solution has been criticized in the recent literature for two reasons: it does not always propose a stable solution where no coalition has an incentive to quit the group and do the project on its own, and it sometimes proposes negative cost shares.

While the first criticism is valid, we feel that the second one is not. In mcst problems, when an agent i joins a coalition S, if we suppose that a coalition can only use the location of its members, then it is possible for the cost of connecting agents in $S \cup \{i\}$ to the source to be smaller than the cost to connect agents in S. It is natural to impose non-negativity of the cost shares for monotonic games, since we do not want agents to pay less than their smallest incremental cost, which are all non-negative. However, for non-monotonic games this natural lower bound is not appropriate as incremental costs can be negative. The assumption that a coalition can only use the locations of its members is natural when there is a notion of property rights over these locations. This is the case in many applications of the mcst model. For example, Russian natural gas producer Gazprom sends gas to Europe through Ukraine. In exchange for allowing European nations to reach the source of natural gas cheaply, Ukraine is compensated with transit fees by Gazprom. Negotiations over these fees have played a role in the many disputes between Ukraine and Gazprom over the past years.

While the folk solution offers a stable and non-negative allocation, it depends only on the irreducible cost matrix. Therefore, a large portion of the information contained in the cost matrix is lost. (See Bogomolnaia and Moulin (2008) for a critique of this Reductionism property.) Similar criticisms can be made of other stable allocation solutions.

This paper offers a way to reconcile the two cost sharing solutions by defining and characterizing a family of solutions that contains both solutions. This is done by introducing some new properties, with the main one being the Weak Problem Separation property. Suppose that the optimal way to connect agents is to have only one of them connected to the source, and that this edge is the most expensive of the minimum cost spanning tree. Then, Weak Problem Separation says that we can split the problem in two: finding who to connect to the source and then connecting all agents together. Applying a cost sharing solution on the mcst problem or separately on these two problems should yield the same cost shares. This property allows one to split the problem into two simpler problems and is, thus, similar in nature to many properties found in the cost sharing literature, most notably the Additivity property.

This property is combined with three known properties: Piecewise Linearity, Group Independence and Symmetry; and two new ones: Independence of Irrelevant Arcs and Free Cycle Consistency. Piecewise Linearity allows one to decompose the problem into elementary problems where edges have a cost of 0 or 1. Group Independence says that if two groups are such that no pair of agents from different groups gain anything from being connected directly together, then we can share cost independently on each group. Symmetry says that if two agents are symmetric with respect to the cost of their edges, then they should have the same cost share. Independence of Irrelevant Arcs states that if the cost of the edge between i and j is larger then the cost to connect i to the source and the cost to connect j to the source, then cost shares should not depend on the cost of the edge between i and j, since this edge will never be used. Free Cycle Consistency applies when there exists a cycle of free arcs between a group of agents. In this case, there are many different mcst. This property says that cost shares should be the average of the cost shares when we remove, one by one, an agent from the cycle by increasing the cost of the arcs connecting him to other agents. It is, therefore, similar in nature to the procedure used to compute the Bird allocation when there are more than one mcst.

The set of solutions satisfying these six properties is composed of solutions that are a weighted sum of the Kar and folk solutions. Since the weights sum to one, we actually have an affine combination of the Kar and folk solutions, leaving only one parameter free. This parameter turns out be the cost share in a simple two-agent problem. This not only shows clearly the difference between the two rules, but it can also ease implementation. Once agents agree that the six properties are desirable, their opinion of what should be the allocations in the simple two-agent problem will be enough to generate the corresponding cost-sharing solution.

We offer new characterizations of the folk and Kar solutions. The folk solution is obtained by adding one of the following properties: Core Selection, Population Monotonicity or Non-Negativity. Core Selection assures that all cost shares are stable allocations. Population Monotonicity states that no agent should be worse off when a coalition grows. Non-Negativity prohibits subsidizing an agent. The Kar solution is obtained by either strengthening the Weak Problem Separation property or by adding a new property called Source Connection Appropriation, which says that if an agent who has the smallest connection cost to the source sees that cost decrease, then he should get all of this cost reduction.

The set also contains another natural solution, which had not been studied previously. In opposition to the folk solution, which severely limits the ability of an agent to benefit from his location, this new solution gives much more credit to an agent who has an advantageous location in the network that allows others to connect to the source at a lower cost. It is characterized by a Location Dependence property that says that if an agent is the only one that allows others to improve on their stand-alone connection cost, he should extract all of the surplus.

The structure of the paper is as follows. In section 2 we formally define mcst problems as well as the Kar and the folk solutions. Definitions of the main properties used in the paper are given in section 3 together with the main theorem describing the family of solutions characterized by these properties. Section 4 offers characterizations of the Kar and folk solutions. Section 5 introduces and characterizes a new solution in the family. Parts of the proofs are in the appendix.

2 The setting

2.1 Minimum cost spanning tree problems

Let $N = \{1, ..., n\}$ be the set of agents and let 0 denote the source to which agents have to be connected. Let $N_0 = N \cup \{0\}$. For any set Z, define Z^p as the set of all non-ordered pairs (i, j) of elements of Z. In our context, any element (i, j) of Z^p represents the edge between i and j. Let $c = (c_{ij})_{i,j \in N_0^P}$ be a vector in $\mathbb{R}_+^{N_0^P}$ with c_{ij} representing the cost of edge (i, j). Let $\Gamma(n)$ be the set of all cost vectors when N contains n agents, with $n \in \mathbb{N}$. Let Γ be the set of all cost vectors, for all possible values of n. Since c assigns cost to all edges (i, j), we often abuse language and call c a cost matrix. A minimum cost spanning tree problem is a triple $(0, c, N)$. Since 0 does not change, we omit it in the following and simply identify a mcst problem as (c, N).

A spanning tree is a non-orientated graph without cycles that connects all elements of N_0. A spanning tree t is identified by the set of its edges. Its associated cost is $\sum_{e \in t} c_e$.

Let p^{lm} be a path between l and m. It is a set of K edges (i_k, i_{k+1}), with $k \in [0, K-1]$, containing no cycle and such that $i_0 = l$ and $i_K = m$. Let $P^{lm}(N_0)$ be the set of all such paths between l and m. For a set of edges $Y \in N_0^p$, we say that Y is in $S \subseteq N_0$ if for all $(i, j) \in Y$, $i, j \in S$. We say that Y contains a cycle in S if, for all $i \in S$, there exists a path p^{ii} in S that contains at least three edges and such that all elements of p^{ii} are also in Y. We say that a path p^{lm} is a free path if $c_e = 0$ for all $e \in p^{lm}$.

The minimum cost of connecting N to the source and the associated minimum cost spanning tree is obtained using Prim's algorithm, which has n steps. First, pick an edge $(0, i)$ such that $c_{0i} \leq c_{0j}$ for all $j \in N$. We then say that i is connected. In the second step, we choose an edge with the smallest cost connecting an agent in $N \backslash \{i\}$ either directly to the source or to i, which is connected. We continue until all agents are connected, at each step connecting an agent not already connected to an agent already connected or to the source. Let $C(N, c)$ be the associated cost. Note that the mcst might not be unique. Let $t^*(c)$ be a minimum cost spanning tree for the cost matrix c. Let $T^*(c)$ be the set of all minimum cost spanning trees for the cost matrix c.

Let c^S be the restriction of the cost matrix c to the coalition $S_0 \subseteq N_0$. Let $C(S,c)$ be the cost of the mcst of the problem (S, c^S). Given these definitions, we say that C is the stand-alone cost function associated with c.

2.2 Cost sharing solutions

A cost allocation $y \in \mathbb{R}^n$ assigns a cost share to each agent, and the budget balance condition is $\sum_{i\in N} y_i = C(N,c)$. Note that these cost shares can be negative. Since C is not necessarily monotonic, we have ample justification to subsidize an agent.

A cost sharing solution (or rule) assigns a cost allocation $y(c,N)$ to any admissible mcst problem (c,N). We introduce the two solutions that are the focus of the paper.

The Kar solution was explicitly defined and characterized in Kar (2002). It is the Shapley value of the game C. More precisely,

$$y_i^k(c,N) = Sh_i(C) = \sum_{S\subseteq N\setminus\{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(C(S\cup\{i\}, c) - C(S,c)\right)$$

for all $i \in N$, with $C(\emptyset, c) = 0$. See Winter (2002) for a review of the broad applications and appeal of the Shapley value.

As mentioned in the introduction, the so-called folk solution has been obtained in different ways. We focus on the approach of Bergantinos and Vidal-Puga (2007a), which uses the Shapley value, thus allowing a clear comparison with the Kar solution.

From any cost matrix c, we can define the irreducible cost matrix c^* as follows:

$$c_{ij}^* = \min_{p^{ij}\in P^{ij}(N_0)} \max_{e\in p^{ij}} c_e.$$

The folk solution is the Shapley value of the stand-alone cost function associated to c^*, defined as $C^*(S,c) = C(S,c^*)$ for all $S \subseteq N$.

Bogomolnaia and Moulin (2008) offer a closed-form expression of the folk solution. Fix i and rearrange the cost c_{ij}^* of the $n-1$ edges connecting agent i to other agents in increasing order as c_i^{*k} such that $c_i^{*1} \le c_i^{*2} \le ... \le c_i^{*(n-1)}$. Then, the folk solution $y^f(N,c)$ can be written as

$$y_i^f(c,N) = \frac{1}{n}c_{0i}^* + \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \min\left\{c_i^{*k}, c_{0i}^*\right\}.$$

Another interpretation, found in Bergantinos and Vidal-Puga (2007b), uses the notion of an optimistic game. This game assigns to any coalition S the cost of connecting its members to the source under the assumption that agents in $N\setminus S$ are already connected and that agents in S can use their locations. Formally, for any $c \in \Gamma$ and $S \subset N$, let $\bar{c}^{N\setminus S} \in \mathbb{R}_+^{S_0^p}$ be the cost matrix such that for all $i \in S$, $\bar{c}_{0i}^{N\setminus S} = \min_{j\in N_0\setminus S} c_{ij}$ and $\bar{c}_{ij}^{N\setminus S} = c_{ij}$ for all $j \in S$. Then, the optimistic game is C^o, where $C^o(S,c) = C(S, \bar{c}^{N\setminus S})$. We can then define the folk solution as $y^f(c,N) = Sh(C^o)$.

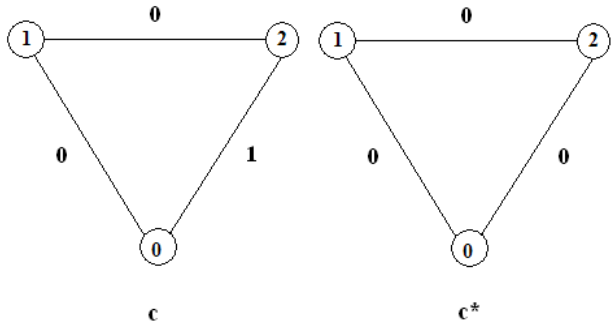
By contrast, the Kar solution is the Shapley value of the stand-alone game where a coalition assumes that others are not connected and that it cannot use their locations, also called the pessimistic stand-alone game.

Even on simple games, the two solutions often propose allocations that are quite different. Consider the following two player example.

Example 1 $N = \{1,2\}$, *c and c^* are represented below. Agents are identified in the circles and the costs are next to the edges.*

We have the following values for $C(\cdot,c)$ and $C(\cdot,c^*)$.

Figure 1: Two-agent example



S	$C(S,c)$	$C(S,c^*)$
$\{1\}$	*0*	*0*
$\{2\}$	*1*	*0*
$\{1,2\}$	*0*	*0*

We obtain the following allocations: $y_1^k(c,\{1,2\}) = -\frac{1}{2}$, $y_2^k(c,\{1,2\}) = \frac{1}{2}$, $y_1^f(c,\{1,2\}) = 0$ *and* $y_2^f(c,\{1,2\}) = 0$.

3 Problem Separation and solutions satisfying it

Before introducing the main new property, we start with a new but very weak property stating that cost shares should not depend on the cost of edges that are never used. An edge (i,j) is irrelevant if $c_{ij} > \max[c_{0i}, c_{0j}]$. Such an edge is never used, as it is always preferable to connect agents i and j through the source.

Let $\bar{\Gamma}$ be set of cost matrices such that there are no irrelevant edges; i.e. $c_{ij} \leq \max[c_{0i}, c_{0j}]$ for all $i,j \in N$. Let $\bar{c} \in \bar{\Gamma}$ be the cost matrix with no unused edges associated with c. For all $i,j \in N$, $\bar{c}_{ij} = \min[c_{ij}, \max[c_{0i}, c_{0j}]]$, while $\bar{c}_{0i} = c_{0i}$ for all $i \in N$.

Independence of Irrelevant Edges: For any $c \in \Gamma(|N|)$, $y(c,N) = y(\bar{c},N)$.

Therefore, if a solution satisfying Independence of Irrelevant Edges is well defined on $\bar{\Gamma}$, it is also uniquely defined on Γ. This very mild property is satisfied by all usual cost sharing solutions.

Notice that for $c \in \bar{\Gamma}$, there is always a mcst such that only one agent is connected to the source. Therefore, the minimum cost spanning tree problem contains two sub problems: connecting one agent to the source and connecting that agent to all others. We introduce a new property based on this observation that we are able to split a minimum cost spanning tree problem into these two problems. Therefore, applying a cost sharing solution to the whole problem or independently to the sub-problems should yield the same result.

Formally, let $\hat{c}$ be the source connection problem associated with $c:$ for all $i \in N$, $\hat{c}_{0i} = c_{0i}$, while $\hat{c}_{ij} = 0$ for all $i,j \in N$. Then, all that is left are the costs to connect agents to the source. The mcst is such that one agent is connected to the source (i such that $\hat{c}_{0i} \leq \hat{c}_{0j}$ for all $j \in N$) and all others are connected to him (at no cost since $\hat{c}_{ij} = 0$ for all $i,j \in N$).

Let $\tilde{c}$ be the agent connection problem associated with c : for all $i,j \in N$, $\tilde{c}_{ij} = c_{ij}$, while $\tilde{c}_{0i} = \max_{e \in N_0^P} c_e$ for all $i \in N$. Then, all agents have the same (high) cost to connect to the source, so the mcst is such that only one (random) agent is connected to the source, and all other agents are connected through this agent.

We have this relationship between stand alone costs on $c, \hat{c}$ and $\tilde{c}:$

Lemma 1 *For all* $c \in \bar{\Gamma}(|N|)$ *and* $S \subseteq N$, $C(S,c) = C(S,\hat{c}) + C(S,\tilde{c}) - \max_{e \in N_0^P} c_e$.

Proof. Since for all $c \in \bar{\Gamma}$ there is always a mcst such that only one agent is connected to the source, this is also true for all c^S, with $S \subset N$. Then, there is always a mcst for the problem (c^S, S) such that one agent is connected to the source and all other agents are connected to this agent. Let w^S be (one of) the cheapest way to connect agents in S using only the edges between agents in S (not using the source). Therefore, $C(S,c) = \min_{i \in S} c_{0i} + \sum_{e \in w^S} c_e$.

Consider now the cost matrix $\hat{c}$. Clearly, $C(S, \hat{c}) = \min_{i \in S} \hat{c}_{0i} = \min_{i \in S} c_{0i}$. Consider the cost matrix $\tilde{c}$. There always is a mcst for the problem $(\tilde{c}^S, S)$ such that one agent is connected to the source and all other agents are connected to this agent. Therefore, $C(S, \tilde{c}) = \max_{e \in N_0^P} c_e + \sum_{e \in w^S} c_e$.

Therefore, $C(S, \hat{c}) + C(S, \tilde{c}) - \max_{e \in N_0^P} c_e = \min_{i \in S} c_{0i} + \max_{e \in N_0^P} c_e + \sum_{e \in w^S} c_e - \max_{e \in N_0^P} c_e$ which simplifies to $\min_{i \in S} c_{0i} + \sum_{e \in w^S} c_e$. Therefore, $C(S,c) = C(S, \hat{c}) + C(S, \tilde{c}) - \max_{e \in N_0^P} c_e$. ■

Since we have this relationship between a mcst problem and its associated source connection and agent connection problems, it becomes natural to introduce a property linking the cost shares of the mcst problem to those of the source connection and agent connection problems.

Problem Separation: For any $c \in \bar{\Gamma}(|N|)$, $y_i(c,N) = y_i(\hat{c}, N) + y_i(\tilde{c}, N) - \frac{1}{|N|} \max_{e \in N_0^P} c_e$ for all $i \in N$.

This property, however, might be too strong. In fact, the folk solution does not satisfy it. While we have the relationship of Lemma 1, notice that in both the source connection and agent connection problems, we can find the optimal tree by connecting an agent to the source and all others through that agent. While for any $c \in \bar{\Gamma}$ there exists such an optimal tree, we will limit the scope of Problem Separation to a subset of problems where it is natural to connect only one agent to the source. This case is as follows: suppose that the smallest cost to connect an agent to the source is x. If for any pair of agents $i, j \in N$, we can construct a path p_{ij} in N such that the cost of all edges is no larger than x, then clearly the optimal way to go is to connect only one agent to the source. In many examples associated to mcst problems, it is expected that costs to connect to the source are large compared to costs to connect agents between themselves.[1]

Weak Problem Separation: For any $c \in \bar{\Gamma}(|N|)$, if $c_e \leq \min_{i \in N} c_{0i}$ for all $e \in t^*(c)$, all $t^* \in T^*(c)$, then $y_i(c,N) = y_i(\hat{c}, N) + y_i(\tilde{c}, N) - \frac{1}{|N|} \max_{e \in N_0^P} c_e$ for all $i \in N$.

We therefore restrict the scope of Problem Separation to the set of problems where there is no edge used in a mcst that is more expensive than the cheapest edge connecting an agent to the source.

Weak Problem Separation and Independence of Irrelevant Edges will be used with four other properties, with the next three being already known in the literature.

We use a symmetry property, found in Bergantinos and Vidal-Puga (2009) and Bogomolnaia and Moulin (2008):

Symmetry: For any $c \in \Gamma(|N|)$, if i, j are such that $c_{ik} = c_{jk}$ for all $k \in N_0 \backslash \{i,j\}$, then $y_i(c,N) = y_j(c,N)$.

We also use one of the properties used to characterize the Kar solution (Kar (2002)), that says that if we can split our agents into two groups that can be connected independently to the source, then we can do the cost sharing separately on these two groups. More precisely, two groups S and $N \backslash S$ can be connected independently to the source if for all $i \in S$ and $j \in N \backslash S$, $c_{ij} \geq \max[c_{0i}, c_{0j}]$. Then, it is always as costly to connect two agents in distinct groups directly one to the other than to connect them both to the source.

Group Independence: For any $c \in \Gamma(|N|)$, if $S \subset N$ is such that for all $i \in S$ and $j \in N \backslash S$, $c_{ij} \geq \max[c_{0i}, c_{0j}]$, then, $y_i(c,N) = \begin{cases} y_i(c^S, S) \text{ if } i \in S \\ y_i(c^{N \backslash S}, N \backslash S) \text{ if } i \in N \backslash S \end{cases}$.

Kar (2002) actually uses a weaker version, where S and $N \backslash S$ are considered distinct if for all $i \in S$ and $j \in N \backslash S$, $c_{ij} > \max[c_{0i}, c_{0j}]$. Removing the strict inequality adds the case where we are indifferent between connecting agents from distinct groups to each other or independently. The mere fact that the groups can be connected independently seems a sufficient reason to consider the groups

[1] In their study of Monotonicity and Ranking properties, Bogomolnaia and Moulin (2008) use a stronger but similar restriction. They apply their properties to the set of cost matrices such that for all $i, j \in N$, $c_{ij} \leq \min_{k \in N} c_{0k}$. That is, they suppose that all connection costs between agents are lower than all connection costs to the source.

as independent.

Note that when we apply Group Independence to a problem (c,N) with $c \in \bar{\Gamma}(N)$, there is no edge (i,j) such that $c_{ij} > \max\left[c_{0i}, c_{0j}\right]$. Groups S and $N \backslash S$ are considered independent if for all $i \in S$ and $j \in N \backslash S$, $c_{ij} = \max\left[c_{0i}, c_{0j}\right]$. Then we are indifferent between connecting agents in S alone or with agents in $N \backslash S$. More importantly, even in this case, there is no gain for coalition S (or any of its subsets) to cooperate with agents in $N \backslash S$ (or any of its subsets), which justifies their cost shares being computed independently.

Next, we define Piecewise Linearity, which says that if we can decompose a cost matrix into submatrices where the cost of all edges are ordered in the same manner as the original matrix, then the cost allocation on the original cost matrix should equal the sum of the cost allocations on the submatrices. This property (or similar versions), a weaker version than the classical Additivity property in the general setting, has been used in Branzei et al. (2004), Tijs et al. (2005, 2006), Bergantinos and Vidal-Puga (2009) and Bogomolnaia and Moulin (2008). Piecewise Linearity generates a rich class of solutions having a simple structure. Cost shares can be defined on simple elementary matrices where costs of all edges are either 0 or 1, making it particularly appealing. In addition, many normative properties easily defined on those elementary matrices automatically extend to arbitrary matrices.

To formally define the Piecewise Linearity property we need the following notation. Suppose $N = \{1, ..., n\}$ and denote arbitrarily the $p = \frac{n(n+1)}{2}$ distinct edges in N_0^P, such that $c = (c_{e^1}, ..., c_{e^p})$. For any permutation σ of $\{1, ..., p\}$, define $K_\sigma(n) = \{c \in \Gamma(n) \mid c_{e^{\sigma(1)}} \leq c_{e^{\sigma(2)}} \leq ... \leq c_{e^{\sigma(p)}}\}$ to be the cone in $\Gamma(n)$ containing all cost matrices with a given increasing ordering of connection costs. Note that $\Gamma(n) = \cup K_\sigma(n)$.

Piecewise Linearity: a cost sharing solution y is piecewise linear if it is linear in K_σ for all σ.

More precisely, denote by Γ^e the set of elementary cost matrices where all connection costs are either 0 or 1 : $\Gamma^e(|N|) = \left\{c \in \Gamma(|N|) : c_e \in \{0,1\} \text{ for all } e \in N_0^P\right\}$.

For any cone $K_\sigma(|N|)$ and any $k \in \{1, ..., p\}$, let $b^k \in \Gamma^e(|N|)$ be such that $b^k_{e^{\sigma(1)}} = ... = b^k_{e^{\sigma(k-1)}} = 0$ while $b^k_{e^{\sigma(k)}} = ... = b^k_{e^{\sigma(p)}} = 1$. Then, a cost sharing solution is piecewise linear if

$$y(c,N) = \sum_{k=1}^{p} \left(c_{e^{\sigma(k)}} - c_{e^{\sigma(k-1)}}\right) y(b^k, N) \text{ for any } \sigma \text{ and any } c \in K_\sigma(|N|).$$

Therefore, if a solution satisfying Piecewise Linearity is well defined on Γ^e, it is also uniquely defined on Γ. Let y be a solution defined over Γ^e. The piecewise linear extension of y is a solution y^L such that for all $c \in \Gamma$, $y^L(c,N) = \sum_{k=1}^{p} \left(c_{e^{\sigma(k)}} - c_{e^{\sigma(k-1)}}\right) y(b^k, N)$.

Having defined the set of elementary matrices, we define the subsets of elementary matrices that correspond to source connection and agent connection problems. The set of elementary cost matrices generating source connection problems is $\hat{\Gamma}^e$, the set of elementary matrices c with no irrelevant edges and such that $c_{ij} = 0$ for all $i, j \in N$. The set of elementary cost matrices generating agent connection problems is $\tilde{\Gamma}^e$, the set of elementary matrices c with no irrelevant edges and such that $c_{0i} = 1$ for all $i \in N$.

We introduce a final property that puts restrictions on how cost sharing solutions should behave in presence of free cycles. Suppose that because many edges have a cost of zero, there exist many different mcst t^*. The union of those trees contains at least one cycle such that all edges in the cycle are free. Then, Free Cycle Consistency says that cost shares for c should be the average of the cost shares when we remove, one by one, an agent from the cycle, replacing the costs of its edges connecting him to others in the cycle by the next highest cost (with an adjustment made to maintain budget balance). If the free cycle contains m agents, there are multiple mcst that all contain $m-1$ free arcs. Free Cycle Consistency is somewhat similar to the procedure used to compute the Bird solution when there are multiple mcst. The Bird solution takes a particular t^* and assigns to each agent the cost of the edge that connects him to the network. If there are multiple mcst, we take the average over all of them. The procedure in Free Cycle Consistency is therefore similar, as we modify the cost matrix to eliminate many mcst.

We need to define the following notation. Let $F(c) = \left\{ i \in N \mid \text{there exists a free cycle } p^{ii} \text{ in } N \right\}$. For all $i \in F(c)$, let $F^i(c) = \left\{ j \in N \backslash \{i\} \mid \text{there exists a free cycle } p^{ii} \text{ that go through } j \right\}$.

Free Cycle Consistency: For any $c \in \Gamma(|N|)$, if $F(c) \neq \emptyset$, then, $y(c, N) = \frac{\sum_{j \in F(c)} y(c^{-j}, N) - \alpha}{|F(c)|}$ with $\alpha = \min_{\substack{e \in N_0^P \\ c_e > 0}} c_e$ being the smallest positive cost in the network[2] and c^{-i} such that

$$c_{kl}^{-i} = \begin{cases} max(\alpha, c_{kl}) \text{ if } k = i \text{ and } l \in F^i(c) \\ c_{kl} \text{ else} \end{cases}.$$

Compared to the problem (c, N), in all problems (c^{-j}, N), the cost to connect agent j goes from 0 to α. Therefore, $C(N, c^{-j}) = C(N, c) + \alpha$. Since this is true for all j, this cost difference is simply divided equally among members of S.

$\tilde{\Gamma}_{NC}^e(|N|)$ is the set of elementary matrices c with no irrelevant edges, with $c_{0i} = 1$ and such that there are no free cycles p_{ii} in N, for all $i \in N$.

Let y be a solution defined over $\tilde{\Gamma}_{NC}^e$. The cycle consistency extension of y is a solution y^C such that for all $c \in \tilde{\Gamma}^e$, if $F(c) \neq \emptyset$, $y^C(c, N) = \frac{\sum_{j \in S} y(c^{-j}, N) - \alpha}{|S|}$.

We first show that under these properties, a cost sharing solution is uniquely defined by its values on a small subset of problems.

Lemma 2 *If a solution satisfies Free Cycle Consistency, Group Independence, Weak Problem Separation, Independence of Irrelevant Edges and Piecewise Linearity, it is uniquely defined by its values on problems (c, N) with c in $\tilde{\Gamma}_{NC}^e$ or $\hat{\Gamma}^e$.*

Proof. Suppose that y is uniquely defined for problems on $\tilde{\Gamma}_{NC}^e$ and $\hat{\Gamma}^e$ and satisfies Free Cycle Consistency, Group Independence, Weak Problem Separation, Independence of Irrelevant Edges and Piecewise Linearity.

Suppose that $N = \{1, ..., n\}$ and c is an elementary cost matrix such that $\min_{i \in N} c_{0i} = 1$, that is $c \in \tilde{\Gamma}^e(n)$. If $c \in \tilde{\Gamma}_{NC}^e(n)$, then by assumption, $y(c, N)$ is defined. If $c \notin \tilde{\Gamma}_{NC}^e(n)$, then $F(c) \neq \emptyset$ and it contains at least one free cycle. By Free Cycle Consistency, $y(c, N) = \frac{\sum_{j \in F(c)} y(c^{-j}, N) - \alpha}{|F(c)|}$. We either have that $F(c^{-i}) = \emptyset$ or $F(c^{-i}) \neq \emptyset$. In the first case, $c \in \tilde{\Gamma}_{NC}^e(n)$ and $y(c^{-i}, N)$ is well defined. In the second case, $c \notin \tilde{\Gamma}_{NC}^e(n)$ and still contains a free cycle. By Group Independence, $y_i(c, N) = y_i(c^{\{i\}}, \{i\})$ and $y_j(c, N) = y_j(c^{N \backslash \{i\}}, N \backslash \{i\})$ for all $j \in N \backslash \{i\}$. The cost matrix $c^{N \backslash \{i\}}$ is in $\tilde{\Gamma}^e(n-1)$. Since $\tilde{\Gamma}^e(2)$ is well defined (as there cannot be a free cycle in $\{1, 2\}$), recursively, we can find values for all c in $\tilde{\Gamma}^e(k)$, $k \in \mathbb{N}$. Therefore $y(c, N)$ is well defined for any $c \in \tilde{\Gamma}^e(|N|)$.

Suppose that $c \in \bar{\Gamma}^e$ is an elementary cost matrix such that $\min_{i \in N} c_{0i} = 0$. If there exists a free path linking all agents in N, then by Weak Problem Separation, $y(\hat{c}, N) + y(\tilde{c}, N) - \frac{1}{|N|} \max_{e \in N_0^P} c_e$, with $\hat{c} \in \hat{\Gamma}^e$ and $\tilde{c} \in \tilde{\Gamma}^e$. Therefore, $y(c, N)$ is well defined.

If there does not exist a free path linking all agents in N, since $c \in \bar{\Gamma}^e$, there exists a partition of N, $\left\{ N^1, ..., N^K \right\}$ such that for all $i \in N^k$ and $j \in N^l$, with $l \neq k$, $c_{ij} = 1 \geq \max \left[c_{0i}, c_{0j} \right]$. Therefore, by Group Independence, if $i \in N^k$, $y_i(c, N) = y_i(c^{N^k}, N^k)$. By definition, the mcst of problem (c^{N^k}, N^k) consists of picking an arc $(0, i)$ such that $i \in \arg\min_{j \in N_k} c_{0j}$, and then selecting a free path in N^k. Therefore, we can apply Weak Problem Separation and obtain $y(c^{N^k}, N^k) = y(\hat{c}^{N^k}, N^k) + y(\tilde{c}^{N^k}, N^k) - \frac{1}{|N^k|} \max_{e \in N_0^P} c_e$, with $\hat{c}^{N^k} \in \hat{\Gamma}^e$ and $\tilde{c}^{N^k} \in \tilde{\Gamma}^e$. Therefore, $y(c^{N^k}, N^k)$ is well defined for all $k \in 1, ..., K$.

Putting everything together, we have that y is uniquely defined on $\bar{\Gamma}^e$. By Independence of Irrelevant Edges, y is uniquely defined on Γ^e. By Piecewise Linearity, y is uniquely defined on Γ. ■

This result allows us to characterize a solution by its values for problems on $\tilde{\Gamma}_{NC}^e$ and $\hat{\Gamma}^e$. Before moving on to the main theorem, we define some additional notation. Let $R(c) \subseteq N$ be the set of agents

[2] If $c_e = 0$ for all $e \in N_0^P$, let $\alpha = 1$.

such that $c_{0i}=1$. Any $c\in\hat{\Gamma}^e(|N|)$ is uniquely defined by $R(c)$. Let $Z^i(c)=\{j\in N\backslash\{i\}\mid c_{ij}=0\}$ be the set of agents to which i has a free connection. Any $c\in\tilde{\Gamma}^e(|N|)$ is uniquely defined by $Z^1,Z^2,...,Z^{|N|}$. Let $N_i(c)=\{j\in N\mid \text{there exists a free path } p_{ij} \text{ in } N\}$. Of course, $Z^i(c)\subseteq N_i(c)$. Also, since $i\in N_i(c)$, we always have that $N_i(c)\neq\emptyset$.

The following theorem shows that there is a family of solutions satisfying the set of properties defined in the previous section. These solutions are the affine combination of the Kar and folk solutions. We write the weights such that it depends on a, the cost share of agent 2 in Example 1. Therefore, by determining what we deem fair as a solution in Example 1, we can determine the weight to put on the Kar and folk solutions.

Theorem 1 *A solution y satisfies Weak Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Free Cycle Consistency and Independence of Irrelevant Edges if and only if $y=2a\left(y^k-y^f\right)+y^f$, with $a\in\mathbb{R}$ being the cost share of agent 2 in Example 1.*

Proof. In the appendix, it is shown that all solutions of this form satisfy the six properties. By Lemma 2, if a solution satisfies Weak Problem Separation, Group Independence, Piecewise Linearity, Free Cycle consistency and Independence of Irrelevant Edges, it is uniquely defined by its values on $\hat{\Gamma}^e$ and $\tilde{\Gamma}^e_{NC}$. Therefore, the proof contains the following steps. First, we show that for $c\in\hat{\Gamma}^e$, the properties imply cost shares $y_i^a(c,N)=\begin{cases}\frac{2a}{|N|} \text{ if } i\in R(c) \text{ and } R(c)\neq N\\ \frac{1}{|N|} \text{ if } R(c)=N\\ -\frac{|R(c)|2a}{(|N|-|R(c)|)|N|} \text{ if } i\in N\backslash R(c)\end{cases}$, with $a\in\mathbb{R}$.

Next, we show that for $c\in\tilde{\Gamma}^e_{NC}$, the properties imply $y_i^a(\tilde{c},N)=\frac{1}{|N_i(c)|}+\frac{a(|N_i(c)|-2)}{|N_i(c)|}-\left(\left|Z^i\right|-1\right)a$. Finally, we show that for any a, these cost shares are equivalent to $y^a=2a\left(y^k-y^f\right)+y^f$.

For $c\in\hat{\Gamma}^e(|N|)$, if $R(c)=N$, then $C(N,c)=1$. By Symmetry and budget balance, $y_i(c,N)=\frac{1}{|N|}$

Let $R\subsetneq N$ and $c^{N,R}\in\Gamma^e$ be such that $c_{0i}^{N,R}=\begin{cases}1 \text{ if } i\in R\\ 0 \text{ if } i\in N\backslash R\end{cases}$. Then, we define $a_{|R|}^{|N|}\equiv y_i(\hat{c}^{N,R},N)$ for all $i\in R$. By budget balance and Symmetry, we have $y_i(\hat{c}^{N,R},N)=-\frac{|R|a_{|R|}^{|N|}}{|N|-|R|}$ for $i\in N\backslash R$.

Suppose that $|N|=n$. Fix $i\in N$ and let $c\in\Gamma^e$ be such that

$$\begin{aligned}\text{for all } j,k &\in N\backslash\{i\},c_{jk}=0\\ \text{we have } S &\subset N\backslash\{i\} \text{ such that } c_{ij}=0 \text{ if } j\in S \text{ and } c_{ij}=1 \text{ else, with } |S|=m\\ c_{0j} &= 0 \text{ for all } j\in N\backslash\{i\},\ c_{0i}=1.\end{aligned}$$

By Weak Problem Separation, $y_l(c,N)=y_l(\hat{c},N)+y_l(\tilde{c},N)-\frac{1}{n}$ for all $l\in N$, with $y(\hat{c},N)$ such that $y_i(\hat{c},N)=a_1^n$ and $y_j(\hat{c},N)=-\frac{a_1^n}{n-1}$ for all $j\in N\backslash\{i\}$.

Notice that for $k\in S\cup\{i\}$ and $l\in N\backslash(S\cup\{i\})$, $c_{kl}\geq\max[c_{0k},c_{0l}]$. Therefore, by Group Independence, $y_k(c,N)=y_k(c^{S\cup\{i\}},S\cup\{i\})$ for all $k\in S\cup\{i\}$ and $y_l(c,N)=y_l(c^{N\backslash(S\cup\{i\})},N\backslash(S\cup\{i\}))$ for all $l\in N\backslash(S\cup\{i\})$.

The problem $(c^{S\cup\{i\}},S\cup\{i\})$ is such that $c_{kl}^{S\cup\{i\}}=0$ for all $k,l\in S\cup\{i\}$, $c_{0j}^{S\cup\{i\}}=0$ for $j\in S$, and $c_{0i}^{S\cup\{i\}}=1$. Therefore, $y_i(c^{S\cup\{i\}},S\cup\{i\})=a_1^{m+1}$ and $y_j(c^{S\cup\{i\}},S\cup\{i\})=-\frac{a_1^{m+1}}{m}$.

The problem $(c^{N\backslash(S\cup\{i\})},N\backslash(S\cup\{i\}))$ is such that $c_{kl}^{N\backslash(S\cup\{i\})}=0$ for all $k,l\in N_0\backslash(S\cup\{i\})$. Therefore, by Symmetry, $y_k(c^{N\backslash(S\cup\{i\})},N\backslash(S\cup\{i\}))=0$ for all $k\in N\backslash(S\cup\{i\})$.

Combining these results, we obtain

$$\begin{aligned}y_i(\tilde{c},N) &= \frac{1}{n}+a_1^{m+1}-a_1^n\\ y_j(\tilde{c},N) &= \frac{1}{n}+\frac{a_1^n}{n-1}-\frac{a_1^{m+1}}{m} \text{ for } j\in S\\ y_j(\tilde{c},N) &= \frac{1}{n}+\frac{a_1^n}{n-1} \text{ for } j\in N\backslash(S\cup\{i\})\end{aligned}\tag{1}$$

Consider c' such that $c'_{kl} = c_{kl}$ if $k, l \neq 0$, $c'_{0i} = 0$, $c'_{0j} = 0$ if $j \in S$ and $c'_{0j} = 1$ if $j \in N \backslash (S \cup \{i\})$.

By Weak Problem Separation, $y_l(c', N) = y_l(\hat{c}', N) + y_l(\tilde{c}', N) - \frac{1}{n}$ for all $l \in N$, with $y(\hat{c}', N)$ such that $y_j(\hat{c}', N) = a^n_{n-m-1}$ if $j \in N \backslash (S \cup \{i\})$ and $y_j(\hat{c}', N) = -\frac{(n-m-1)a^n_{n-m-1}}{m+1}$ for all $j \in S \cup \{i\}$. Also, notice that $\tilde{c}' = \tilde{c}$, so $y(\tilde{c}', N) = y(\tilde{c}, N)$.

Notice also that for $j \in N \backslash \{i\}$, $c'_{ij} \geq \max\left[c'_{0i}, c'_{0j}\right]$. Therefore, by Group Independence, $y_j(c', N) = y_j(c'^{N \backslash \{i\}}, N \backslash \{i\})$ for all $j \in N \backslash \{i\}$ and $y_i(c', N) = y_i(c'^{\{i\}}, \{i\})$.

The problem $(c'^{N \backslash \{i\}}, N \backslash \{i\})$ is such that $c'^{N \backslash \{i\}}_{kl} = 0$ for all $k, l \in N \backslash \{i\}$, $c'^{N \backslash \{i\}}_{0j} = 0$ for $j \in S$, and $c'^{N \backslash \{i\}}_{0j} = 1$ for $j \in N \backslash (S \cup \{i\})$. Therefore, $y_j(c'^{N \backslash \{i\}}, N \backslash \{i\}) = a^{n-1}_{n-m-1}$ for all $j \in N \backslash (S \cup \{i\})$ and $y_j(c'^{N \backslash \{i\}}, N \backslash \{i\}) = -\frac{(n-m-1)a^{n-1}_{n-m-1}}{m}$ for all $j \in S$.

By budget balance, $y_i(c'^{\{i\}}, \{i\}) = c'_{0i} = 0$.

Putting these results together with the values of $y(\tilde{c}, N)$ found in (1), we obtain

$$
\begin{aligned}
y_i(c', N) &= -\frac{(n-m-1)a^n_{n-m-1}}{m+1} + a^{m+1}_1 - a^n_1 = 0 \\
y_j(c', N) &= -\frac{(n-m-1)a^n_{n-m-1}}{m+1} + \frac{a^n_1}{n-1} - \frac{a^{m+1}_1}{m} = -\frac{(n-m-1)a^{n-1}_{n-m-1}}{m} \text{ for } j \in S \\
y_j(c', N) &= a^n_{n-m-1} + \frac{a^n_1}{n-1} = a^{n-1}_{n-m-1} \text{ for } j \in N \backslash (S \cup \{i\})
\end{aligned}
$$

This gives us the following three equations:

$$
\begin{aligned}
-\frac{(n-m-1)a^n_{n-m-1}}{m+1} + a^{m+1}_1 - a^n_1 &= 0 \\
-\frac{(n-m-1)a^n_{n-m-1}}{m+1} + \frac{a^n_1}{n-1} - \frac{a^{m+1}_1}{m} + \frac{(n-m-1)a^{n-1}_{n-m-1}}{m} &= 0 \\
a^n_{n-m-1} + \frac{a^n_1}{n-1} - a^{n-1}_{n-m-1} &= 0
\end{aligned}
$$

Combining these and simplifying, we obtain

$$
\begin{aligned}
\frac{(m+1)}{m} a^{m+1}_1 &= \frac{n}{n-1} a^n_1 + \frac{(n-m-1)}{m} a^{n-1}_{n-m-1} \\
a^n_{n-m-1} + a^{m+1}_1 &= \frac{n}{m+1} a^n_{n-m-1} + \frac{n}{n-1} a^n_1 \\
\frac{n}{m+1} a^n_{n-m-1} + \frac{1}{m} a^{m+1}_1 &= \frac{n-1}{m} a^{n-1}_{n-m-1}
\end{aligned}
$$

From the second equation, we obtain $a^{m+1}_1 = \frac{n}{m+1} a^n_{n-m-1} + \frac{n}{n-1} a^n_1 - a^n_{n-m-1}$. Using this in the first or third equation gives us

$$a^n_{n-m-1} + \frac{1}{n-1} a^n_1 = a^{n-1}_{n-m-1}. \tag{2}$$

Reinserting this in the second equation gives

$$\frac{n-m-1}{m+1} a^n_{n-m-1} + a^n_1 = a^{m+1}_1. \tag{3}$$

Notice by our assumptions, $1 \leq m \leq n-2$.

In (2), let $m = n-2$. We obtain

$$\frac{n}{n-1} a^n_1 = a^{n-1}_1. \tag{4}$$

By iteration, we obtain

$$a^k_1 = \frac{n}{k} a^n_1 \text{ for } 2 \leq k \leq n.$$

Replacing this in (3), we obtain

$$\frac{n-m-1}{m+1}a^n_{n-m-1}+a^n_1=a^{m+1}_1=\frac{n}{m+1}a^n_1$$

which simplifies to $a^n_{n-m-1}=a^n_1$ for $1\leq m\leq n-2$, which can be restated as $a^n_1=a^n_k\equiv a^n$ for $2\leq k\leq n-2$. It remains to show that $a^n_{n-1}=a^n$.

In (3), let $m=1$. We obtain

$$\frac{n}{n-1}a^n_1=a^{n-1}_{n-2}.$$

Therefore, $a^{n-1}_{n-2}=a^{n-1}_k=a^{n-1}$. These results hold for any values of n, so we can conclude that for all $k\geq 2$, $1\leq l\leq k-1$, $a^k_l=a^k$. Furthermore, let $a\equiv a^2$. Then, by iteration on (4), we get $a^k=\frac{2a}{k}$.

Therefore, for $c\in\hat{\Gamma}^e$, $y^a_i(c,N)=\begin{cases}\frac{2a}{|N|}\text{ if } i\in R(c)\text{ and } R(c)\neq N\\ \frac{1}{|N|}\text{ if } R(c)=N\\ -\frac{|R(c)|2a}{(|N|-|R(c)|)|N|}\text{ if } i\in N\backslash R(c)\end{cases}$, with $a\in\mathbb{R}$.

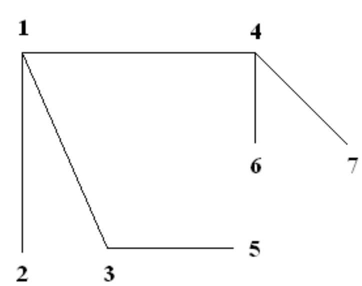
Next, we show that for $\tilde{c}\in\tilde{\Gamma}^e_{PNC}$, $y^a_i(\tilde{c},N)=\frac{1}{|N|}+\frac{a(|N|-2)}{|N|}-\left(\left|Z^i(\tilde{c})\right|-1\right)a$, with $\tilde{\Gamma}^e_{PNC}$ the set of elementary matrices c with no irrelevant edges generating agent connection problems and such that it does not contain a free cycle but, rather, contains a free path between any agent i and j. It is therefore the subset of $\tilde{\Gamma}^e_{NC}$ for which the cost of connecting all agents to the source is equal to 1.

For all $\tilde{c}\in\tilde{\Gamma}^e_{PNC}$, there is a unique free path $p^f_{ij}(\tilde{c})$ between any agents i and j. For $\tilde{c}\in\tilde{\Gamma}^e_{PNC}$, $i,j\in N$, let $N^{-i}_j(\tilde{c})=\left\{k\in N\mid p^f_{jk}(\tilde{c})\text{ is in } N\backslash\{i\}\right\}$. N^{-i}_j represents the agents to which j can connect to freely without i. Then, for $S\subseteq N\backslash\{i\}$, $D^S_i(\tilde{c})=\cup_{j\in S}N^{-i}_j(\tilde{c})$ represent the players in S plus the agents k for which there is a free path p_{jk} in $N\backslash\{i\}$ connecting it to an agent $j\in S$. It represents the set of agents to which agent i can connect freely with the help of agents in S. By definition, $S\subseteq D^S_i(\tilde{c})$. Since $Z_i(\tilde{c})$ is the set of players for which the direct connection to i is free, we have that $D^{Z_i(\tilde{c})}_i(\tilde{c})=N\backslash\{i\}$. Let $d^S_i=\left|D^S_i\right|$.

In the following, we consider c,c' and c'' such that $\tilde{c}=\tilde{c}'=\tilde{c}''$. In order to simplify, we write D^S_i and d^S_i instead of $D^S_i(\tilde{c})$ and $d^S_i(\tilde{c})$.

To make this notation clear, consider the example in Figure 2. It is such that $N=\{1,2,3,4,5,6,7\}$ and the free edges are represented in the figure. Since everyone is connected and there are no free cycles, then the cost matrix is in $\tilde{\Gamma}^e_{PNC}$. Consider agent 1. We have $Z^1=\{2,3,4\}$ and $N^{-1}_2=\{2\}$, $N^{-1}_3=N^{-1}_5=\{3,5\}$, $N^{-1}_4=N^{-1}_6=N^{-1}_7=\{4,6,7\}$. Therefore, we have, among others, $D^{\{2,3\}}_1=\{2,3,5\}$, $D^{\{3,4\}}_1=\{3,4,5,6,7\}$ and $D^{\{2,3,4\}}_1=N\backslash\{1\}$.

Figure 2: Example of cost matrix in $\tilde{\Gamma}^e_{PNC}$.



We now return to the general case. Suppose that $\left|Z^i(\tilde{c})\right|=m$ and that $|N|=n$. Take any ordering of the agents in $Z^i(\tilde{c})$, $j_1,j_2,...,j_m$. The proof contains m steps, one for each agent in $Z^i(\tilde{c})$.

Step 1: Suppose that $c_{0i} = c_{0j_1} = 0$ and $c_{0k} = 1$ else. By Weak Problem Separation, we have that $y_i(c, N) = -\frac{a(n-2)}{n} + y_i(\tilde{c}, N) - \frac{1}{n}$. By definition of $\tilde{\Gamma}^e_{PNC}$, there is only one free path in n. Therefore, for all $k \in D_i^{\{j_1\}}(c)$, $c_{ik} \geq c_{0k}$. We can therefore apply Group Independence. We have that $y_i(c, N) = y_i\left(c^{N\setminus D_i^{\{j_1\}}}, N\setminus D_i^{\{j_1\}}\right)$. By Weak Problem Separation, $y_i\left(c^{N\setminus D_i^{\{j_1\}}}, N\setminus D_i^{\{j_1\}}\right) = -\frac{2a\left(n-d_i^{\{j_1\}}-1\right)}{\left(n-d_i^{\{j_1\}}\right)} + y_i\left(\tilde{c}^{N\setminus D_i^{\{j_1\}}}, N\setminus D_i^{\{j_1\}}\right) - \frac{1}{\left(n-d_i^{\{j_1\}}\right)}$.

Therefore, combining these results, we obtain

$$y_i(\tilde{c}, N) = y_i\left(\tilde{c}^{N\setminus D_i^{\{j_1\}}}, N\setminus D_i^{\{j_1\}}\right) + \frac{a(n-2)}{n} - \frac{2a\left(n-d_i^{\{j_1\}}-1\right)}{\left(n-d_i^{\{j_1\}}\right)} + \frac{1}{n} - \frac{1}{\left(n-d_i^{\{j_1\}}\right)}$$

Step 2: Define c' such that $c'_{kl} = c_{kl}$ if $k, l \in N$, $c'_{0i} = c'_{0j_2} = 0$, $c'_{0k} = 1$ else. By definition, $\tilde{c}' = \tilde{c}$ and $j_2 \notin D_i^{\{j_1\}}$. Consider the problem $\left(c'^{N\setminus D_i^{\{j_1\}}}, N\setminus D_i^{\{j_1\}}\right)$. By Weak Problem Separation, we have

$$y_i\left(c'^{N\setminus D_i^{\{j_1\}}}, N\setminus D_i^{\{j_1\}}\right) = -\frac{a\left(n-d_i^{\{j_1\}}-2\right)}{\left(n-d_i^{\{j_1\}}\right)} + y_i\left(\tilde{c}^{N\setminus D_i^{\{j_1\}}}, N\setminus D_i^{\{j_1\}}\right) - \frac{1}{\left(n-d_i^{\{j_1\}}\right)}.$$

By the definition of $\tilde{\Gamma}^e_{PNC}$, there is only one free path in $N\setminus D_i^{\{j_1\}}$. Therefore, for all $k \in D_i^{\{j_2\}}(c)$, $c_{ik} \geq c_{0k}$. We can therefore apply Group Independence. Therefore, $y_i\left(c'^{N\setminus D_i^{\{j_1\}}}, N\setminus D_i^{\{j_1\}}\right) = y_i\left(c'^{N\setminus D_i^{\{j_1,j_2\}}}, N\setminus D_i^{\{j_1,j_2\}}\right)$. By Weak Problem Separation,

$$y_i\left(c'^{N\setminus D_i^{\{j_1,j_2\}}}, N\setminus D_i^{\{j_1,j_2\}}\right) = -\frac{2a\left(n-d_i^{\{j_1,j_2\}}-1\right)}{\left(n-d_i^{\{j_1,j_2\}}\right)} + y_i\left(\tilde{c}^{N\setminus D_i^{\{j_1,j_2\}}}, N\setminus D_i^{\{j_1,j_2\}}\right) - \frac{1}{\left(n-d_i^{\{j_1,j_2\}}\right)}.$$

Combining these results, we obtain

$$\begin{aligned} y_i\left(\tilde{c}^{N\setminus D_i^{\{j_1\}}}, N\setminus D_i^{\{j_1\}}\right) &= y_i\left(\tilde{c}^{N\setminus D_i^{\{j_1,j_2\}}}, N\setminus D_i^{\{j_1,j_2\}}\right) + \frac{a\left(n-d_i^{\{j_1\}}-2\right)}{\left(n-d_i^{\{j_1\}}\right)} \\ &\quad -\frac{2a\left(n-d_i^{\{j_1,j_2\}}-1\right)}{\left(n-d_i^{\{j_1,j_2\}}\right)} + \frac{1}{\left(n-d_i^{\{j_1\}}\right)} - \frac{1}{\left(n-d_i^{\{j_1,j_2\}}\right)} \end{aligned}$$

and

$$y_i(\tilde{c}, N) = y_i\left(\tilde{c}^{N\setminus D_i^{\{j_1,j_2\}}}, N\setminus D_i^{\{j_1,j_2\}}\right) + \frac{a(n-2)}{n} - a + \frac{1}{n} - \frac{1}{\left(n-d_i^{\{j_1,j_2\}}\right)}.$$

Step l. Define c'' such that $c''_{kl} = c_{kl}$ for $k, l \in N$, $c_{0i} = c_{0j_l} = 0$, $c_{0k} = 1$ else. By definition, $\tilde{c}' = \tilde{c}$ and $j_l \notin D_i^{\{j_1,\ldots,j_{l-1}\}}$. Consider the problem $\left(c''^{N\setminus D_i^{\{j_1,\ldots,j_{l-1}\}}}, N\setminus D_i^{\{j_1,\ldots,j_{l-1}\}}\right)$. By Weak Problem Separation, we have that

$$\begin{aligned} y_i\left(c''^{N\setminus D_i^{\{j_1,\ldots,j_{l-1}\}}}, N\setminus D_i^{\{j_1,\ldots,j_{l-1}\}}\right) &= -\frac{a(n-d_i^{\{j_1,\ldots,j_{l-1}\}}-2)}{\left(n-d_i^{\{j_1,\ldots,j_{l-1}\}}\right)} + y_i\left(\tilde{c}^{N\setminus D_i^{\{j_1,\ldots,j_{l-1}\}}}, N\setminus D_i^{\{j_1,\ldots,j_{l-1}\}}\right) \\ &\quad -\frac{1}{\left(n-d_i^{\{j_1,\ldots,j_{l-1}\}}\right)}. \end{aligned}$$

By definition of $\tilde{\Gamma}^e_{PNC}$, there is only one free path in $N\backslash D_i^{\{j_1,\ldots,j_{l-1}\}}$. Therefore, for all $k\in D_i^{\{j_l\}}$, $c_{ik}\geq c_{0k}$. We can therefore apply Group Independence. We obtain $y_i\left(c''^{N\backslash D_i^{\{j_1,\ldots,j_{l-1}\}}}, N\backslash D_i^{\{j_1,\ldots,j_{l-1}\}}\right)= y_i\left(c''^{N\backslash D_i^{\{j_1,\ldots,j_l\}}}, N\backslash D_i^{\{j_1,\ldots,j_l\}}\right)$. By Weak Problem Separation,

$$\begin{aligned} y_i\left(c''^{N\backslash D_i^{\{j_1,\ldots,j_l\}}}, N\backslash D_i^{\{j_1,\ldots,j_l\}}\right) &= -\frac{2a\left(n-d_i^{\{j_1,\ldots,j_l\}}-1\right)}{\left(n-d_i^{\{j_1,\ldots,j_l\}}\right)}+y_i\left(\tilde{c}^{N\backslash D_i^{\{j_1,\ldots,j_l\}}}, N\backslash D_i^{\{j_1,\ldots,j_l\}}\right)\\ &\quad -\frac{1}{\left(n-d_i^{\{j_1,\ldots,j_l\}}\right)}. \end{aligned}$$

Combining these results, we obtain

$$\begin{aligned} y_i\left(\tilde{c}^{N\backslash D_i^{\{j_1,\ldots,j_{l-1}\}}}, N\backslash D_i^{\{j_1,\ldots,j_{l-1}\}}\right) &= y_i\left(\tilde{c}^{N\backslash D_i^{\{j_1,\ldots,j_l\}}}, N\backslash D_i^{\{j_1,\ldots,j_l\}}\right)+\frac{a\left(n-d_i^{\{j_1,\ldots,j_{l-1}\}}-2\right)}{\left(n-d_i^{\{j_1,\ldots,j_{l-1}\}}\right)}\\ &\quad -\frac{2a\left(n-d_i^{\{j_1,\ldots,j_l\}}-1\right)}{\left(n-d_i^{\{j_1,\ldots,j_l\}}\right)}+\frac{1}{\left(n-d_i^{\{j_1,\ldots,j_{l-1}\}}\right)}-\frac{1}{\left(n-d_i^{\{j_1,\ldots,j_l\}}\right)} \end{aligned}$$

and

$$y_i(\tilde{c},N)=y_i\left(\tilde{c}^{N\backslash D_i^{\{j_1,\ldots,j_l\}}}, N\backslash D_i^{\{j_1,\ldots,j_l\}}\right)+\frac{a(n-2)}{n}-(l-1)a+\frac{1}{n}-\frac{1}{\left(n-d_i^{\{j_1,\ldots,j_l\}}\right)}.$$

Step m. By definition, $N\backslash D_i^{\{j_1,\ldots,j_m\}}=\{i\}$ and $\tilde{c}^{N\backslash D_i^{\{j_1,\ldots,j_m\}}}$ is such that $c_{0i}=1$. Therefore, $y_i\left(\tilde{c}^{N\backslash D_i^{\{j_1,\ldots,j_m\}}},\{i\}\right)=1$. The term $\frac{1}{\left(n-d_i^{\{j_1,\ldots,j_m\}}\right)}$ is also equal to 1. Therefore, we have that for all $\tilde{c}\in\tilde{\Gamma}^e_{PNC}$

$$\begin{aligned} y_i^a(\tilde{c},N) &= \frac{a(n-2)}{n}-(m-1)a+\frac{1}{n}\\ &= \frac{1}{|N|}+\frac{a(|N|-2)}{|N|}-\left(\left|Z^i\right|-1\right)a \end{aligned}$$

By Group Independence we can easily extend the result to any $\tilde{c}\in\tilde{\Gamma}^e_{NC}$. We obtain

$$y_i^a(\tilde{c},N)=\frac{1}{|N_i(c)|}+\frac{a(|N_i(c)|-2)}{|N_i(c)|}-\left(\left|Z^i\right|-1\right)a$$

if $|N_i(c)|>1$. If $|N_i(c)|=1$, then agent i is alone in his group. By Group Independence and budget balance, $y_i^a(\tilde{c},N)=1$.

Finally, we prove that a solution y^a that satisfies the six properties, and is thus defined as above on $\hat{\Gamma}^e$ and $\tilde{\Gamma}^e_{NC}$, can be rewritten, for any c, as $y^a=2a(y^k-y^f)+y^f$.

First, consider $c\in\hat{\Gamma}^e$. We can check that $y_i^k(c,N)=\begin{cases}\frac{1}{|N|}\text{ if } i\in R(c)\text{ and } R(c)\neq N\\ \frac{1}{|N|}\text{ if } R(c)=N\\ -\frac{|R(c)|}{(|N|-|R(c)|)|N|}\text{ if } i\in N\backslash R(c)\end{cases}$ and $y_i^f(c,N)=\begin{cases}0\text{ if } i\in R(c)\text{ and } R(c)\neq N\\ \frac{1}{|N|}\text{ if } R(c)=N\\ 0\text{ if } i\in N\backslash R(c)\end{cases}$. Therefore, for $c\in\hat{\Gamma}^e$, $2a(y^k(c,N)-y^f(c,N))+y^f(c.N)=$

$$\left\{\begin{array}{c}\frac{2a}{|N|} \text{ if } i\in R(c) \text{ and } R(c)\neq N\\ \frac{1}{|N|} \text{ if } R(c)=N\\ -\frac{|R(c)|2a}{(|N|-|R(c)|)|N|} \text{ if } i\in N\backslash R(c)\end{array}\right. = y^a(c,N).$$

Next, consider $c\in\tilde{\Gamma}^e_{NC}$. We can check that $y^k_i(c,N)=\frac{1}{|N_i(c)|}+\frac{(|N_i(c)|-2)}{2|N_i(c)|}-\frac{\left(\left|Z^i\right|-1\right)}{2}$ and $y^f_i(c,N)=\frac{1}{|N_i(c)|}$. Therefore, for $c\in\tilde{\Gamma}^e_{NC}$, $2a(y^k(c,N)-y^f(c,N))+y^f(c,N)=\frac{1}{|N_i(c)|}+\frac{a(|N_i(c)|-2)}{|N_i(c)|}-\left(\left|Z^i\right|-1\right)a=y^a(c,N)$.

Since it has been shown that all solutions in the form $2a(y^k(c,N)-y^f(c,N))+y^f(c,N)$ satisfy the six properties and that all solutions satisfying the six properties can be written in this form, the proof is complete. ■

This theorem can ease implementation among agents who do not necessarily have strong economic backgrounds. Once they are convinced that the properties of the theorem are acceptable, we can ask for their preferred value of a in Example 1 and obtain the corresponding cost sharing solution.

While a can take any value in $\mathbb{R}$, if $a<0$, then in Example 1, agent one has a cost share that is higher than 0, his stand-alone cost. Similarly, if $a>1$, it is agent 2 that has a cost share that is higher than his stand-alone cost. In those cases, agents would be better off not cooperating with each other. This weak stability property, called Individual Rationality (or Stand-Alone property) is formally defined as follows.

Individual Rationality: For any $c\in\Gamma(|N|)$ and $i\in N$, $y_i(c,N)\leq C(\{i\},c)=c_{0i}$.

Lemma A.2 in appendix shows formally that this property is satisfied if and only if $a\in[0,1]$.

4 New characterizations of the Folk and Kar solutions

Quite obviously, the folk and Kar solutions are part of the family of solutions defined in the previous section, respectively, when $a=0$ and $a=\frac{1}{2}$. We provide new characterizations for these solutions by adding additional properties to those presented in the previous section.

The first property we add is the usual Core Selection property, which assures stability. A stronger version of Core Selection has been used by Bogomolnaia and Moulin (2008) to characterize the folk solution.

Core Selection: for all problems (c,N) and all $S\subseteq N$, $\sum_{i\in S}y_i(c,N)\leq C(S,c)$.

Theorem 2 *A solution y satisfies Weak Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Independence of Irrelevant Edges, Free Cycle Consistency and Core Selection if and only if y is the folk solution.*

Proof. Theorem A.1 shows that the folk solution satisfies Weak Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Independence of Irrelevant Edges and Free Cycle Consistency. Lemma A.1 shows that the folk solution satisfies Core Selection and that the Kar solution does not satisfy it.

Theorem 1 shows that if a solution satisfies Weak Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Independence of Irrelevant Edges and Free Cycle consistency, then it can be written as $y=2a\left(y^k-y^f\right)+y^f$, with $a\in\mathbb{R}$. Since the Kar solution does not satisfy Core Selection, it is easy to see that any solution that puts a non-zero weight on the Kar solution will not satisfy Core Selection. Therefore, we must have $a=0$ and $y=y^f$. ■

The following Corollary is obtained, as Core Selection implies Population Monotonicity (Bergantinos and Vidal-Puga (2007a)). Population Monotonicity requires that no agent be made worse off when agents join the coalition. This eliminates possibilities that some agents would veto the addition of members to the coalition.

Population Monotonicity: Let $c\in\Gamma(|N|)$ and $i\in S\subset N$. Then, $y_i(c,N)\leq y_i(c^S,S)$.

Corollary 1 *A solution y satisfies Weak Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Free Cycle Consistency, Independence of Irrelevant Edges and Population Monotonicity if and only if y is the folk solution.*

Using the same proof as for Theorem 2, we can see that the folk solution is also the only solution in the family described in the previous section that always guarantees non-negative cost shares.

Non-Negativity: For all problems (c, N) and all $i \in N$, $y_i(c, N) \geq 0$.

Theorem 3 *A solution y satisfies Weak Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Free Cycle Consistency, Independence of Irrelevant Edges and Non-Negativity if and only if y is the folk solution.*

If we compare this with other characterizations of the folk solution, we see for instance that like the characterization found in Bogomolnaia and Moulin (2008), we use Piecewise Linearity and Symmetry, and replace Strong Core Selection by the weaker property of Core Selection. While they use Reductionism, that says that cost shares should only depend on the irreducible cost matrix, we use Weak Problem Separation, Group Independence, Free Cycle consistency and Independence of Irrelevant Edges. This shows how strong the Reductionism property is. Similarly, Bergantinos and Vidal-Puga (2009) characterize the folk solution with Symmetry, a stronger version of Piecewise Linearity and Separability, a stronger version of Group Independence that states that if $C(S, c) + C(N\backslash S, c) = C(N, c)$ then we can compute cost shares of S and $N\backslash S$ independently. This is stronger than Group Independence as it applies to cases where $C(R, c) + C(T, c) > C(R \cup T, c)$ where $R \subset S$ and $T \subseteq N\backslash S$, whereas the Group Independence property only applies when two groups are completely disjoint. It seems like this strengthening of the Group Independence axiom is significant, because, in our case, we need to compensate by adding Weak Problem Separation, Free Cycle consistency, Independence of Irrelevant Edges and either Core Selection, Population Monotonicity or Non-Negativity.

The stronger version of Problem Separation is satisfied by the Kar solution. It turns out that it is the only solution in the family described in the previous section that satisfies it.

Theorem 4 *A solution y satisfies Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Free Cycle Consistency and Independence of Irrelevant Edges if and only if y is the Kar solution.*

Proof. Theorem A.1 shows that the Kar solution satisfies Weak Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Independence of Irrelevant Edges and Free Cycle consistency. Lemma A.1 shows that the Kar solution satisfies Problem Separation and that the folk solution does not satisfy it.

Theorem 1 shows that if a solution satisfies Weak Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Independence of Irrelevant Edges and Free Cycle consistency, then it can be written as $y = 2a\left(y^k - y^f\right) + y^f$, with $a \in \mathbb{R}$. Since the folk solution does not satisfy Problem Separation, it is easy to see that any solution that puts a non-zero weight on the folk solution will not satisfy Problem Separation. Therefore, we must have $a = \frac{1}{2}$ and $y = y^k$. ■

We can also characterize the Kar solution by a new property called Source Connection Appropriation, which says that if the agent with the lowest connection cost to the source sees this cost decrease, then he should capture all of the cost reduction. This property makes an agent responsible for the cost reductions generated by his location. The same proof as in the previous characterization can be used.

Source connection appropriation. If $c_{0i} \leq c_{0j}$ for all $j \in N\backslash\{i\}$ and $c'_{0i} = c_{0i} - x$ and $c'_e = c_e$ else, then $y_i(c', N) = y_i(c, N) - x$.

Theorem 5 *A solution y satisfies Weak Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Free Cycle Consistency, Independence of Irrelevant Edges and Source Connection Appropriation if and only if y is the Kar solution.*

The only characterization of the Kar solution appears in Kar (2002). It uses a weaker version of Group Independence (strict inequalities) and Absence of Cross Subsidization, which together act as the version of Group Independence used here. They are used with Equal Treatment, that says that if the cost of the edge (i, j) changes, then the cost shares of agents i and j should change by the same amount.

This strong property is replaced in our characterization by Piecewise Linearity, Symmetry, Free Cycle consistency, Independence of Irrelevant Edges and either Problem Separation or the combination of Weak Problem Separation and Source Connection Appropriation.

Comparing the characterizations of the folk and Kar solutions, we see that the folk solution is characterized by properties that assure stability (making sure that no group wants to leave in the case of Core Selection and that no agent will be vetoed from joining in the case of Population Monotonicity). The Kar solution is characterized using a property that allows simplification of the problem or a property that makes the agent with the lowest connection cost to the source fully responsible for this cost.

We now consider two properties making agents responsible for where they are located in the network.

Strict Cost Monotonicity: Suppose that $c_{ij} \leq \max[c_{0i}, c_{0j}]$. If c, c' such that $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else, then $y_k(c', N) < y_k(c, N)$ for $k \in \{i, j\}$.

Strict Cost Monotonicity says that for a relevant edge (i, j), if its cost decreases and the cost of all other edge stay the same, then both agents i and j see their cost allocations strictly decrease.

Strict Ranking: Suppose that $c_{ik} \leq c_{jk}$ for all $j \in N_0 \setminus \{i, j\}$ and $c_{il} < c_{jl}$ for some $l \in N_0 \setminus \{i, j\}$, with $c_{il} < \max[c_{0i}, c_{0l}]$. Then $y_i(c, N) < y_j(c, N)$.

Strong Ranking says that the location of agent i is strictly better than the location of agent j, then the cost allocation of i is strictly less than the cost allocation of j.

Lemma A.2 in appendix shows that members of the family that satisfy these properties are such that $a > 0$. Therefore, both eliminate the folk solution.

We can see that the choice between the folk and Kar solutions is essentially a choice between stability properties and properties making agents responsible for the cost of the edges adjacent to their locations.

5 A new solution and the Location Dependence property

Example 1 has a third natural value, $a = 1$. In that case, agent 1 is able to extract all of the surplus from cooperation with agent 2. It is thus a case where we value highly the position of a player in a network and punish severely those who are badly located, up to the point where they are just indifferent between cooperating or not. It is thus the mirror image of the folk solution, which does not allow an agent to be subsidized for his location.

A value of $a = 1$ generates the cost sharing solution $y^1 = 2y^k - y^f$. While this looks like an unusual solution, we know that it satisfies Weak Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Independence of Irrelevant Edges and Free Cycle consistency.

To complete the characterization, we add this property:

Location Dependence: For any $c \in \Gamma(|N|)$, if $i \in N$ is such that $c_{0i} = 0$, $c_{ij} = 0$ for all $j \in N \setminus \{i\}$ and $c_e = \beta$ else, with $\beta > 0$, then $y_j(c, N) = \beta$ for all $j \in N \setminus \{i\}$ and $y_i(c, N) = -(|N| - 1)\beta$.

The property generalizes the notion that if everyone in a set of agents depends on a single agent to improve on their stand-alone connection, then this agent should be able to extract all of the surplus as he is in a power position.

We obtain the following characterization:

Theorem 6 *A solution y satisfies Weak Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Free Cycle Consistency, Independence of Irrelevant Edges and Location Dependence if and only if $y = y^1$.*

Proof. Example 1 is a case where Location Dependence applies, with two players and $\beta = 1$. It is satisfied only when $a = 1$. y^1 is therefore our only candidate. Lemma A.1 shows that y^1 satisfies Weak Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Independence of Irrelevant Edges and Free Cycle Consistency. It remains to show that it satisfies Location Dependence in general.

By Piecewise Linearity, the value of β has no importance, so we set it to 1. Let c be such that $c_{0i} = 0$, $c_{ij} = 0$ for all $j \in N \setminus \{i\}$ and $c_e = 1$ else. We can apply Weak Problem Separation and consider the problems $(\hat{c}, N)$ and $(\tilde{c}, N)$. By Theorem 1, we have $y_l^f(\hat{c}, N) = 0$ for all $l \in N$, $y_i^k(\hat{c}, N) = -\frac{|N|-1}{|N|}$ and $y_j^k(\hat{c}, N) = \frac{1}{|N|}$ for all $j \in N\setminus\{i\}$. We also have $y_l^f(\tilde{c}, N) = \frac{1}{|N|}$ for all $l \in N$, $y_i^k(\tilde{c}, N) = \frac{1}{|N|} + \frac{|N|-2}{2|N|} - \frac{|N|-2}{2}$ and $y_j^k(\tilde{c}, N) = \frac{1}{|N|} + \frac{|N|-2}{2|N|}$ for all $j \in N\setminus\{i\}$. Applying Weak Problem Separation, we have $y_l^f(c, N) = 0$ for all $l \in N$ while $y_i^k(c, N) = -\frac{|N|-1}{2}$ and $y_j^k(c, N) = \frac{1}{2}$ for all $j \in N\setminus\{i\}$.

Since $y^1(c, N) = 2y^k(c, N) - y^f(c, N)$, we have $y_i^1(c, N) = -(|N| - 1)$ and $y_j^1(c, N) = 1$ for all $j \in N\setminus\{i\}$. ■

Notice that the Kar solution is the average of the folk solution where, in situations were Location Dependence applies, we split the surplus evenly between agents in $N\setminus\{i\}$ and the solution y^1 where, in the same situation, we give all of the surplus to agent i.

In addition, we have already mentioned that if we add an Individual Rationality constraint we need to restrict a to the set $[0, 1]$. Therefore, the main result of the paper can be restated as follows, providing a new justification for the Kar solution: the set of rules corresponding to the convex combination of the folk solution and y^1 are the only ones satisfying Weak Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Free Cycle consistency, Independence of Irrelevant Edges and Individual Rationality. Moreover, the weight put on y^1 is equal to the cost share of agent 2 in Example 1. The average of these rules is the Kar solution.

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Appendix

Theorem A.1 *All solutions defined in Theorem 1 satisfy Weak Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Free Cycle Consistency and Independence of Irrelevant Edges.*

Proof. Notice that the weight on the Kar solution is $2a$ while the weight on the folk solution is $1-2a$. Let $\theta = 2a$ and rewrite the family of solutions as $\theta y^k + (1-\theta)y^f$. The family of solutions is a weighted sum of the Kar and folk solutions, with the weights summing to one, but the weights being possibly negative. Since both solutions are budget balanced and the weights sum to one, all of the solutions of Theorem 1 are also budget balanced.

Since the Kar and folk solutions are part of the family, we first show that they satisfy all properties. This result then trivially extends to solutions $y^a = 2a(y^k - y^f) + y^f$, the weighted sums of these two solutions.

We start with the Kar solution. Symmetry is proven in Bergantinos and Vidal-Puga (2007a). Group Independence is proven in Kar (2002) (The proof easily extends when we remove strict inequality). Piecewise Linearity is proven in Bogomolnaia and Moulin (2008).

We show that the Kar solution satisfies the stronger property of Problem Separation: By Lemma 1, $C(S,c) = C(S,\hat{c}) + C(S,\tilde{c}) - \max_{e\in N_0^P} c_e$. Consider a game $\check{c}$ such that $\check{c}_{0i} = \max_{e\in N_0^P} c_e$ for all $i \in N_0$ and $c_e = 0$ otherwise. Then, $C(S,\check{c}) = \max_{e\in N_0^P} c_e$. Clearly, by Symmetry, $y_i^k(\check{c},N) = \frac{1}{|N|}\max_{e\in N_0^P} c_e$ for all $i \in N$.

We then have $C(S,\hat{c}) + C(S,\tilde{c}) - C(S,\check{c}) = C(S,c)$. By the properties of the Shapley value, $y_i^k(c,N) = y_i^k(\hat{c},N) + y_i^k(\tilde{c},N) - y_i^k(\check{c},N)$, or $y_i^k(c,N) = y_i^k(\hat{c},N) + y_i^k(\tilde{c},N) - \frac{1}{|N|}\max_{e\in N_0^P} c_e$ for all $i \in N$.

Free Cycle Consistency: We need to show that when $i \in F(c)$, $y_i^k(c,N) = \frac{\sum_{j\in F(c)} y_i^k\left(c^{-j},N\right)-\alpha}{|F(c)|}$.

We have

$$C(T\cup\{i\},c^{-i}) = \begin{cases} C(T\cup\{i\},c) + \alpha \text{ if } T\cap F_i(c) \neq \emptyset \\ C(T\cup\{i\},c) \text{ if } T\cap F_i(c) = \emptyset \end{cases}$$

as when $T \cap F_i(c)$, the incremental cost of agent i is 0 when the cost matrix is c and α when the cost matrix is c^{-j}. All other costs are the same. In particular, we have $C(T,c^{-j}) = C(T,c)$ for all $T \subseteq N\backslash\{i\}$. Let C^{α_i} be a coalitional cost function such that $C^{\alpha_i}(S) = C(S,c) - C(S,c^{-i})$ for all $S \subseteq N$. By the properties of the Shapley value, $Sh(C(c^{-i},N)) = Sh(C(c,N)) - Sh(C^{\alpha_i})$.

Consider $Sh(C^{\alpha_i})$. By the properties of the Shapley value, $Sh_j(C^{\alpha_i}) = 0$ if $j \notin F_i(c)$ (as those agents are dummies). We also have $Sh_i(C^{\alpha_i}) = \frac{|F_i(c)|}{(|F_i(c)|+1)}\alpha$ and $Sh_j(C^{\alpha_i}) = \frac{1}{(|F_i(c)|+1)|F_i(c)|}\alpha$ for all $j \in F_i(c)$. This result is obtained as the incremental cost of agent i is equal to α for all coalitions that include members of $F_i(c)$ and is zero for the empty set.

Notice that if $j \in F_i(c)$, then $i \in F_j(c)$ and $|F_i(c)| = |F_j(c)|$. Therefore, $\sum_{j\in F(c)} Sh_i(C^{\alpha_i}) = \frac{|F_i(c)|}{(|F_i(c)|+1)}\alpha + |F_i(c)|\frac{1}{(|F_i(c)|+1)|F_i(c)|}\alpha = \alpha$.

Thus, we obtain $\frac{\sum_{j\in F(c)} y_i^k\left(c^{-j},N\right)-\alpha}{|F(c)|} = \frac{\sum_{j\in F(c)} y_i^k(c,N)+\alpha-\alpha}{|F(c)|} = y_i^k(c,N)$.

Independence of Irrelevant Edges: Clearly, $C(S,c) = C(S,\bar{c})$ for all $S \subseteq N$. Therefore, $y^k(c,N) = y^k(\bar{c},N)$.

Next, we show that the folk solution satisfies all of these properties.

Piecewise Linearity and Symmetry were proved in Bergantinos and Vidal-Puga (2007a).

Weak Problem Separation: Suppose $c \in \bar{\Gamma}(|N|)$ such that $c_e \leq \min_{i\in N} c_{0i}$ for all $e \in t^*$, all $t^* \in T^*$. This implies that there is an optimal tree where one agent is connected to the source (say agent i), and all other agents are connected, directly or not, to this agent. The cost of connection to the source is the highest. Therefore, we obtain that $c_{0j}^* = c_{0i}$ for all $j \in N$ (as the cost of the most expensive arc between i and j is smaller than c_{0i}) and $c_{kl}^* \leq c_{0i}$ for all $k,l \in N$. Now consider $\hat{c}$. Since $\hat{c}_{kl} = 0$ for all $k,l \in N$, $\hat{c}_{0j}^* = c_{0i}$ for all $j \in N$. Consider $\tilde{c}$. Since $c_{kl}^* \leq c_{0i}$ for all $k,l \in N$, and neither c_{kl}^* nor $\tilde{c}_{kl}^*$ depends on the values of c_{0j} or $\tilde{c}_{0j}$, then $c_{kl}^* = \tilde{c}_{kl}^*$ for all $k,l \in N$. Then, clearly, $y_i^f(c,N) = y_i^f(\hat{c}^*,N) + y_i^f(\tilde{c}^*,N) - \frac{1}{|N|}\max_{e\in N_0^P} c_e$ for all $i \in N$.

Group Independence: Bergantinos and Vidal-Puga (2007a) show that the folk solution satisfies Separability, which says that if $C(S,c)+C(N\backslash S,c)=C(N,c)$, then $y_i(c,N)=y_i(c^S,S)$ if $i\in S$. This is a stronger property than Group Independence.

Free Cycle Consistency: We can see that the only differences between $\bar{c}$ and $\bar{c}^{-i}$ are that $\bar{c}_{ij}=0$ and $\bar{c}_{ij}^{-i}=\alpha$ for all $j\in F_i(c)$. Therefore,

$$C(T\cup\{i\},c^{-i})=\begin{cases} C(T\cup\{i\},c)+\alpha \text{ if } T\cap F_i(c)\neq\emptyset \\ C(T\cup\{i\},c) \text{ if } T\cap F_i(c)=\emptyset \end{cases}.$$

The rest of the proof is identical to the one for the Kar solution.

Independence of Irrelevant Edges: Suppose c and c' are such that $c_{ij}=\max(c_{0i},c_{0j})$, $c'_{ij}=c_{ij}+1$ and $c'_e=c_e$ otherwise. Clearly, $c^*=c'^*$ and therefore $y^f(c,N)=y^f(c',N)$.

It is easy to show that Weak Problem Separation, Group Independence, Piecewise Linearity, Symmetry, Free Cycle consistency and Independence of Irrelevant Edges extend to any solution $y^a=2a(y^k-y^f)+y^f$ since they are satisfied by both y^k and y^f. ■

Lemma A.1 *i) The folk solution satisfies Core Selection, Population Monotonicity and Non-Negativity, but it does not satisfy Problem Separation or Source Connection Appropriation.*

ii) The Kar solution satisfies Problem Separation and Source Connection Appropriation, but it does not satisfy Core Selection, Population Monotonicity or Non-Negativity.

Proof. i) Bergantinos and Vidal-Puga (2007a) show that the folk solution satisfies Core Selection, Population Monotonicity and Non-Negativity.

Take $c\in\bar{\Gamma}^e$ such that $c_{0i}=c_{0j}=c_{ij}=0$ and $c_e=1$ else. We have $y_l^f(c,N)=0$ if $l=\{i,j\}$ and 1 else. Also, we have $y_l^f(\hat{c},N)=0$ for all $l\in N$ and $y_l^f(\tilde{c},N)=\frac{1}{2}$ if $l=\{i,j\}$ and 1 else. Therefore, $y_l^f(\hat{c},N)+y_l^f(\tilde{c},N)-\frac{1}{|N|}\neq y_l^f(c,N)$. Problem Separation is not satisfied.

Consider c' such that $c'_{ij}=0$ for all $j\in N\backslash\{i\}$ and $c'_e=1$ else. We have $y_l^f(c',N)=\frac{1}{|N|}$ for all $l\in N$.

Consider c'' such that $c''_{0i}=0$ and $c''_e=c'_e$ else. We obtain $y_l^f(c'',N)=0$ for all $l\in N$. Source Connection Appropriation is not satisfied.

ii) Problem Separation was proved in Theorem A.1.

Suppose $c_{0i}\leq c_{0j}$ for all $j\in N\backslash\{i\}$, $c'_{0i}=c_{0i}-x$ and $c'_e=c_e$ else. Then, for all $S\subseteq N\backslash\{i\}$, $C(S,c')=C(S,c)$ and $C(S\cup\{i\},c')=C(S\cup\{i\},c)-x$. By the properties of the Shapley value, $y_i^k(c',N)=y_i^k(c,N)-x$. Thus, the Kar solution satisfies Source Connection Appropriation.

Consider the problem c such that $c_{jk}=0$ for all $j,k\in N$, $c_{0j}=0$ for all $j\in N\backslash\{i\}$ and $c_{0i}=1$, with $|N|>2$. Then, $y_i^k(c,N)=\frac{1}{|N|}$ and $y_j^k(c,N)=-\frac{1}{|N|(|N|-1)}$ for all $j\in N\backslash\{i\}$. We have $C(S,c)=0$ for all $S\neq\{i\}$. Take $l\in N\backslash\{i\}$. We obtain $y_i(c,N)+y_l(c,N)=\frac{(|N|-2)}{|N|(|N|-1)}>0$. The Kar solution fails to satisfy Core Selection.

Consider the addition of agent m, so that the problem becomes $(c^m,N\cup\{m\})$. We have $c_{lm}^m=0$ for all $l\in N$, $c_{0m}^m=0$ and $c_e^m=c_e$ else. Then, for all $j\in N\backslash\{i\}$, $y_j(c^m,N\cup\{m\})=-\frac{1}{|N|+1(|N|)}<-\frac{1}{|N|(|N|-1)}=y_j^k(c,N)$. Population Monotonicity is not satisfied.

Example 1 shows that the Kar solution does not satisfy Non-Negativity. ■

Lemma A.2 *Suppose that* $y^a(c,N)=2ay^k(c,N)+(1-2a)y^f(c,N)$ *with* $a\in\mathbb{R}$. *We have that:*

i) $y^a(c,N)$ *satisfies Individual Rationality if and only if* $a\in[0,1]$.

ii) $y^a(c,N)$ *satisfies Strict Cost Monotonicity if and only if* $a>0$.

iii) $y^a(c,N)$ *satisfies Strict Ranking if and only if* $a>0$.

Proof. i) Example 1 shows that Individual Rationality is not satisfied if $a<0$ or $a>1$. We show that $a\in[0,1]$ is a sufficient condition for Individual Rationality.

Suppose that $c \in \hat{\Gamma}^e$. Suppose that $i \in R(c)$ and $R(c) \neq N$. Then, $c_{0i} = 1$ and $y_i = \frac{2a}{|N|}$. We need $\frac{2a}{|N|} \leq 1$. Since $R(c) \neq N$, we have $|N| \geq 2$. Therefore, Individual Rationality is satisfied if $a \leq 1$ for all such c. If $R(c) = N$, then $y_i = 1 = c_{0i}$.

If $i \notin R(c)$, then $y_i = -\frac{|R|2a}{(|N|-|R|)|N|}$ and $c_{0i} = 0$. We have $-\frac{|R|2a}{(|N|-|R|)|N|} \leq 0$ if and only if $a \geq 0$.

Suppose that $c \in \tilde{\Gamma}^e_{NC}$. If $|N_i(c)| = 1$, then $y_i = 1 = c_{0i}$. If $|N_i(c)| > 1$, then $y_i = \frac{1}{|N_i(c)|} + \frac{a(|N_i(c)|-2)}{|N_i(c)|} - \left(\left|Z^i\right| - 1\right) a$, with $\left|Z^i\right| \geq 1$ and $c_{0i} = 1$. We need $\frac{1}{|N_i(c)|} + \frac{a(|N_i(c)|-2)}{|N_i(c)|} - \left(\left|Z^i\right| - 1\right) a \leq 1$, which simplifies to $a\left(2(|N_i(c)| - 1) - \left|Z^i\right| |N_i(c)|\right) \leq |N_i(c)| - 1$. Suppose that $2(|N_i(c)| - 1) - \left|Z^i\right| |N_i(c)| \geq 0$. Then, the condition becomes $a \leq \frac{|N_i(c)|-1}{2(|N_i(c)|-1)-|Z^i||N_i(c)|}$. However, we have that $\frac{|N_i(c)|-1}{2(|N_i(c)|-1)-|Z^i||N_i(c)|} \geq 1$ if and only $\left|Z^i\right| \geq 1 - \frac{1}{|N_i(c)|}$. This is always satisfied, since $\left|Z^i\right| \geq 1$. Therefore, for $a \leq 1$, Individual Rationality is satisfied on such c.

Suppose that $2(|N_i(c)|-1)-\left|Z^i\right| |N_i(c)| \leq 0$. Then, the condition becomes $a \geq \frac{|N_i(c)|-1}{2(|N_i(c)|-1)-|Z^i||N_i(c)|} \leq 0$. Therefore, for $a \geq 0$, Individual Rationality is satisfied on such c.

For $c \in \tilde{\Gamma}^e \backslash \tilde{\Gamma}^e_{NC}$, we can apply Free Cycle Consistency. Thus, $y_i(c, N) = \frac{\sum_{j \in F(c)} y_i(c^{-j},N)-\alpha}{|F(c)|}$. Suppose that all $c^{-j} \in \tilde{\Gamma}^e_{NC}$. Then, we have shown that $y_i(c^{-j}, N) \leq c^{-j}_{0i} = c_{0i}$ which results in $y_i(c, N) \leq \frac{|F(c)|c_{0i}-\alpha}{|F(c)|} < c_{0i}$. If some $c^{-j} \notin \tilde{\Gamma}^e_{NC}$, we obtain the same result by iteration on c^{-j}. Therefore, Individual Rationality is satisfied for all $c \in \tilde{\Gamma}^e$ if $a \in [0, 1]$.

For $c \in \Gamma^e$, we can apply Problem Separation. We have $y_i(c, N) = y_i(\hat{c}, N) + y_i(\tilde{c}, N) - \frac{1}{|N|}$, with $\hat{c} \in \hat{\Gamma}^e$ and $\tilde{c} \in \tilde{\Gamma}^e$. Therefore, $y_i(\hat{c}, N) \leq \hat{c}_{0i}$ and $y_i(\tilde{c}, N) \leq \tilde{c}_{0i}$. Since $c_{0i} = \hat{c}_{0i} + \tilde{c}_{0i} - \frac{1}{|N|}$, we also have that $y_i(c, N) \leq c_{0i}$. Individual Rationality is satisfied for all $c \in \Gamma^e$ if $a \in [0, 1]$.

For $c \in \Gamma$, we can apply Piecewise Linearity. We have $y_i(c, N) = \sum_{k=1}^p \left(c_{e^{\sigma(k)}} - c_{e^{\sigma(k-1)}}\right) y_i(b^k, N)$ with all $b^k \in \Gamma^e$. Therefore, $y_i(b^k, N) \leq b^k_{0i}$ for all k. Since $c_{0i} = \sum_{k=1}^p \left(c_{e^{\sigma(k)}} - c_{e^{\sigma(k-1)}}\right) b^k_{0i}$ and all the weights are positive, we have $y_i(c, N) \leq c_{0i}$. Individual Rationality is satisfied for all $c \in \Gamma$ if $a \in [0, 1]$.

ii) Suppose that $c_{ij} \leq \max\left[c_{0i}, c_{0j}\right]$ and c, c' such that $c'_{ij} < c_{ij}$ and $c'_e = c_e$ else.

Define $\Delta y^k_l = y^k_l(c, N) - y^k_l(c, N)$ and $\Delta y^f_l = y^f_l(c, N) - y^f_l(c, N)$. By the properties of the Kar and folk solutions, $\Delta y^k_l > 0$ and $\Delta y^f_l \geq 0$ for $l \in \{i, j\}$. We have Strict Cost Monotonicity if $2a\Delta y^k_l + (1 - 2a)\Delta y^f_l > 0$ for $l \in \{i, j\}$.

For the following, suppose that $l \in \{i, j\}$. Suppose that $\Delta y^f_l = 0$ (which happens when $c_{ij} > \bar{c}_{ij}$). We have $2a\Delta y^k_l > 0$ if and only if $a > 0$.

We need to show that $2a\Delta y^k_l + (1 - 2a)\Delta y^f_l > 0$ when $\Delta y^f_l > 0$. If $0 < a \leq \frac{1}{2}$, we have $2a > 0$ and $1 - 2a \geq 0$. Combined with $\Delta y^k_l > 0$ and $\Delta y^f_l \geq 0$, it assures that $2a\Delta y^k_l + (1 - 2a)\Delta y^f_l > 0$.

We can see that $\Delta y^k_l \geq \Delta y^f_l$. Therefore, if $a > \frac{1}{2}$, we have $2a\Delta y^k_l + (1 - 2a)\Delta y^f_l \geq 2a\Delta y^k_l + (1 - 2a)\Delta y^k_l = \Delta y^k_l > 0$.

Therefore, Strict Cost Monotonicity is satisfied if and only if $a > 0$.

iii) Suppose that $c_{ik} \leq c_{jk}$ for all $j \in N_0 \backslash \{i, j\}$ and $c_{il} < c_{jl}$ for some $l \in N_0 \backslash \{i, j\}$, with $c_{il} < \max\left[c_{0i}, c_{0l}\right]$.

Define $\Delta y^k_{ij} = y^k_j(c, N) - y^k_i(c, N)$ and $\Delta y^f_{ij} = y^f_j(c, N) - y^f_i(c, N)$. By the properties of the Kar and folk solutions, $\Delta y^k_{ij} > 0$ and $\Delta y^f_{ij} \geq 0$. We have Strict Ranking if $2a\Delta y^k_{ij} + (1 - 2a)\Delta y^f_{ij} > 0$. Suppose that $\Delta y^f_{ij} = 0$ (which happens when $\bar{c}_{il} = \bar{c}_{jl}$). We have $2a\Delta y^k_{ij} > 0$ if and only if $a > 0$.

We need to show that $2a\Delta y^k_{ij} + (1 - 2a)\Delta y^f_{ij} > 0$ when $\Delta y^f_{ij} > 0$. If $0 < a \leq \frac{1}{2}$, we have $2a > 0$ and $1 - 2a \geq 0$. Combined with$\Delta y^k_{ij} > 0$ and $\Delta y^f_{ij} \geq 0$, it assures that $2a\Delta y^k_{ij} + (1 - 2a)\Delta y^f_{ij} > 0$.

We can see that $\Delta y^k_{ij} \geq \Delta y^f_{ij}$. Therefore, if $a > \frac{1}{2}$. we have $2a\Delta y^k_{ij} + (1 - 2a)\Delta y^f_{ij} \geq 2a\Delta y^k_{ij} + (1 - 2a)\Delta y^k_{ij} = \Delta y^k_{ij} > 0$.

Therefore, Strict Ranking is satisfied if and only if $a > 0$. ■