Aggregate uncertainty and learning in a search model*

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March 15, 2009

Abstract

We present the first equilibrium search model with learning about an important characteristic of the aggregate market, tightness. In a steady state dynamic matching and bargaining game, the inflow of new buyers ("Demand") can be either high or low, depending on the state of the world. Buyers participate in sealed bid second price auctions. Bidders learn about the state through unsuccessful bids. There is no other source of information, i.e., all learning is endogenous. Beliefs depend on the bidding history, inducing endogenous heterogeneity into a population of ex ante identical agents.

We show that there exists a unique equilibrium in symmetric, increasing bidding strategies. We show that a seller has no incentive to reveal information about bids after the auction, since this extra information increases continuation values and thus depresses bids in the current auction. This result differs markedly from static second price auctions where the linkage principle implies that revealing any information the seller has (or obtains from the bidders) increases revenues.

Preliminary version.

JEL Classifications: D44, D82, D83

Keywords: Dynamic Matching and Bargaining, Decentralized Markets,
Foundations of Competitive Equilibrium, Search Theory, Learning, Duration Dependent Hazard Rate

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1 Introduction

In most markets agents learn about the market environment as they interact with other agents. Important aggregate variables like market tightness and the size of search frictions may be not completely known to the participating agents. As an agent learns new information about market conditions he changes his behavior, but so do other agents. Therefore, upon receiving new information an agent also updates his belief about how the others behave. This feature of learning introduces endogenous dispersion in beliefs about the state of the market, and also about how the other agents behave. This dispersion then translates to an endogenous dispersion of transaction prices for agents with different beliefs.

To sidestep the difficulties that stem from the endogenous determination of strategies, beliefs and price, we modify the dynamic matching and bargaining model of Satterthwaite and Shneyerov (2008) in a way that learning about market conditions is introduced, but the model still remains tractable.[1] Just like in Satterthwaite and Shneyerov (2008), there is an infinite sequence of (discrete) time periods, and in everyone of them a continuum of buyers and buyers arrive in the market. All buyers are randomly matched to one of the sellers and each seller conducts a sealed bid auction. After the auction she gives notice to the winner and tells her how much to pay. However, no buyer observes the bids or even the number of her respective opponents. At the end of each round all agents who transacted leave the economy. An unsuccessful agent stays for the next period with probability δ, and has to leave the market without trading with probability $1 - \delta$.

The defining feature of our model is that the mass of incoming buyers can be either large or small; it does not change over time, though. Competition among buyers is more intense in the high state, so the continuation value from participating in future auctions is lower. Therefore, the buyers would like to bid more in the high state than in the low one. They are, however, assumed not to know the state in which they are. Upon birth, every buyer forms an initial belief about the state of the world. Moreover, after every auction the losing buyers obtain additional information regarding the state, because they are able to make some inference about the bidding behavior of their opponents. Thus, the buyers' beliefs are endogenous and evolve over time. While the sellers do not exhibit any strategic behavior (they do not take any decision, like setting a reserve price), the buyers aim at maximizing expected utility which is their valuation minus the payment if they win an auction and zero if they do not obtain an object. This simplifying assumption about the sellers is made to keep tractability. If the sellers were able to choose an open reserve price, then they would be able to signal their beliefs and there would be multiple equilibrium induced by off equilibrium beliefs. If the sellers chose a secret reserve price, i.e. decide ex post whether to trade, then the entire bid distribution would influence the decision of the seller and the analysis would

[1] We also adopt some simplifying assumptions like no reserve price, no buyer heterogeneity in valuations and second price auction as opposed to a first price auction studied there.

also become intractable. To simplify matters we choose to completely ignore the incentives of the seller, and focus only on the other side of the market, still achieving endogenous price formation through buyer competition.

The interaction between bidding and learning in this economy has two important features. First, the agents learn about a bid distribution that is not exogenously given, but is determined by equilibrium forces. Second, the assumption that there are a continuum of agents implies that each buyer takes the equilibrium bid distribution in each state as given. Therefore, from the point of view of a single buyer, the learning process has similar features as in Burdett and Vishwanath (1988) where the price distribution to learn is exogenous. These two features provide one with a tractable framework where learning and endogenous price formation occurs at the same time.

Our equilibrium model can shed light on phenomena that are not possible to study if one takes the price (offer) distribution as given, like in Burdett and Vishwanath (1988). For example with a given offer distribution an agent is strictly better off if he is able to sample more often, i.e. the search friction parameter $1-\delta$ decreases. However, in our general equilibrium approach this may not be the case. For example suppose that regardless of the state of the economy there are more buyers than sellers. On one hand, as frictions vanish ($\delta \to 1$) buyers are able to sample more, which allows them to experiment and obtain a better price. On the other hand, if all buyers follow the same strategy, then there are more buyers on the market, which reduces the equilibrium utility of the buyers. In equilibrium, the second effect dominates as $\delta \to 1$, overturning the intuition that can be gained from a partial equilibrium analysis.

To account for benefits from participating in future auctions, the buyers' optimal strategy in our setting is to shade their bids, i.e. to bid less than they would do in the absence of any future buying opportunities.[2] More precisely, the buyers should simply subtract the opportunity cost of winning from their valuation. In the case of a second-price sealed-bid auction, the optimal bidding function then takes the form $b = v - \delta V$, where V denotes the value of bidding in future auctions. We study a monotone equilibrium where buyers with more pessimistic beliefs bid more aggressively. The mechanics of the learning process in such an equilibrium is simple: if a buyer has lost many times, then he holds it more likely that there are many other buyers and is inclined to bid higher, because his continuation utility V is low. This effect is similar to the discouragement effect of Burdett and Vishwanath (1988). They establish that as the agent obtains unfavorable news over time (i.e. losing an auction), he learns that the state of the world is more likely to be unfavorable. The agent is thus discouraged from future search, preferring to accept an offer that he would have rejected earlier on.

However, matters are more complicated, because the continuation value of a bidder depends on his future strategy, which in turn is affected by the information available to the bidder *after losing*

[2] See also Milgrom and Weber (1999) for this result in dynamic auction games with a fixed number of objects and participants.

the current auction. Therefore, a seller can influence the bidding strategies of agents by changing the information revealed *after* the auction is concluded. An important question is then how the revenue of a seller would change if she revealed the bids or the number of buyers after her auction (and the buyers knew this beforehand). We show that, for any given distribution of buyers' types, announcing the winning bid has an adverse effect on revenue, unlike in a standard static common value model with affiliated signals. The intuition behind this result is that any (ex-post) revelation of information only serves to increase the buyers' continuation payoff, thereby lowering all bids.[3] This result is different from the linkage principle that states that in a (static) common value auction the auctioneer should reveal all that he knows to maximize his revenue. The key difference from static common value auctions is that here information revelation *after* the auction is important as it influences continuation values and thus current bids, an effect absent in a static auction with common values where the linkage principle is usually applied.

To establish the existence of a monotone stationary equilibrium, we establish that the bid distribution in the high state stochastically dominates the bid distribution in the low state, if the other buyers use monotone strategies. More precisely, we show that the two distributions satisfy the monotone likelihood ratio property. This implies that the continuation value of a given buyer is lower when the state is high, making him bid more aggressively if he believes that the state is more likely to be high. Using this key insight, one can use the contraction mapping theorem on value functions to prove existence of a monotone equilibrium. However, establishing that the monotone likelihood ratio property holds is non-trivial. We show it in Appendix B that the first order statistics of a random number of i.i.d. random variables may not satisfy the MLRP property even if the original random variable does. Therefore, we cannot rely on existing results and have to establish the property directly in the context of our model.

An interesting feature of our setup is that while the value function, as in other models with learning, is convex, but proving that a unique interior best reply exists is much simpler than in Gonzalez and Shi (2008). The key idea is that while the continuation value is convex, but the actual bid function is such that the utility achieved in the current period is concave making the entire objective function (quasi-)concave in the maximization variable. It is an open question whether in other equilibrium search models with learning this insight could be used to simplify analysis. Our set-up is related to Duffie and Manso (2007) who assume that all the information about current bids (and thus beliefs) are revealed and study a model of information percolation. As long as there are trading frictions, their model also generates dispersion in beliefs and trading prices, since a buyer only learns the information that other buyers have who participate in the *same auction* as he does. Naturally, the speed of learning is much faster under their assumptions.

The paper is organized as follows: in Section 2 we formalize the ideas laid out in this introduction

[3] This result is reminiscent of the findings of Mezzetti, Pekec, and Tsetlin (2004) who make a similar point in the framework of a more stylized, two-period model.

by setting up a model of sequential auctions with uncertainty about the ratio of buyers to sellers in the market ("aggregate demand uncertainty"). In Section 3 we conduct preliminary analysis, and Section 4 shows that the model has a unique steady state equilibrium and characterize some of its properties. Section 5 asks whether sellers could increase their revenue by employing a more transparent informational regime than the one assumed in Section 3. Finally, a discussion of the model and conclusions are presented in Section 6.

2 Model

We provide a dynamic model of decentralized trading market where agents learn about aggregate market conditions over time. There are a continuum of buyers and a continuum of sellers present in the market. Each period a homogeneous, indivisible good is traded between the market participants. Each agent who traded leaves the market and with probability $1 - \delta$ each agent who did not trade in the previous period, is removed from the market. Finally, each period a number of buyers and sellers are born to participate in the market.

Let us now describe the matching and trading procedure. Each buyer is matched independently and randomly with a single seller each period. (As a consequence, a seller may have any (integer) number of buyers - including zero - matched with him in a given period.) To simplify analysis we assume that all sellers run a *second price sealed-bid auction* without any reserve price and thus they do not have any strategic choice. Each buyer chooses a bid to submit in the second-price auction. The winner pays his bid and obtains the object, while the losers stay in the pool with probability δ and are matched with a new seller in the next period.

There are two states of the world that occur with equal probability. In both states of the world there is a given number of sellers born each period, which we normalize to 1. In the low state the measure of buyers born each period is d_L, while in the high state it is d_H. Whether the state is low or high is not known to any buyer when born and the buyers only observe their own trade history, so the state cannot be deduced by them with certainty at any point of the game. They only observe their own bid each period and whether they managed to trade or not. The mere fact of being born allows each buyer to conclude that the probability of the high state is $d_H/(d_L + d_H)$, which is then the belief each new born buyer has. The motion of beliefs of the buyers are described in the next section.

Finally, each buyer who trades at price p obtains a surplus of $v - p$, where v is common among all buyers and it is also a common knowledge among all agents. Each buyer is assumed risk neutral and maximizes the expected surplus with no discounting beyond the one that follows from the possibility of dying exogenously in any given period.

2.1 Steady State Equilibrium

We restrict attention to steady state equilibria in which the distribution of bids depends only on the state, not on (calendar) time. An immediate consequence is that in any period the optimal bid of a buyer depends only on her current belief about the likelihood of being in the high state; we denote her belief by $\theta \in [0,1]$.

For tractability, we would like to restrict attention to pure strategy equilibria in which the bid is an increasing function of the belief of the buyer. However, this does not work, because the distribution of buyers' beliefs will have an atom at the prior of the newborns and we cannot ensure existence of a pure strategy for them, see the Appendix. To deal with this problem in a convenient way, we add a payoff irrelevant variable to the newborn which will allow us to express a mixed strategy equilibrium as a pure equilibrium in distributional strategies. Since the beliefs will otherwise be distributed without atoms, we need to use the variable only at the prior. Formally, we let the type of an agent be $z = \{\theta, i\}$ where θ is her belief and a payoff irrelevant variable $i \in [0,1]$ serves as a purification variable. We set $i = 0$ for all agents with belief below θ_0 and we set $i = 1$ for all agents with belief above θ_0. Thus, the set of types is

$$Z = \left\{(i,\theta) \mid \theta < \theta_0, i = 0;\ \theta = \theta_0, i \in [0,1\ ];\ \theta > \theta_0,\ i = 1\right\}.$$

Given this construction, the set of types Z is a connected line in $[0,1]^2$. Types are naturally ordered on the line by $(i,\theta) \geq (i',\theta')$ if and only if $\theta > \theta'$ or $\theta = \theta' = \theta_0$ (the prior) and $i \geq i'$ (the purification variable). Let $\theta(z)$ denote the belief of a type z and $z(\theta)$ denote the (generalized) inverse, with $z(\theta) = sup\{z|\theta(z) = \theta\}$.[4] The distribution of types in state $w \in \{L, H\}$ is denoted by Γ^w. To avoid technical difficulties, we restrict attention to distributions that have a continuous density on their support; we verify that the equilibrium distributions have this property. Let $\Gamma^\theta = \theta\Gamma^H + (1-\theta)\Gamma^L$ denote the distribution of types if it is believed that the high state has probability θ. The mass of buyers is denoted by M_B^w and the mass of sellers is denoted by M_S^w. A pure bidding strategy is a function $\beta : Z \to [p_0, v]$, i.e. the bid is determined by the belief of the bidder except for the initial cohort where the belief is equal to θ_0 and thus the purification variable i is needed to represent the strategy of the bidder as a pure strategy equilibrium in the space of types Z.

A symmetric steady state equilibrium can be characterized by the distributions of types in the two states, the masses M_S^w and M_B^w, the bidding strategy β, and a transition function for the types (the posterior after loosing). We will now characterize the equilibrium requirements for these objects. A formal definition of equilibrium follows at the end of this Section. Importantly, we will

[4] Since z is effectively a one-dimensional variable, we will take derivates like $\frac{\partial}{\partial z}\tilde{\theta}$, with $\frac{\partial}{\partial z}\tilde{\theta} = \lim_{z_N \to z} \frac{\tilde{\theta}(z) - \tilde{\theta}(z_N)}{d(z_N, z)}$, $z_N \in Z$ and with $d(z_N, z)$ being the usual euclidian distance on $[0,1]^2$.

restrict attention to equilibria in which the bid is a strictly increasing function of the type and every buyer employs the same strategy β.

We first derive the transition function, given a symmetric strategy β and a population. We want to know the posterior of a type z who bid b and lost. The fact of loosing implies that the buyer must have been in a match in which the highest type bid above b. Thus, there must haven been a competing buyer (at the same seller) with a type larger $\beta^{-1}(b)$, where β^{-1} is the (generalized) inverse, $\beta^{-1}(b) = \inf\{z|\beta(z) \geq b\}$.[5] To derive the posterior, we need to know the probability that the highest competing type is larger than $\beta^{-1}(b)$ in each state. To this end, denote by $Z_{(1)}$ the first order statistic among types in any given match and let $\Gamma^w_{(1)}$ denote its cdf, so $\Gamma^w_{(1)}(z')$ is the probability that the highest type in the auction is below z'. A convenient implication of Poisson matching is that this is also the probability that the competitors of a given bidder have types below z', hence $\Gamma^w_{(1)}(z')$ characterizes also the distribution of a buyer's competitors' types. Thus, Bayes' rule (if it applies) requires that the posterior of type z after loosing with a bid b is

$$\tilde{\theta}\left(z, Z_{(1)} \geq \beta^{-1}(b)\right) = \frac{\theta(z)\left(1 - \Gamma^H\left(\beta^{-1}(b)\right)\right)}{1 - \Gamma^\theta\left(\beta^{-1}(b)\right)}. \tag{1}$$

The new type is denoted by $\tilde{z}(z,b) = z\left(\tilde{\theta}\right)$. Note, that the posterior defines an *endogenous* transition function: given a type z_n and an action b today, tomorrow's type will be $z_{n+1} = \tilde{z}(z_n, b)$.

Note, also, that the event in which the highest type is below z' includes the event in which there is no buyer at all in the match. Hence, $\Gamma^w_{(1)}(0)$ is the probability that there is no bidder in a given auction[6] and $\Gamma^w_{(1)}(0)$ is the probability that a given bidder does not have any competitor. Let $\mu^w = \frac{M^w_B}{M^w_S}$ measure market tightness in state w, and note, that the fact that Γ^w has the Poisson distribution implies that

$$\Gamma^w_{(1)}(0) = e^{-\mu^w}.$$

Given the transition function, we can derive the steady state condition for the population. Suppose the size of the population of sellers is M^w_S today; tomorrow's population of sellers will consists of those sellers who did not get matched with a buyer and the newly entering sellers. In steady state, these two populations must be identical,

$$M^w_S = 1 + \delta\Gamma^w_{(1)}(0)\, M^w_S. \tag{2}$$

Denote the inflow of buyers with type less than z as $d^w(z)$. Recalling that the distribution of the

[5] We adopt the convention that $\beta^{-1}(b) = (1,1)$ if $\beta(z) < b$ for all z.

[6] In equilibrium, there is no mass of types $z = 0$.

purification variable i is uniform yields that:

$$d^w(z) = \begin{cases} 0 & z < (0, \theta_0) \\ i(z)\, d^w & z \in [(0,\theta_0), (1,\theta_0)] \\ d^w & z > (1, \theta_0) \end{cases}$$

Given a population of buyers today, the steady state mass of buyers with type below z is equal to $M_B^w \Gamma^w(z)$. This mass has to be equal to the inflow with type less than z plus the mass of buyers who lose, survive, and update to some type less than z. Using our notation from above, a type z' satisfies that his updated type upon losing is less than z if and only if

$$\tilde{z}\left(z', \beta(z')\right) \leq z.$$

In a symmetric equilibrium with increasing bidding strategies, the probability that type z' loses is $1 - \Gamma_{(1)}^w(z')$, since she loses if and only if there is another bidder with a higher type (and therefore, a higher bid). Therefore, the stationarity condition can be written as

$$M_B^w \Gamma^w(z) = d^w(z) + \delta M_B^w \int_{\{z' : \tilde{z}(z', \beta(z')) \leq z\}} \left(1 - \Gamma_{(1)}^w(t)\right) d\Gamma^w(t). \tag{3}$$

Given a symmetric strategy β used by the other buyers and a distribution $\Gamma_{(1)}^\theta$, let us derive the expected payoff of a type z who uses a bidding strategy β'. The strategy, together with the transition function $\tilde{z}$, determines a sequence of bids. The first bid is $b_1 = \beta'(z)$. After losing, the new type is $z_2 = \tilde{z}\left(z, \beta'(z)\right)$ and the second bid is $b_2 = \beta'(z_2)$. The third bid is $b_3 = \beta'(\tilde{z}(z_2, b_2))$ and in general $b_n' = \beta'\left(\tilde{z}\left(z_{n-1}, b_{n-1}'\right)\right)$. Conditional on state w, we can define the expected payoff of a bidder with initial type z and bidding strategy β' recursively as

$$\begin{aligned} EU\left(z, \beta' | w\right) &= (v - p_0)\, \Gamma_{(1)}^w(0) + \\ &+ \int_{\{t : 0 < t < \beta^{-1}\beta'(z)\}} (v - \beta(t))\, d\Gamma_{(1)}^w(t) + \delta \left(1 - \Gamma_{(1)}^w\left(\beta^{-1}\left(\beta'(z)\right)\right)\right) EU\left(\tilde{z}\left(z, \beta'(z)\right), \beta' | w\right). \end{aligned} \tag{4}$$

If the probability of the high state is θ, then expected payoffs are

$$EU\left(z, \beta' | \theta\right) = \theta EU\left(z, \beta' | H\right) + (1 - \theta)\, EU\left(z, \beta' | L\right).$$

Note, that we use a general belief θ and not necessarily the belief $\theta(z)$ of the type z to evaluate expected payoffs.

The maximal payoff depends only on the belief $\theta(z)$ and is denoted by

$$V(\theta(z)) \equiv sup_{\beta'} EU\left(z, \beta'|\theta(z)\right)$$

The optimality condition can be written very compactly, observing that, since all other bidders use a strictly increasing bidding function β, choosing a bid b implies winning against all types with $z \leq \beta^{-1}(b)$. Instead of maximizing with respect to the bid, we can therefore think of maximizing with respect to a cutoff type x such that the bidder wins whenever he is matched with types below x, paying the expected (second-) highest bid. Thus, with a belief θ the optimality condition is

$$V(\theta) = \tag{5}$$

$$= sup_x \quad (v - p_0)\,\Gamma^{\theta}_{(1)}(0) + \int_{\{t:0<t<x\}} (v - \beta(t))\, d\Gamma^{\theta}_{(1)}(t) + \delta\left(1 - \Gamma^{\theta}_{(1)}(x)\right) V\left(\tilde{\theta}\left(\theta, Z_{(1)} \geq x\right)\right).$$

A steady state equilibrium in symmetric, strictly increasing bidding strategies (an equilibrium from now on) consists of distribution functions Γ^H, Γ^L such that the steady state conditions hold, a transition function $\tilde{z}$ that is consistent with Bayes' rule whenever applicable, and a strictly increasing bidding function β that is optimal, $EU(z, \beta|\theta(z)) = V(\theta(z))$.

3 Characterization

We start the analysis by showing that the total mass of buyers and sellers is uniquely determined in equilibrium. The proof is in the appendix:

Lemma 1 *There is a unique mass of buyers M_B^w and sellers M_S^w in equilibrium. The market is tighter in the high state, $\frac{M_B^H}{M_S^H} > \frac{M_B^L}{M_S^L}$.*

The proof the above Lemma utilizes the fact that, within each match of a seller with $N \geq 1$ buyers, one buyer will win, independent of the equilibrium bid functions and equilibrium type distributions. This allows us to simplify the analysis and describe the steady state masses of buyers and sellers by two simple equilibrium conditions. It is also intuitive that in the high state the market is tighter for the sellers than in the low state. Algebraically this follows from the fact that a higher inflow in the high state implies that more buyers stay on the market, because each buyer has a lower chance to transact. Moreover, each seller has a higher chance to transact in the high state, so they leave the market quicker, and thus there are less sellers on the market in the high state.

For a type z, a first order statistic above her own type, $Z_{(1)} \geq z$, is bad news since a) this implies that the bidder lost the auction and b) among the other bidders, some bidder has a higher type. The second observation implies that the buyer becomes more pessimistic upon losing the auction. To see this, note that in the high state the expected number of other bidders is higher (since the market is tighter) and, for each given bidder, the probability that the type is higher than z is higher, $1-\Gamma^H(z) > \left(1-\Gamma^L(z)\right)$. Thus, $\left(1-\Gamma^H_{(1)}(z)\right) > \left(1-\Gamma^L_{(1)}(z)\right)$ for all z and therefore, using formula (1) implies that

$$\tilde{\theta}\left(z, Z_{(1)} \geq z\right) > \theta(z)$$

and thus

$$\widetilde{z}\left(z, \beta(z)\right) > z$$

Hence, a bidder z who lost today with equilibrium bid $\beta(z)$, will have a higher type $(\widetilde{z})$ tomorrow and, by strict monotonicity of the equilibrium strategy, will bid higher tomorrow, i.e. $\beta(\widetilde{z}) > \beta(z)$.

The fact of losing an auction entails bad news; and the fact of loosing an auction with a higher bid entails even worse news, if the first order statistic $\Gamma^w_{(1)}(z)$ has the monotone likelihood property. The underlying distribution of beliefs Γ^w must necessarily have the monotone likelihood ratio.[7] But, since the number of bidders is random, $\Gamma^w_{(1)}(z)$ is the first order statistic of a *random* number of random variables and, which each of these random variable satisfies the MLRP, $\Gamma^w_{(1)}(z)$ itself does not need to, see the counterexample in the Appendix.[8] The following Lemma first shows that the monotone likelihood ratio property holds. Therefore, bidders who hold a higher belief today and bid higher, have a higher belief tomorrow. An important implication is that for each generation in the market (bidders who have been in the market for n periods), the beliefs are within one common interval. Older generations have beliefs in a higher interval. This allows to construct the distribution $\Gamma^w(z)$ from the bottom up: We know the mass of buyers with types $z = (i, \theta)$, from where we can construct the next generation as the set of types who lost with their bid and did not exit. Importantly, for each generation, the distribution Γ^w is continuously differentiable on the support of equilibrium beliefs. The proof is in the Appendix:

Lemma 2 *If bidding strategies are strictly increasing, then there exists a unique distribution* Γ^w *that satisfies the steady state conditions.* Γ^w *is twice continuously differentiable almost everywhere. The distribution* Γ^w *and its density (where it exists) satisfy the Monotone Likelihood Ratio Property. On the support of* Γ^w*, the posteriors* $\tilde{\theta}\left(z, Z_{(1)} = z'\right)$ *and* $\tilde{\theta}\left(z, Z_{(1)} \geq z'\right)$ *are strictly increasing in* z' *and weakly increasing in* z *(strictly increasing in* z *when* $\theta(z) > \theta_0$*).*

Given uniqueness of Γ^w, from now on we can restrict attention to the equilibrium distribution and the equilibrium posteriors.[9] Therefore, to characterize equilibrium we only need to characterize

[7] Given a uniform prior, the posterior after having a type z is $\frac{d\Gamma^H(z)}{d\Gamma^H(z)+d\Gamma^L(z)}$ which must be equal to $\theta(z)$. Rewriting implies that when θ increases (and thus z goes up as well), $\frac{d\Gamma^H(z)}{d\Gamma^L(z)}$ increases. If the density of z has the MLRP, it follows from standard arguments that the cdf of z has the MLRP property as well.

[8] The first order statistic of a fixed number of random variables, each of which having the MLRP, has the MLRP as well.

[9] The proof of the Lemma also implies that Γ^H and Γ^L have the same support.

equilibrium bid incentives. We start the analysis by characterizing the value function $V(\theta)$. We show following a standard revealed preference argument, that the value function is convex, because information has value. This is stated in the next Lemma together with the envelope formula. The first equation states that payoffs depend on the belief only through the direct effect. The second equation uses the linearity of expected payoffs in beliefs. The envelope formula is immediate and the proof is omitted:

Lemma 3 *(Envelope Formula) Given any increasing bidding strategy β used by the other bidders, the value function $V(\theta)$ of a bidder is convex. Given an optimal strategy β', we can write $V(\theta) = EU\left(z, \beta'|\theta\right)$ at $\theta = \theta(z)$ and at all differentiability points of V,*

$$\begin{aligned}
\frac{\partial}{\partial \theta} V(\theta) &= \frac{\partial}{\partial \hat{\theta}} EU\left(z, \beta'|\hat{\theta}\right)|_{\hat{\theta}=\theta} \\
EU\left(z, \beta'|\hat{\theta}\right) &= V(\theta) + \left(\hat{\theta} - \theta\right) \frac{\partial}{\partial \theta} V(\theta)
\end{aligned}$$

Proof: The value function must be convex because, with $\theta^{\alpha} = \left(\alpha\theta^1 + (1-\alpha)\theta^2\right)$

$$\begin{aligned}
V\left(\alpha\theta^1 + (1-\alpha)\theta^2\right) &= \alpha\theta^1 EU\left(z(\theta^{\alpha}), \beta'|\theta^1\right) + (1-\alpha)\theta^2 EU\left(z(\theta^{\alpha}), \beta'|\theta^2\right) \\
&\leq \alpha\theta^1 EU\left(z\left(\theta^1\right), \beta'|\theta^1\right) + (1-\alpha)\theta^2 EU\left(z\left(\theta^2\right), \beta'|\theta^2\right) \\
&= \alpha\theta^1 V\left(\theta^1\right) + (1-\alpha)\theta^2 V\left(\theta^2\right). \qquad \text{QED.}
\end{aligned}$$

Thus, the derivative of the value function at $\theta(z)$ allows us to derive the expected payoff from following the bid sequence used by type z, given any probability θ', not necessarily equal to the belief of the type z, $\theta(z)$. Figure 1 (see the last page) illustrates what we have discussed above.

The next Lemma characterizes all equilibria in strictly increasing strategies. For almost all z in the support of Γ^w, the bid $\beta(z)$ must be equal to the valuation minus the expected continuation payoff from tomorrow. This continuation value is calculated assuming that the strategy adopted from tomorrow on is the optimal strategy given the updated belief, which belief can be written as

$$\theta_1(z) = \tilde{\theta}\left(z, Z_{(1)} \geq z\right).$$

It is important that the utility stream accruing from this strategy is evaluated using the posterior conditional on being tied, i.e. according to belief

$$\theta_2(z) = \tilde{\theta}\left(z, Z_{(1)} = z\right).$$

The intuition is similar to that for bidding in a standard, static second price common value auction: changing the bid affects profits only conditional on being tied and thus the optimal bid is the expected gain from winning conditional on the second highest type being equal to the winner's type. Therefore, the expected continuation payoff is calculated by evaluating the utility derived from the bidding sequence of a bidder with type $\tilde{z} = z(\theta_1)$, assuming that the probability of the high state is equal to θ_2.

Lemma 4 *Suppose the bidding strategy β is strictly increasing and part of an equilibrium and let θ_1 and θ_2 be defined as above. Then, for (Γ^w) almost all types z it holds that*

$$\begin{aligned}\beta &= v - \delta \left(V(\theta_1) + (\theta_2 - \theta_1) \frac{\partial}{\partial \theta} V(\theta_1) \right) \\ &= v - \delta EU(\tilde{z}, \beta | \theta_2).\end{aligned}$$

The above result makes it necessary to construct continuation payoffs that are evaluated at beliefs that are different than the belief the buyer has when he actually executes his bidding sequence (θ_2 vs θ_1). Therefore, we need to construct the expected utility function evaluated according to any belief, which can be done using the results from Lemma 3. More precisely, we calculate $EU(\tilde{z}, \beta|\theta_2)$ from the derivative of the value function:

$$EU(\tilde{z}, \beta|\theta_2) = V(\theta_1) + (\theta_2 - \theta_1)\frac{\partial}{\partial \theta}V(\theta_1).$$

Interestingly, the standard formulation of the optimality condition (5), maximizing current payoffs plus a continuation payoff $V(\theta)$, does not contain information about continuation payoffs conditional on being tied, since the value $V(\theta)$ itself contains information only about $EU(\tilde{z}, \beta|\theta_1) = V(\theta_1)$, and not about $EU(\tilde{z}, \beta|\theta_2)$. Therefore, the bid function *cannot* be written as a simple formula only using the value function, which makes it necessary for us to use the function EU.

4 Existence and Uniqueness of Equilibrium

In this section we prove that the exogenous parameters - δ, d^H, d^L - determine the market outcome in an essentially unique way: As we have seen before, the distribution of beliefs is unique. We show that payoffs $V(\theta)$ are uniquely determined as well. Using Lemma 4, this implies that the bids are uniquely determined for almost all types:

Proposition 5 *There exists an almost everywhere unique symmetric equilibrium in strictly increasing bidding strategies; payoffs $V(\theta)$ are unique and for almost all z,*

$$\beta = v - \delta EU(\tilde{z}, \beta|\theta_2).$$

The proposition is proven through a sequence of Lemmas which are combined at the end of this section. In other papers of search models with learning it has been shown that convexity of value function may create problems for proving existence of equilibrium, since it implies that a first order approach may not be used. However, in our model this property turns out to be helpful and key in the analysis that follows below. As we have seen in Lemma 3, the envelope formula implies a close connection between the value function $V(\theta)$ and the payoffs of any type z, $EU(z,\beta|\theta)$ given any prior θ, even if θ is not the belief of type z. Note that, for every given type z, expected payoffs are linear in the beliefs; hence, $EU(z,\beta|\cdot)$ is a family of linear functions. By definition, $V(\theta)$ is the upper envelope of this family.

The fact that convex functions can be characterized as the upper envelope of linear functions will be used throughout. For this, we define a **regular** function $W\left(\theta,\widehat{\theta}\right)$ analogously to $EU\left(z(\theta),\beta|\widehat{\theta}\right)$. A function $W\left(\theta,\widehat{\theta}\right)$ is called **regular** if θ parameterizes a family of nonincreasing, linear functions $W(\theta,\cdot)$ which are the tangents of a nonincreasing convex functions, i.e.,

R1. $W\left(\theta,\widehat{\theta}\right)=\widehat{\theta}W_H(\theta)+(1-\widehat{\theta})W_L(\theta)$ for some functions W_H, W_L

R2. for all $\hat{\theta}$, θ it holds that $W(\widehat{\theta},\widehat{\theta}))\geq W(\theta,\widehat{\theta})$

R3. $W(x,x)$ is nonincreasing in x

An immediate **observation** about a regular function $W\left(\theta,\widehat{\theta}\right)$ is that

R4 the function $W\left(\theta,\widehat{\theta}\right)$ is (weakly) decreasing in $\widehat{\theta}$ for all θ

R5 function W_H is (weakly) increasing, while W_L is (weakly) decreasing in θ.

R6 $W(\theta_h,\theta_{hh})\leq W(\theta_l,\theta_{ll})$ when $\theta_h\geq\theta_l$, $\theta_{hh}\geq\theta_{ll}$ and $\theta_{hh}\leq\theta_h$ and $\theta_{ll}\leq\theta_l$.

Figure 2 (see the last page) illustrates the main idea behind property R6; a formal proof appears in the Appendix.

We know from Lemma 4 that the only candidate for an equilibrium bidding strategy is a bid equal to the value of the object minus the expected continuation value, conditional on being tied. Let $W\left(\theta,\widehat{\theta}\right)$ denote the continuation value from following the bid sequence (now and forever in the future) employed by type $z(\theta)$ conditional on a prior $\hat{\theta}$. With this notation one can rewrite the candidate equilibrium bid function as

$$\beta=v-\delta W(\theta_1,\theta_2).$$

However, we still need to show that the necessary first order conditions identified in Lemma 4 are sufficient for the optimization problem of a single bidder. This result is established in the next Lemma after introducing some notation. Given W, let $U(x,z)$ denote the payoff of type z from winning against all types less than x today and following the equilibrium strategy in the future. Let

$$\tilde{\theta}_1(x,z)=\tilde{\theta}\left(z,Z_{(1)}\geq x\right)$$

denote the posterior belief for type z who lost with a bid $\beta(x)$. Note, that by construction

$$\theta_1(z) = \tilde{\theta}_1(z, z)$$

is the updated belief of a bidder with type z who bids according to the (candidate) equilibrium strategy. For later use, it is also useful to introduce the analogous notation for beliefs conditional on tieing:

$$\tilde{\theta}_2(x, z) = \tilde{\theta}\left(z, Z_{(1)} = x\right),$$

with

$$\theta_2(z) = \tilde{\theta}_2(z, z).$$

For convenience let us define z_0 as being an arbitrary type below the lowest type in the support, $(0, \theta_0)$, and above zero. With this notation in place, one can write the expected utility $U(x, z)$ as

$$U(x, z) \equiv (v - p_0)\, \Gamma^{\theta(z)}(0) + \int_{\{t: z_0 < t \leq x\}} (v - \beta(t))\, \gamma_{(1)}^{\theta(z)}(t) + \delta\left(1 - \Gamma_{(1)}^{\theta(z)}(x)\right) W\left(\tilde{\theta}_1(x, z), \tilde{\theta}_1(x, z)\right).$$

Lemma 6 *Let W be regular and suppose that $\beta(z) = v - \delta W(\theta_1(z), \theta_2(z))$ for almost all z. Then $z \in \arg\max_x U(x, z)$.*

The following Corollary states an important consequence of the above Lemma:

Corollary 7 *If function $\beta = v - \delta W(\theta_1, \theta_2)$ is strictly increasing, then it is a best reply for each bidder to use bid function β, if all the other bidders use β and the continuation values are given by function W.*

Proof : If β is strictly increasing than with a bid $\beta(x)$ a bidder indeed wins against all types less than x in the current auction. Therefore, $U(x, z)$ is the utility from bidding $\beta(x)$ today and then reverting back to the equilibrium from tomorrow. The above lemma then states that bidding $\beta(z)$ is optimal today. QED

We now define an operator T on the space of regular functions. Let $TW\left(\theta, \widehat{\theta}\right)$ be the expected payoff of a type $z\left(\widehat{\theta}\right)$ who follows the bidding sequence of type $z(\theta)$ now and forever in the future. To calculate the implied function TW, take a regular function W and the corresponding candidate bidding function

$$\beta = v - \delta W(\theta_1, \theta_2).$$

Note, that conditional on losing with a bid $\beta(z(\theta))$, and having an initial type $z\left(\widehat{\theta}\right)$, the induced belief is $\tilde{\theta}_1\left(z(\theta), z(\widehat{\theta})\right)$. Therefore, conditional on losing with a strategy that follows the bidding sequence of type $z(\theta)$ now and forever in the future yields a continuation payoff of

$$W\left(\tilde{\theta}_1\left(z(\theta), z(\theta)\right), \tilde{\theta}_1\left(z(\theta), z(\widehat{\theta})\right)\right).$$

Then $TW\left(\theta, \widehat{\theta}\right)$ can be calculated as follows:

$$TW\left(\theta, \widehat{\theta}\right) =$$

$$(v-p_0)\Gamma^{\hat{\theta}}(0) + \int_{z_0}^{z(\theta)} (v-\beta(t))\,\gamma_{(1)}^{\hat{\theta}}(t)\,dt + \delta\left(1-\Gamma_{(1)}^{\hat{\theta}}(z(\theta))\right) W\left(\tilde{\theta}_1\left(z(\theta), z(\theta)\right), \tilde{\theta}_1\left(z(\theta), z(\widehat{\theta})\right)\right).$$

The next lemma shows that there exists a unique regular solution of equation $TW = W$. The Lemma follows after checking that T satisfies Blackwell's sufficient condition for a contraction and that if W is regular, then TW is regular.

Lemma 8 *There is a unique regular function such that $W^* = TW^*$ and the induced bid function $\beta^* = v - \delta W^*(\theta_1, \theta_2)$ is strictly increasing in z.*

Proof of Proposition 5:

Using the above result, we turn to proving existence of an equilibrium. By Lemma 8, the candidate bidding function $\beta^* = v - \delta W^*(\theta_1, \theta_2)$ is strictly increasing in z. Corollary 7 implies that β^* is a best response for a bidder, if all other bidders follow the same strategy and the continuation values are given by function W^*. Finally, by being a fixed point of operator T, W^* is indeed the value function if all bidders employ strategy β^*. Therefore, an equilibrium with strictly increasing bidding strategies exists. Uniqueness of the equilibrium bid function β^* follows directly from the uniqueness result of Lemma 8. QED

5 Seller's optimal information policy

In the previous Section we have made the assumption that a buyer who unsuccessfully participates in an auction does not learn anything about her competitors, except the fact that one of them bid higher than she did. The sellers had no chance to provide buyers with any additional information. In this Section we relax this assumption by letting any individual seller to choose between conducting his auction in a nontransparent way (as in the previous Section) or reveal the winning bid after the auction to the respective participants.[10] The question we ask is whether a single seller has a unilateral incentive to announce the winning bid.

[10] Note that since we consider second-price auctions, the winning bid is not identical to the price. Assuming that the latter is revealed would lead to asymmetric learning on the part of the losing bidders, because each losing bidder would learn something different given their own bids.

The analysis yields that in the auction of the deviating seller the buyers use (symmetric) equilibrium bid function

$$\beta^{rev} = v - \delta W^*(\theta_2, \theta_2).$$

The intuition is that now a buyer with type y who ties against someone with the same type (but loses after the coin flip), knows that the highest type in the auction was z and thus he acts based on this information. Then the above bidding function guarantees that winning or losing against a type like yourself yields the same payoff, as it should in a private values second-price auction. We are now in the position to state the major result of this Section:

Theorem 9 *The buyers always bid (weakly) less under the transparent regime than under the nontransparent regime. Therefore, it never pays for a seller to reveal the winning bid in his auction unilaterally.*

Proof. Let β^* denote the equilibrium bid function in an auction with no revelation and, as above, β^{rev} denote the bid function when the seller reveals the winning bid unilaterally. Property *R2* of the value function implies that for all z it holds that $W^*(\theta_2, \theta_2) \geq W^*(\theta_1, \theta_2)$ and thus

$$\beta^{rev}(z) \leq \beta^*(z)$$

for all z, and thus the deviating seller cannot gain (and in fact will generically lose) from revealing the winning bid. ■

This Proposition might come as a surprise to those whose intuition follows the linkage principle, which holds that the seller of common value goods should always reveal as much information about the good as possible. The contrast between our result and the linkage principle is explained by the fact that in our model the information is only revealed after the auction has already taken place. Thus the only effect of announcing the winning bid is on the continuation value of the losing buyers. Since equilibrium bids depend negatively on that value, no seller ever wants to provide information about the state of the world, if the only information he can provide is that the highest bid is not so high in the relevant event when a buyer ties with the winner.[11] Hence it is an equilibrium strategy for each seller to reveal as little information about the state of the world as possible.

Finally, suppose that all other sellers reveal the winning bid. In this case the form of the equilibrium is somewhat different, and thus our equilibrium analysis has to be somewhat modified. However, if a steady state equilibrium exists, then a similar argument can be made that shows that it is *profitable* for any single seller to unilaterally deviate and not announce the winning bid. Therefore, it seems a stable feature of our model that the sellers would like to reveal as little information as possible *independently* from the information policy of the other sellers.

[11] Mezzetti et al. (2004) find a similar result in a two-period model.

6 Conclusions

We present the first equilibrium search model with learning about an important characteristic of the aggregate market, tightness. In a steady state dynamic matching and bargaining game, the inflow of new buyers ("Demand") can be either high or low, depending on the state of the world. Buyers participate in sealed bid second price auctions. Bidders learn about the state through unsuccessful bids. There is no other source of information, i.e., all learning is endogenous. Beliefs depend on the bidding history, inducing endogenous heterogeneity into a population of ex ante identical agents.

We show that there exists a unique equilibrium in symmetric, increasing bidding strategies. We show that a seller has no incentive to reveal information about bids after the auction, since this extra information increases continuation values and thus depresses bids in the current auction. This result differs markedly from static second price auctions where the linkage principle implies that revealing any information the seller has (or obtains from the bidders) increases revenues.

7 Appendix A

7.1 Proof of Lemma 1

Proof: First, if there are M_B^w and M_S^w buyers and sellers, then the probability that each particular seller trades is equal to the probability that he meets with any buyer, which is equal to (using that the number of meets follows a Poisson distribution) $1 - e^{-M_B^w/M_S^w} = 1 - e^{-\mu_i}$. Therefore, the measure of overall trades is equal to

$$M_S^w(1 - e^{-M_B^w/M_S^w}).$$

For the number of buyers to be in steady state, the number of newcomers has to be equal to the number of "survivors" or

$$M_B^w = d_i + \delta(M_B^w - M_S^w(1 - e^{-M_B^w/M_S^w})) \tag{6}$$

and similarly for the sellers

$$M_S^w = 1 + \delta(M_S^w - M_S^w(1 - e^{-M_B^w/M_S^w})). \tag{7}$$

Using the last two equations it follows that

$$M_S^w = M_B^w - \frac{d_i - 1}{1 - \delta}. \tag{8}$$

Equation (7) can be rewritten as

$$M_S^w = 1 + \delta M_S^w e^{-M_B^w/M_S^w}. \tag{9}$$

The last equation implies that $M_S^w > 1$ and that

$$M_B^w = \tau(M_S^w) = -M_S^w \log \frac{M_S^w - 1}{\delta M_S^w}.$$

Differentiating function τ yields that $\tau'(\infty) \leq 0$ and $\tau''(t) \geq 0$ for all $t \geq 1$. Therefore, for all $t \geq 1$ it holds that $\tau'(t) < 0$ and thus τ is a decreasing function. Using (8) and the last formula implies that

$$\beta(M_S^w) = M_S^w - \tau(M_S^w) + \frac{d_i - 1}{1 - \delta} = 0. \tag{10}$$

Since τ is decreasing, the right hand side of (10), β is strictly increasing in M_S^w and thus there is at most one solution to this equation. Moreover, $\tau(1) = \infty$ and thus $\beta(1) < 0$. Also, $\lim_{t\to\infty} \beta(t) = \infty$ and thus, by continuity of β, a (unique) solution of (10) exists.

Differentiating (10) by d_i yields

$$(1-\tau'(M_S^w))\frac{\partial M_S^w}{\partial d_i}+\frac{1}{1-\delta}=0$$

or

$$\frac{\partial M_S^w}{\partial d_i}=\frac{-1}{(1-\delta)(1-\tau'(M_S^w))}<0$$

and thus $d_H > d_L$ implies that $M_S^H < M_S^L$. Using that τ is decreasing, it also follows that

$$\frac{\partial M_B^w}{\partial d_i}=\frac{\partial M_S^w}{\partial d_i}\tau'(M_S^w)>0,$$

and thus $M_B^H > M_B^L$ follows. Thus, the market is tighter in the high state, $\frac{M_B^H}{M_S^H} > \frac{M_B^L}{M_S^L}$. **QED**

7.2 Proof of Lemma 2

Proof: **(Step 1)** We show that the MLRP holds for the first order statistics, $\frac{1-\Gamma_{(1)}^H(z)}{1-\Gamma_{(1)}^L(z)}$ and $\frac{\gamma_{(1)}^H}{\gamma_{(1)}^L}$ in Appendix B. **(Step 2)** The MLRP implies that the posteriors are nondecreasing on the support (and strictly increasing if $\theta(z_h) > \theta_0$)

$$\tilde{\theta}\left(z_h, Z_{(1)}=z_h\right)=\frac{1}{1+\frac{\theta(z_h)\gamma_{(1)}^H(z_h)}{(1-\theta(z_h))\gamma_{(1)}^L(z_h)}}>\frac{1}{1+\frac{\theta(z_l)\gamma_{(1)}^H(z_l)}{(1-\theta(z_l))\gamma_{(1)}^L(zl)}}=\tilde{\theta}\left(z_l, Z_{(1)}=z_l\right)$$

and the same for $\tilde{\theta}\left(z, Z_{(1)} \geq z\right)$.

(Step 3) Let $\tilde{z}_0^n$ be the updated belief of the $z=(0,\theta_0)$ type after n losses and let $\tilde{z}_1^n$ be the updated belief of the $z=(1,\theta_0)$ type after n losses. We want to show that for all $z=(i,\theta_0)$, the updated beliefs after n losses are in the interval $(\tilde{z}_0^n, \tilde{z}_1^n)$ and the boundaries of the interval are increasing, $\tilde{z}_0^n, \tilde{z}_1^n < \tilde{z}_0^{n+1}, \tilde{z}_1^{n+1}$. Each interval corresponds to a generation of bidders. Since the updated belief after a loss is higher than the prior, $\tilde{\theta}\left(z, Z_{(1)} \geq z\right) > \theta(z)$, it must be that $\tilde{z}_0^{n=1} > \tilde{z}_1^{n=0}$. Furthermore, since $\tilde{\theta}\left(z, Z_{(1)} \geq z\right)$ is strictly increasing in z, the updated beliefs are ordered $\tilde{z}_1^{n=1} > \tilde{z}_0^{n=1}$ and $\left(\tilde{z}_0^{n=1}, \tilde{z}_1^{n=1}\right)$ is indeed an interval. Suppose that, for all $n-1$ and n, we have $z_0^n > z_1^{n-1}$, then $z_0^{n+1} > z_1^n$. This is because of the fact that the upper bound z_1^n is the update $\tilde{\theta}\left(z_1^{n-1}, Z_{(1)} \geq z_1^{n-1}\right)$ and from $z_0^n > z_1^{n-1}$, $z_0^{n+1}=\tilde{\theta}\left(z_0^n, Z_{(1)} \geq z_0^n\right) > z_1^n$ follows as claimed. Together with the earlier observation for $n=1$, the claim about the ordering of the intervals $(\tilde{z}_0^n, \tilde{z}_1^n)$ follows by induction.

(Step 4) Let us construct the distribution Γ^H. The mass of types with believes below the prior, $z \leq (i,\theta_0)$ is given by the mass of their inflow, id^H (recall that i is uniformly distributed and the mass of the inflow is d^H) since after losing bidders update towards a belief strictly higher than θ_0. The share of the types is therefore simply

$$\Gamma^H(z) = \frac{id^H}{M_B^H} \quad z \in \left(z_0^{n=1}, z_1^{n=1}\right). \tag{11}$$

Suppose we know $\Gamma^H(z)$ for all $z \leq z_1^{n-1}$. This will imply that we know $\Gamma^H(z)$ for all $z \leq z_1^n$. Take some $z' \in (z_0^n, z_1^n)$ and let $\tilde{z}^{-1}(z')$ be its predecessor, $z' = \tilde{\theta}\left(\tilde{z}^{-1}(z'), Z_{(1)} \geq \tilde{z}^{-1}(z')\right)$ (which exists by continuity of the posterior $\tilde{\theta}$). Then the mass of types in $[z_0^n, z']$ must be such that

$$M_B^H \Gamma^H\left(z'\right) - M_B^H \Gamma^H\left(z_1^{n-1}\right) = M_B^H \int_{z_0^{n-1}}^{\tilde{z}^{-1}(z')} \delta\left(1 - \Gamma^H(t)\right) \gamma^H(t)\, dt. \tag{12}$$

Induction of n concludes the construction of Γ.

Step 5. Clearly, Γ^w is twice continuously differentiable on the support of the types of the first generation bidders, $[(0, \theta_0), (1, \theta_0)]$, see (11). Furthermore, inspection of (12) shows that, if Γ^w is twice continuously differentiable on the support of the types of the $(n-1)\,th$ generation, then it is twice continuously differentiable on the support of the $(n)\,th$ generation. By induction on n, Γ^w is continuously differentiable on its support. Γ^w is not differentiable at the countably many boundaries of the generations, $\{(\tilde{z}_0^n, \tilde{z}_1^n)\}_{n=1}^{\infty}$. QED

7.3 Proof of Lemma 4

Recall that with an increasing bidding function, choosing a bid is equivalent to choosing the highest type against whom to win. Given a bidding strategy β by the other buyers, bidding $b = \beta(z)$ is therefore optimal for type z only if

$$\begin{aligned} z &\in \arg\max_x U(x, z) \\ &\equiv \Gamma^{\theta(z)}(0)\, v + \int_{(0,\theta_0)}^{x} (v - \beta(t))\, \gamma_{(1)}^{\theta(z)}(t)\, dt + \delta\left(1 - \Gamma_{(1)}^{\theta(z)}(x)\right) V\left(\tilde{\theta}\left(z, Z_{(1)} \geq x\right)\right). \end{aligned}$$

Given that $\beta(z)$ is increasing, all types must be bidding in the interior. (And note that the first order condition trivially holds at all z outside the support of Γ.) The derivative $\frac{\partial}{\partial x}\tilde{\theta}|_{x=z}$ exists at all z. By Lemma 3, $V(\theta)$ is convex in θ and thus, $V(\theta)$ is differentiable for almost all θ. By Lemma 2, $\tilde{\theta}\left(z, Z_{(1)} \geq x\right)$ is weakly increasing in z and strictly increasing in x. Hence, $V\left(\tilde{\theta}\left(z, Z_{(1)} \geq x\right)\right)$ is almost everywhere differentiable in x. Given differentiability of V for almost every z (and hence almost every θ), the first order condition must hold almost everywhere with equality. The necessary first order condition at interior points where the derivative exists is

$$\begin{aligned}&\left(v-\beta\left(z\right)\right)\gamma_{(1)}^{\theta(z)}\left(z\right)-\delta\gamma_{(1)}^{\theta(z)}\left(z\right)V\left(\tilde{\theta}\left(z,Z_{(1)}\geq z\right)\right)\\&+\delta\left(1-\Gamma_{(1)}^{\theta(z)}\left(z\right)\right)\frac{\partial}{\partial x}V\left(\tilde{\theta}\left(z,Z_{(1)}\geq x\right)\right)|_{x=z}\frac{\partial}{\partial x}\tilde{\theta}|_{x=z} = 0.\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}\tilde{\theta}\left(z,Z_{(1)}\geq x\right) &= \frac{\left(1-\Gamma_{(1)}^{H}\left(x\right)\right)\theta}{1-\Gamma_{(1)}^{\theta}\left(x\right)}\\ \Rightarrow\frac{\partial}{\partial x}\tilde{\theta} &= \frac{-\theta\gamma_{(1)}^{H}\left(x\right)\left(1-\Gamma_{(1)}^{\theta}\left(x\right)\right)-\theta\left(1-\Gamma_{(1)}^{H}\left(x\right)\right)\left(-1\right)\left(\theta\gamma_{(1)}^{H}\left(x\right)+\left(1-\theta\right)\gamma_{(1)}^{L}\left(x\right)\right)}{\left(1-\Gamma_{(1)}^{\theta}\left(x\right)\right)^{2}}\\ &= \frac{\gamma_{(1)}^{\theta}\left(x\right)}{\left(1-\Gamma_{(1)}^{\theta}\left(x\right)\right)}\left(\frac{-\theta\gamma_{(1)}^{H}\left(x\right)}{-\left(1-\theta\right)\gamma_{(1)}^{L}\left(x\right)-\theta\gamma_{(1)}^{H}\left(x\right)}-\frac{\theta\left(1-\Gamma_{(1)}^{H}\left(x\right)\right)}{\left(1-\Gamma_{(1)}^{\theta}\left(x\right)\right)}\right)\\ &= \frac{\gamma_{(1)}^{\theta}\left(x\right)}{\left(1-\Gamma_{(1)}^{\theta}\left(x\right)\right)}\left(\tilde{\theta}\left(z,Z_{(1)}=x\right)-\tilde{\theta}\left(z,Z_{(1)}\geq x\right)\right).\end{aligned}$$

Therefore, the necessary condition becomes (for almost every z)

$$\beta\left(z\right)=v-\delta\left(V\left(\tilde{\theta}\left(z,Z_{(1)}\geq z\right)\right)+[\tilde{\theta}\left(z,Z_{(1)}=z\right)-\tilde{\theta}(z,Z_{(1)}\geq z)]\frac{\partial}{\partial\theta}V\left(\tilde{\theta}\left(z,Z_{(1)}=z\right)\right)\right)=$$

$$=v-\delta\left(V\left(\theta_{1}\right)+\left(\theta_{2}-\theta_{1}\right)\frac{\partial}{\partial\theta}V\left(\theta_{1}\right)\right).$$

Finally, the envelope formula (Lemma 3) implies that we can write the first order condition more compactly as

$$\beta=v-\delta EU\left(\tilde{z},\beta|\theta_{2}\right). \qquad \text{QED}$$

7.4 Algebra for Property R6 of a Regular Function

Property R6 is immediate for $\theta_h \geq \theta_{hh} \geq \theta_l$, since $W(\theta_l,\theta_{ll}) \geq W(\theta_l,\theta_l) \geq W(\theta_{hh},\theta_{hh}) \geq W(\theta_h,\theta_{hh})$.

Suppose $\theta_{hh} < \theta_l$, then, by $W\left(\theta,\theta'\right)=W\left(\theta,\theta\right)+\left(\theta'-\theta\right)\left(W_H-W_L\right)$, and

$$W\left(\theta_{l},\theta_{l}\right)\geq W\left(\theta_{h},\theta_{l}\right)=W\left(\theta_{h},\theta_{h}\right)+\left(\theta_{l}-\theta_{h}\right)\left(W_{H}\left(\theta_{h}\right)-W_{L}\left(\theta_{h}\right)\right)$$

and by

$$W(\theta_h,\theta_h) \geq W(\theta_l,\theta_h) = W(\theta_l,\theta_l) + (\theta_h - \theta_l)(W_H(\theta_l) - W_L(\theta_l))$$

and we obtain that $W(\theta_l,\theta_l) \geq W(\theta_h,\theta_h)$implies

$$\begin{aligned}
(\theta_l - \theta_h)(W_H(\theta_h) - W_L(\theta_h)) &\geq (\theta_h - \theta_l)(W_H(\theta_l) - W_L(\theta_l)) \\
(\theta_h - \theta_l)(W_H(\theta_h) - W_L(\theta_h)) &\leq (\theta_h - \theta_l)(W_H(\theta_l) - W_L(\theta_l)) \\
(W_H(\theta_h) - W_L(\theta_h)) &\leq (W_H(\theta_l) - W_L(\theta_l))
\end{aligned}$$

and thus,

$$\begin{aligned}
W(\theta_l,\theta_{ll}) &= W(\theta_l,\theta_l) + (\theta_{ll} - \theta_l)(W_H(\theta_l) - W_L(\theta_l)) \\
&\geq W(\theta_h,\theta_l) + (\theta_{ll} - \theta_l)(W_H(\theta_l) - W_L(\theta_l)) \\
&\geq W(\theta_h,\theta_l) + (\theta_{ll} - \theta_l)(W_H(\theta_h) - W_L(\theta_h)) \\
&= W(\theta_h,\theta_h) + (\theta_l - \theta_h)(W_H(\theta_h) - W_L(\theta_h)) + (\theta_{ll} - \theta_l)(W_H(\theta_h) - W_L(\theta_h)) \\
&= W(\theta_h,\theta_{ll}) \geq W(\theta_h,\theta_{hh}),
\end{aligned}$$

where the last inequality follows using R4.

7.5 Proof of Lemma 6

First, $U(x,z)$ is continuous in both variables and payoffs are differentiable in the first variable for almost all z. Let $U^{(1)}$ the derivative with respect to the first variable whenever it exists. We can write the derivative of payoffs as

$$\begin{aligned}
U^{(1)}(x,z) &= (v - \beta(x))\, d\Gamma^{\theta}_{(1)}(x) - \delta d\Gamma^{\theta}_{(1)}(x)\, W + \delta\left(1 - \Gamma^{\theta}_{(1)}(x)\right)\frac{\partial}{\partial x}\tilde{\theta}(W_1 + W_2) \\
&= (v - \beta(x))\, d\Gamma^{\theta}_{(1)}(x) - \delta d\Gamma^{\theta}_{(1)}(x)\, W + \delta\left(1 - \Gamma^{\theta}_{(1)}(x)\right)\frac{\partial}{\partial x}\tilde{\theta}(0 + (W_H - W_L)) \\
&= (v - \beta(x))\, d\Gamma^{\theta}_{(1)}(x) - \delta d\Gamma^{\theta}_{(1)}(x)\, W + \delta d\Gamma^{\theta}_{(1)}(x)\left(\tilde{\theta}_2 - \tilde{\theta}_1\right)(0 + (W_H - W_L)) \\
&= (v - \beta(x))\, d\Gamma^{\theta}_{(1)}(x) - \delta d\Gamma^{\theta}_{(1)}(x)\left(\tilde{\theta}_1 W_H + \left(1 - \tilde{\theta}_1\right) W_L\right) \\
&\quad + \delta d\Gamma^{\theta}_{(1)}(x)\left(\tilde{\theta}_2 - \theta_1\right)(W_H - W_L) \\
&= d\Gamma^{\theta}_{(1)}(x)\left(v - \beta(x) - \delta\left(\tilde{\theta}_2 W_H + \left(1 - \tilde{\theta}_2\right) W_L\right)\right) \\
&= d\Gamma^{\theta}_{(1)}(x)\left(v - \beta(x) - \delta W\left(\tilde{\theta}_1, \tilde{\theta}_2\right)\right) \\
&= d\Gamma^{\theta}_{(1)}(x)\,\delta\left(W\left(\tilde{\theta}_1(x,x), \tilde{\theta}_2(x,x)\right) - W\left(\tilde{\theta}_1(x,z), \tilde{\theta}_2(x,z)\right)\right),
\end{aligned}$$

with $U^{(1)}(x,z) = 0$ at all x not in the support of Γ^w. On the support, both expression in the bracket are continuous and γ^w is a continuous density. The support consists of a countable number of intervals (generations, see Lemma 2), so that the derivative is discontinuous only at the boundaries of the support and it exists almost everywhere.

Now, we establish that $U^{(1)}(x,z) \le 0$ for almost all $x > z$, which implies that for all $x > z$, $U(x,z) \le U(z,z)$ follows. By definition of function β, we obtain that $U^{(1)}(z,z) = 0$. At any $x > z$,

$$W\left(\tilde{\theta}_1(x,x), \tilde{\theta}_2(x,x)\right) - W\left(\tilde{\theta}_1(x,x), \tilde{\theta}_2(x,z)\right) \le 0$$

by property R4 of a regular function, $W\left(\theta, \hat{\theta}\right)$ being nonincreasing in $\hat{\theta}$. Furthermore,

$$W\left(\tilde{\theta}_1(x,x), \tilde{\theta}_2(x,z)\right) - W\left(\tilde{\theta}_1(x,z), \tilde{\theta}_2(x,z)\right) \le 0$$

by the second defining property, $W\left(\hat{\theta}, \hat{\theta}\right) \ge W\left(\theta, \hat{\theta}\right)$ and thus

$$W\left(\tilde{\theta}_1(x,x), \tilde{\theta}_2(x,x)\right) - W\left(\tilde{\theta}_1(x,z), \tilde{\theta}_2(x,z)\right) \le 0.$$

Therefore, $U^{(1)}(x,z) \le U^{(1)}(z,z) = 0$ for all $x > z$. A similar argument establishes $U^{(1)}(x,z) \ge U^{(1)}(z,z) = 0$ at all $x < z$. Thus, $U(z,z) \ge U(x,z)$ for all x, which concludes the proof. QED

7.6 Proof of Lemma 8

TW is defined on the set of regular functions. The set of regular functions is complete in the sup norm. Below we will show that the operator T maps regular functions into regular functions. Existence and uniqueness of a solution W^* follows then from Blackwell's sufficient conditions. Recall

$$TW\left(\theta, \widehat{\theta}\right) =$$

$$= (v - p_0)\Gamma^{\hat{\theta}}(0) + \int_{z_0}^{z(\theta)} (v - \beta(t))\,\gamma^{\hat{\theta}}_{(1)}(t)\,dt + \delta\left(1 - \Gamma^{\hat{\theta}}_{(1)}(z(\theta))\right) W\left(\tilde{\theta}_1(z(\theta), z(\theta)), \tilde{\theta}_1\left(z(\theta), z(\widehat{\theta})\right)\right).$$

and thus, if $W''\left(\theta, \hat{\theta}\right) > W'\left(\theta, \hat{\theta}\right)$ for all $\left(\theta, \hat{\theta}\right)$, then $TW''\left(\theta, \hat{\theta}\right) > TW'\left(\theta, \hat{\theta}\right)$ (by inspection). Furthermore, $T(W + a) \le TW + \delta a$, again, by inspection. Hence, TW satisfies the two sufficient conditions for a contraction in the sup norm, see Stokey and Lucas (with Prescott) (1989) and a unique solution exists.

We show that TW is regular if W is regular. We first establish that property *R2* holds for function TW. To see this, note that for all $\theta, \hat{\theta}$ it holds that

$$TW\left(\widehat{\theta}, \widehat{\theta}\right) \geq$$

$$\geq (v - p_0)\Gamma^{\hat{\theta}}(0) + \int_{z_0}^{z(\theta)} (v - \beta(t))\, \gamma_{(1)}^{\hat{\theta}}(t)\, dt + \delta \left(1 - \Gamma_{(1)}^{\hat{\theta}}(z(\theta))\right) W\left(\tilde{\theta}_1\left(z(\theta), z(\widehat{\theta})\right), \tilde{\theta}_1\left(z(\theta), z(\widehat{\theta})\right)\right)$$

$$\geq (v - p_0)\Gamma^{\hat{\theta}}(0) + \int_{z_0}^{z(\theta)} (v - \beta(t))\, \gamma_{(1)}^{\hat{\theta}}(t)\, dt + \delta \left(1 - \Gamma_{(1)}^{\hat{\theta}}(z(\theta))\right) W\left(\tilde{\theta}_1(z(\theta), z(\theta)), \tilde{\theta}_1\left(z(\theta), z(\widehat{\theta})\right)\right)$$

$$= TW\left(\theta, \widehat{\theta}\right).$$

The first inequality follows by Lemma 6 that implies that the choice of $x = z$ maximizes the payoff of type z, and the second inequality follows from the fact that W satisfies property *R2*.

To show property *R1* (linearity) of TW, let $z = z(\theta)$ and define

$$\begin{aligned} W_H^T(\theta) &\equiv \int_0^z (v - \beta(t))\, d\Gamma_{(1)}^H(t) + (1 - \Gamma_{(1)}^H(z)) W_H(\theta_1), \\ W_L^T(\theta) &\equiv \int_0^z (v - \beta(t))\, d\Gamma_{(1)}^L(t) + (1 - \Gamma_{(1)}^L(z)) W_L(\theta_1), \end{aligned}$$

where W_H, W_L are given by the regular function W. Noting that

$$\theta_1 = \tilde{\theta}\left(z, Z_{(1)} \geq z\right) = \frac{\theta(1 - \Gamma_{(1)}^H(z))}{1 - \Gamma_{(1)}^{\theta}(z)},$$

we can rewrite the third term in definition of operator T as

$$\left(1 - \Gamma_{(1)}^{\hat{\theta}}(z(\theta))\right) W\left(\tilde{\theta}_1(z(\theta), z(\theta)), \tilde{\theta}_1\left(z(\theta), z(\widehat{\theta})\right)\right) =$$

$$= \theta(1 - \Gamma_{(1)}^H(z)) W_H\left(\tilde{\theta}\right) + (1 - \theta)\left(1 - \Gamma_{(1)}^L(z)\right) W_L\left(\tilde{\theta}\right),$$

and hence

$$TW\left(\theta, \widehat{\theta}\right) = \widehat{\theta} W_H^T(\theta) + \left(1 - \widehat{\theta}\right) W_L^T(\theta).$$

Property *R3* for function TW: First, we show that $\beta = v - \delta W(\theta_1, \theta_2)$ is nondecreasing in z. Property *R6* for function W implies that

$$W(\theta_1(z_l), \theta_2(z_l)) \geq W(\theta_1(z_h), \theta_2(z_h)).$$

Next, we show that function W_H^T is increasing, while function W_L^T is decreasing. Using property

R2, we obtain that for all $\widehat{\theta}, \theta$ it holds that

$$TW\left(\widehat{\theta}, \widehat{\theta}\right) = \widehat{\theta} W_H^T\left(\widehat{\theta}\right) + \left(1 - \widehat{\theta}\right) W_L^T\left(\widehat{\theta}\right) \geq \tag{13}$$

$$\geq TW\left(\theta, \widehat{\theta}\right) = \widehat{\theta} W_H^T(\theta) + \left(1 - \widehat{\theta}\right) W_L^T(\theta).$$

and

$$TW(\theta, \theta) = \theta W_H^T(\theta) + (1 - \theta) W_L^T(\theta) \geq \tag{14}$$

$$\geq TW\left(\widehat{\theta}, \theta\right) = \theta W_H^T\left(\widehat{\theta}\right) + (1 - \theta) W_L^T\left(\widehat{\theta}\right).$$

Using the linearity property, *R1* and after some algebra we obtain

$$(\widehat{\theta} - \theta)[(W_H^T\left(\widehat{\theta}\right) - W_L^T\left(\widehat{\theta}\right)) - (W_H^T(\theta) - W_L^T(\theta))] \geq 0.$$

Let w.l.o.g. assume that $\widehat{\theta} > \theta$ and thus

$$W_H^T\left(\widehat{\theta}\right) - W_L^T\left(\widehat{\theta}\right) \geq W_H^T(\theta) - W_L^T(\theta).$$

Suppose that $W_H^T(\widehat{\theta}) < W_H^T(\theta)$ and thus by the last formula $W_L^T(\widehat{\theta}) < W_L^T(\theta)$. Combining these two strict inequalities yields a contradiction to (13) and thus $W_H^T(\widehat{\theta}) \geq W_H^T(\theta)$ holds. A similar argument (but now using (14)) yields that $W_L^T(\widehat{\theta}) \leq W_L^T(\theta)$.

Note, that $W_H^T(1) < W_L^T(1)$, because

$$W_H^T(1) \equiv \int_{[0,1]} (v - \beta(t))\, d\Gamma_{(1)}^H(t)$$

and

$$W_L^T(1) \equiv \int_{[0,1]} (v - \beta(t))\, d\Gamma_{(1)}^L(t),$$

and $\Gamma_{(1)}^H$ first order stochastically dominates $\Gamma_{(1)}^L$ and function β is increasing as we showed above. Then monotonicity of functions W_H^T and W_L^T implies that for all θit holds that

$$W_H^T(\theta) \leq W_H^T(1) < W_L^T(1) \leq W_L^T(\theta). \tag{15}$$

Therefore, for all $\hat{\theta} > \theta$

$$TW\left(\widehat{\theta}, \widehat{\theta}\right) = \hat{\theta} W_H^T\left(\widehat{\theta}\right) + \left(1 - \hat{\theta}\right) W_L^T\left(\widehat{\theta}\right) <$$

$$< \theta W_H^T\left(\widehat{\theta}\right) + (1 - \theta) W_L^T\left(\widehat{\theta}\right) = TW\left(\widehat{\theta}, \theta\right) \leq TW(\theta, \theta), \tag{16}$$

where the strict inequality follows from (15), and the weak follows from property *R2* of function TW.

Thus TW satisfies property *R3*, and a unique fixed point of map T exists in the space of regular functions. Let W^* denote this fixed point. Since $W^* = TW^*$ is a regular function and TW^* is strictly decreasing in the second variable, for all $z_h > z_l$ it holds (using property *R6*) that

$$W\left(\theta_1\left(z_l\right), \theta_2\left(z_l\right)\right) \geq W\left(\theta_1\left(z_h\right), \theta_2\left(z_l\right)\right) > W\left(\theta_1\left(z_h\right), \theta_2\left(z_h\right)\right).$$

Therefore,

$$\beta^*(z_l) = v - \delta W^*\left(\theta_1\left(z_l\right), \theta_2\left(z_l\right)\right) < v - \delta W\left(\theta_1\left(z_h\right), \theta_2\left(z_h\right)\right) = \beta^*(z_h),$$

and β^* is strictly increasing.

8 Appendix B: Monotone Likelihood Ratio Property

We have already argued that the MLRP holds for the densities of the beliefs. Since all types update upwards, the density of types with $(0, \theta_0)$ is given by $\frac{d_H}{M_B^w}$ and thus

$$\mu^H \gamma^H\left((0, \theta_0)\right) = \frac{d_H}{M_S^H} > \frac{d_L}{M_S^L} = \mu^L \gamma^L\left((0, \theta_0)\right)$$

for all i and since $\frac{\gamma^H(z)}{\gamma^L(z)}$ is weakly increasing in z on the support of Γ^w, (strictly for $\theta\left(z\right) \neq \theta_0$),

$$\mu^H \gamma^H\left(z\right) > \mu^L \gamma^L\left(z\right) \quad \text{for all } z \geq \left(0, \theta_0\right). \tag{17}$$

Since the number of buyers in any given auction is Poisson-distributed, the distribution function of the first order statistic of buyers' types in state $w \in \{L, H\}$ is given by

$$\Gamma_{(1)}^w(z) = e^{-\mu^w} + \sum_{n=1}^{\infty} \frac{e^{-\mu^w}\left(\mu^w\right)^n}{n!}[\Gamma^w(z)]^n.$$

Taking the derivative of this expression, we obtain that

$$\gamma_{(1)}^w(z) = \sum_{n=1}^{\infty} \frac{e^{-\mu^w}\left(\mu^w\right)^n}{(n-1)!}[\Gamma^w(z)]^{n-1}\gamma^w\left(z\right).$$

Therefore,

$$
\begin{aligned}
\gamma^w_{(1)}(z)' &= \sum_{n=1}^{\infty} \frac{e^{-\mu^w}(\mu^w)^n}{(n-1)!}[\Gamma^w(z)]^{n-1}\gamma^w(z)' + \sum_{n=2}^{\infty} \frac{e^{-\mu^w}(\mu^w)^n}{(n-2)!}[\Gamma^w(z)]^{n-2}[\gamma^w(z)]^2 = \\
&= \frac{\gamma^w(z)'}{\gamma^w(z)} \sum_{n=1}^{\infty} \frac{e^{-\mu^w}(\mu^w)^n}{(n-1)!}[\Gamma^w(z)]^{n-1}\gamma^w(z) + \mu^w\gamma^w(z) \sum_{n=2}^{\infty} \frac{e^{-\mu^w}(\mu^w)^{n-1}}{(n-2)!}[\Gamma^w(z)]^{n-2}[\gamma^w(z)] \\
&= \frac{\gamma^w(z)'}{\gamma^w(z)} \sum_{n=1}^{\infty} \frac{e^{-\mu^w}(\mu^w)^n}{(n-1)!}[\Gamma^w(z)]^{n-1}\gamma^w(z) + \mu^w\gamma^w(z) \sum_{n=2}^{\infty} \frac{e^{-\mu^w}(\mu^w)^{n-1}}{(n-2)!}[\Gamma^w(z)]^{n-2}[\gamma^w(z)] \\
&= \frac{\gamma^w(z)'}{\gamma^w(z)}\gamma^w_{(1)}(z) + \mu^w\gamma^w(z)\,\gamma^w_{(1)}(z)
\end{aligned}
$$

Thus

$$\frac{\gamma^w_{(1)}(z)'}{\gamma^w_{(1)}(z)} = \frac{\gamma^w(z)'}{\gamma^w(z)} + \mu^w\gamma^w(z).$$

Under the assumption that the densities γ^w satisfy the MLRP (γ^L/γ^H is decreasing) it must be the case that

$$\frac{\gamma^H(z)'}{\gamma^H(z)} \geq \frac{\gamma^L(z)'}{\gamma^L(z)},$$

where the inequality is weak at $z = (i, \theta_0)$ and strict otherwise. In addition, (17). Therefore, the densities satisfy the MLRP,

$$\frac{\gamma^H_{(1)}(z)'}{\gamma^H_{(1)}(z)} = \frac{\gamma^H(z)'}{\gamma^H(z)} + \mu^H\gamma^H(z) > \frac{\gamma^L(z)'}{\gamma^L(z)} + \mu^L\gamma^L(z) = \frac{\gamma^L_{(1)}(z)'}{\gamma^L_{(1)}(z)}, \tag{18}$$

where the strict inequality follows for all z since $\mu^H > \mu^L$.

Using the MLRP of the density, we can show the MLRP of the cdf,

$$\frac{\partial}{\partial z}\frac{1-\Gamma^H_{(1)}(z)}{1-\Gamma^L_{(1)}(z)} = \frac{\gamma^L_{(1)}(1-\Gamma^H_{(1)}(z)) - \gamma^H_{(1)}(1-\Gamma^L_{(1)}(z))}{(1-\Gamma^L_{(1)}(z))^2} > 0$$

which is trues if and only if

$$\frac{1-\Gamma^H_{(1)}(z)}{1-\Gamma^L_{(1)}(z)} > \frac{\gamma^H_{(1)}}{\gamma^L_{(1)}}.$$

This is established by the following chain of formulas for $z \geq (0, \theta_0)$:

$$\frac{1-\Gamma^H_{(1)}(z)}{1-\Gamma^L_{(1)}(z)} = \int_z^{(1,1)} \frac{\gamma^H_{(1)}(x)\gamma^L_{(1)}(x)}{\gamma^L_{(1)}(x)\left(1-\Gamma^L_{(1)}(z)\right)}dx > \int_z^{\infty} \frac{\gamma^H_{(1)}(z)\gamma^L_{(1)}(x)}{\gamma^L_{(1)}(z)\left(1-\Gamma^L_{(1)}(z)\right)}dx = \frac{\gamma_H(y)}{\gamma_L(y)}, \tag{19}$$

where the inequality follows from the fact that $\gamma_{(1)}^{H}(z)/\gamma_{(1)}^{L}$ is monotone increasing, established in (18).

The posteriors $\tilde{\theta}\left(z, Z_{(1)}=z'\right)$ and $\tilde{\theta}\left(z, Z_{(1)} \geq z'\right)$ are both strictly increasing (on the support), since the inequalities characterizing the MLRP hold strictly, see (18) and (19).

Counterexample. Suppose $\mu_H = \mu_L = \mu$ (so the expected number of bidders is state independent) and suppose for some z^*, the densities are constant for $z \leq z^*$ and for $z > z^*$, with $\gamma_L(z) = C > \gamma_H(z) = c$ if $z \leq z^*$ and $\gamma_L(z) = c < \gamma_H(z) = C$. These densities satisfy the MLRP. However, for $z < z^*$,

$$\frac{\gamma_{(1)}^{H}(z)'}{\gamma_{(1)}^{H}(z)} = 0 + \mu c < 0 + \mu C = \frac{\gamma_{(1)}^{L}(z)'}{\gamma_{(1)}^{L}(z)}$$

and hence, the first order statistic does not satisfy the MLRP. QED

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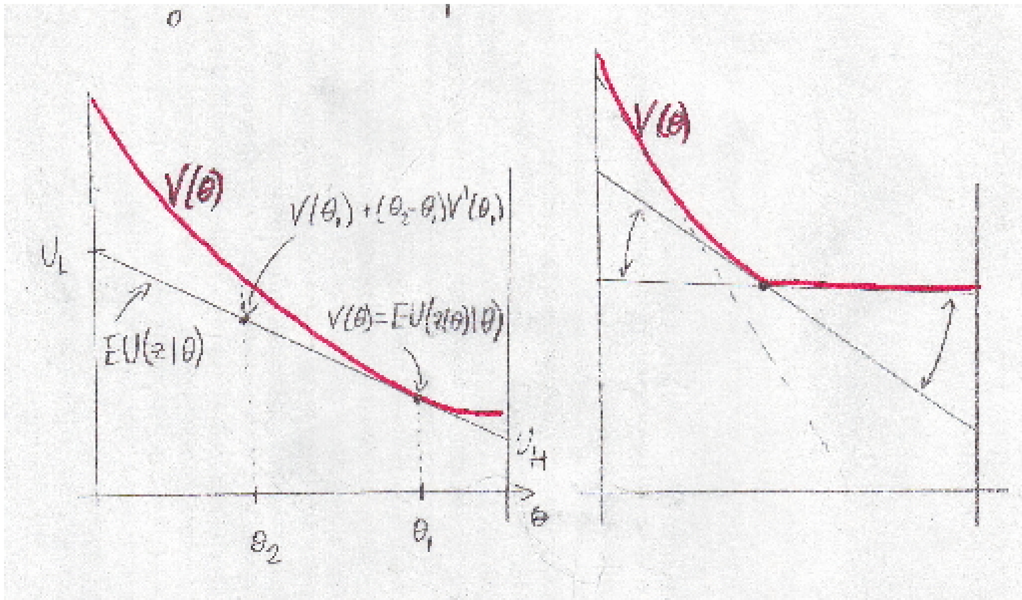

Figure 1.

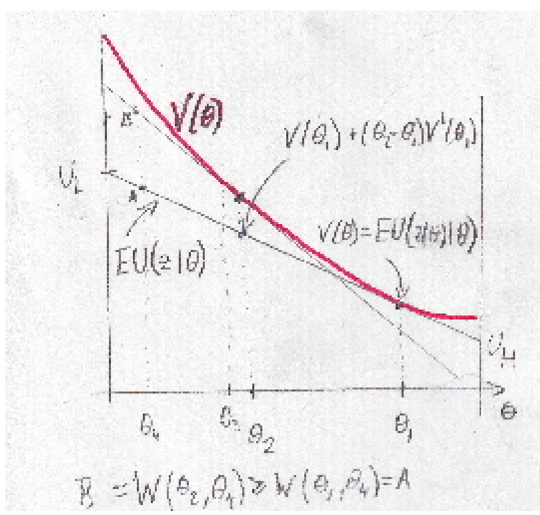

Figure 2