Communication equilibria in all-pay auctions*

PRELIMINARY AND INCOMPLETE

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May 10, 2009

Abstract

I derive a lower bound on the ex-ante surplus in communication equilibria in the all-pay auctions. For the case of two symmetric bidders and no reserve price the communication equilibrium is essentially unique, and is outcome equivalent to the Nash equilibrium of this game.

1 Introduction

In many situations an auctioneer is able to prevent bidders from organizing an explicit cartel that enforces coordinated behavior of the bidders and facilitates exchange of side payments. However it may be very difficult to prevent the bidders from simply engaging in cheap talk before the auction, and such communication can affect the outcomes of the auction. For example, in the second-price auctions pre-play communication allows to sustain the following "phases-of-the-moon" rotation scheme. Before the auction a designated winner is randomly chosen; during the auction the bidders coordinate on the equilibrium where the designated winner obtains the good for free by submitting a very high bid while the other bidders submit zero bids.[1] In the first-price and the all-pay auctions such a scheme would not work, because (under standard assumptions) there is only one equilibrium for any given prior beliefs. However there may exist other communication equilibria. In this paper I

*Acknowledgements to be added.

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[1] McAfee and McMillan (1992) show that such collusive scheme is bidder optimal in many environments as long as side payments are not allowed.

investigate how communication before the auction can affect the outcomes of the sealed bid all-pay auctions, and make some preliminary observations for the sealed bid first-price auctions.

By the revelation principle (Myerson (1982)) the outcome of any communication protocol can be replicated by the procedure whereby the bidders first secretly report their valuations to a neutral trustworthy mediator, who then makes private non-binding recommendations on how to bid to each bidder. Thus a communication equilibrium is a joint probability distribution over the valuations and the bids of the bidders, subject to the constraints that the bidders should find it optimal first to report their true valuations, and then submit the recommended bids. For each communication equilibrium there is an associated outcome function that maps profiles of the bidders' valuations into their probabilities of winning and their expected bids.

I derive a lower bound on the ex-ante surplus in communication equilibria in the all-pay auctions. To show this I consider the constraints on the equilibrium outcome imposed by one particular kind of deviation strategies available to each bidder. At the reporting stage the bidder randomizes over all possible reports according to the prior probability distribution of her valuations; at the bidding stage the bidder ignores the recommended bids and uses a particular deviational bidding strategy. Such behavior at the reporting stage assures that at the bidding stage the deviating bidder is bidding against the ex-ante distribution of the bids of the opponents. Hence, in the equilibrium each type of each bidder should do at least as well as she would do if he was just facing the ex-ante distribution of the bids of the opponents. This observation allows to derive a bound on the ex-ante distribution function of bids and thus on the expected profits from each bidder, that in turn yields a set of constraints on the equilibrium outcome.

For the case of two symmetric bidders and no reserve price there is only a single equilibrium outcome that satisfies these constraints. Thus the communication equilibrium is essentially unique, and is outcome equivalent to the Nash equilibrium of this game. I conjecture that the uniqueness result also holds in other cases, i.e. when the bidders are asymmetric, when there are more than two bidders, and when there is a binding reserve price. I hope that a similar approach can be used to analyze communication equilibria of other related games like all-pay contests with asymmetric information, as well as first-price auctions. I make some preliminary observations for the sealed bid first-price auctions in the last section of the paper.

Most of the studies of the bidder collusion in static auctions focus on a scenario when the bidders organize an explicit cartel that enforces coordinated behavior of the bidders and facilitates exchange of side payments. For example, Graham and Marshall (1987) study collusion in second-price auctions, and McAfee and McMillan (1992) study collusion in first-price auctions.[2] A more recent mechanism design literature, like Laffont and Martimort (1997) and Che and Kim (2006), studies the optimal response of the principal to collusion between the agents. A scenario when the bidders do not organize an explicit cartel is for the most part considered in the context of repeated auctions. For example, Aoyagi (2003) studies self-enforcing collusion with pre-auction communication. The case of tacit collusion without communication is studied by Skrzypacz and Hopenhayn (2004) in the environment with public monitoring and by Blume and Heidhaus (2006) in the environment with private monitoring.

There is also a number of recent papers that study collusion in static auctions when coordinated behavior of the bidders in the auction cannot be enforced. For example, Lopomo, Marshall and Marx (2005) and Garratt, Troger and Zheng (2008) show how a possibility of resale facilitates self-enforcing collusion in English auctions. Marshall and Marx (2007) consider a scenario with no resale when pre-auction side payments between the bidders are allowed, and show that it is much harder to collude in the first-price auction than in the second-price auction. Lopomo, Marx and Sun (2009) further study this model of collusion in the first-price auctions in a discretized framework with two symmetric bidders. Using linear programming techniques they show that the collusive equilibrium is essentially unique, and is outcome equivalent to the Nash equilibrium.

The rest of the paper is organized as follows. The model of the all-pay auction and the discussion of communication and Nash equilibria are in Section 2. The case of two bidders and no reserve price is studied in Section 3. The cases of non-zero reserve price and more than two bidders are analyzed in Section 4. Section 5 contains preliminary results on the first-price auction. The Appendix contains the analysis of the all-pay auction in the discrete model with two types.

[2] McAfee and McMillan (1992) also consider the case of "weak cartels" when side payments are not allowed.

2 Model and preliminaries

2.1 Environment and all-pay auction rules

There are $n \geq 2$ bidders; each bidder i has a valuation v_i for the good which is known only to him. Valuation v_i is distributed according to a continuous cumulative distribution function F_i and everywhere positive density f_i with support $[\underline{v}_i, \overline{v}_i]$, where $0 \leq \underline{v}_i < \overline{v}_i < +\infty$. Valuations $v_1, ..., v_n$ are distributed independently. This information structure is assumed to be common knowledge. Bidder i's utility is $v_i p_i - t_i$, when p_i is his probability of getting the good and t_i is his payment.

The bidders bid in a sealed bid all-pay auction with a reserve price $r \geq 0$. Each bidder i chooses an action from a set $A := [r, \infty) \cup \{\emptyset\}$, he can either submit an "active" bid $b_i \geq r$ or a "null" bid $b_i = \emptyset$. Bidders who submit the null bid do not receive the good and there is no payment. If bidder i with valuation v_i submits an active bid $b_i \geq r$, while the other bidders submit $b_{-i} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_n)$, then his payoff is

$$\begin{cases} v_i - b_i & \text{if} \quad b_i > \max\{r, \max_{j\neq i} b_j\} \\ -b_i & \text{if} \quad b_i < \max\{r, \max_{j\neq i} b_j\} \\ \frac{1}{\#\{k:\ b_k = b_i\}} v_i - b_i & \text{if} \quad b_i = \max_{j\neq i} b_j \geq r \end{cases}$$

2.2 Communication equilibria

According to the revelation principle (Myerson (1982)), any equilibrium outcome of any communication protocol can be replicated by the procedure whereby the bidders first privately report their valuations to a neutral trustworthy mediator, who then makes non-binding private recommendations (possibly stochastic) to each bidder of what bid to play.

Formally define a *communication rule* μ to be a family of probability distributions $\mu(\cdot|v_1, ..., v_n)$ over the action profiles of the bidders (A^n), indexed by the profile of the valuation reports submitted to the mediator ($(v_1, ..., v_n) \in \prod_{i=1}^{n} [\underline{v}_i, \overline{v}_i]$). Suppose first that all players truthfully report their valuations and obey the mediator's recommendations. Denote the expected payoff of bidder i with valuation v_i as follows:

$$U_i(v_i) = \int_{v_{-i}} \left(\int_{b_i, b_{-i}} (\Pr\{i \text{ wins} \mid b_i, b_{-i}\} v_i - b_i)\, d\mu(b_i, b_{-i}|v_i, v_{-i}) \right) dF_{-i}(v_{-i})$$

where $\Pr\{i \text{ wins} \mid b_i, b_{-i}\}$ is determined by the rules of the auction:

$$\Pr\{i \text{ wins} \mid b_i, b_{-i}\} = \mathbf{1}\left\{b_i > \max\left\{r, \max_{j\neq i} b_j\right\}\right\} + \frac{1}{\#\{k: \ b_k = b_i\}}\mathbf{1}\left\{b_i = \max_{j\neq i} b_j \geq r\right\} \tag{1}$$

Next suppose that all bidders other than i are still truthful and obedient. Bidder i with valuation v_i reports $\widehat{v}_i \in [\underline{v}_i, \overline{v}_i]$, and is bidding according to a strategy which is a function of the mediator's recommendation $\widehat{b}_i : A \to A$. Denote the expected payoff of bidder i as follows:

$$U_i\left(v_i; \widehat{v}_i, \widehat{b}_i(\cdot)\right) = \int_{v_{-i}}\left(\int_{b_i,b_{-i}}\left(\Pr\left\{i \text{ wins} \mid \widehat{b}_i(b_i), b_{-i}\right\} v_i - \widehat{b}_i(b_i)\right) d\mu(b_i, b_{-i}|\widehat{v}_i, v_{-i})\right) dF_{-i}(v_{-i})$$

The revelation principle implies that without loss of generality reporting the true valuation should be optimal for each bidder, and obeying the mediator's recommendation should be optimal for each bidder.

Definition 1 *A communication rule μ is called a **communication equilibrium** if each type of each player finds it optimal to report the truth and obey the mediator's recommendations, i.e.*

$$U_i(v_i) \geq U_i\left(v_i; \widehat{v}_i, \widehat{b}_i(\cdot)\right) \text{ for every } i,\ v_i,\ \widehat{v}_i \text{ and } \widehat{b}_i(\cdot).$$

Suppose bidder i with valuation v_i reports $\widehat{v}_i \neq v_i$ but obeys the mediator's recommendations (i.e. $\widehat{b}_i(b_i) = b_i$ for every $b_i \in A$). Denote by $P_i(\widehat{v}_i)$ and $T_i(\widehat{v}_i)$ the expected probability of winning and expected payment of bidder i, respectively, from using such strategy in a given communication rule μ:

$$\begin{aligned} P_i(\widehat{v}_i) &= \int_{v_{-i}}\left(\int_{b_i,b_{-i}} \Pr\{i \text{ wins} \mid b_i, b_{-i}\}\, d\mu(b_i, b_{-i}|\widehat{v}_i, v_{-i})\right) dF_{-i}(v_{-i}) \text{ and} \\ T_i(\widehat{v}_i) &= \int_{v_{-i}}\left(\int_{b_i,b_{-i}} b_i d\mu(b_i, b_{-i}|\widehat{v}_i, v_{-i})\right) dF_{-i}(v_{-i}) \end{aligned}$$

Then we can derive a set of incentive constraints which is familiar from the mechanism design literature with hidden types. The result is stated without proof since the argument is standard.[3]

Lemma 1 *In every communication equilibrium μ*

[3] See for example Myerson (1981).

(i) $U_i(v_i) = P_i(v_i)\,v_i - T_i(v_i) \geq P_i(\widehat{v}_i)\,v_i - T_i(\widehat{v}_i)$ *for every* i *and* $v_i, \widehat{v}_i \in [\underline{v}_i, \overline{v}_i]$.

(ii) P_i *is non-decreasing for every* i, *and* $U_i(v_i) = U_i(\underline{v}_i) + \int_{\underline{v}_i}^{v_i} P_i(\widetilde{v}_i)\,d\widetilde{v}_i$ *for every* i *and* $v_i \in [\underline{v}_i, \overline{v}_i]$.

Let us also consider another type of deviation which will be repeatedly used in the remainder of the paper. Denote by $G_{-i} : [0,\infty) \to [0,1]$ the ex-ante cumulative distribution function of the maximal bid among the opponents of bidder i, in the given communication equilibrium μ:

$$G_{-i}(\beta_i) = \int_{v_i}\int_{v_{-i}} \left(\int_{b_i,b_{-i}} \mathbf{1}\left\{ \beta_i > \max_{j\neq i} b_j \right\} d\mu(b_i, b_{-i}|\widehat{v}_i, v_{-i}) \right) dF_{-i}(v_{-i})\,dF_i(v_i) \text{ for every } \beta_i \in [r,\infty) \tag{2}$$

Lemma 2 *In every communication equilibrium* μ

$$U_i(v_i) \geq G_{-i}(\beta_i)\,v_i - \beta_i, \text{ for every } v_i \in [\underline{v}_i, \overline{v}_i] \text{ and } \beta_i \in [r,\infty).$$

Proof. Suppose bidder i with valuation v_i randomizes over reports $\widehat{v}_i$ according to his ex-ante distribution function F_i, and bids $\beta_i \in [r,\infty)$ regardless of the mediator's recommendation (i.e. $\widehat{b}_i(b_i) = \beta_i$ for every $b_i \in A$). The expected payoff from such deviation is as follows:

$$\begin{aligned}
& \int_{v_i}\int_{v_{-i}} \left(\int_{b_i,b_{-i}} \left(\Pr\{i \text{ wins} \mid \beta_i, b_{-i}\}\,v_i - \beta_i\right) d\mu(b_i, b_{-i}|\widehat{v}_i, v_{-i}) \right) dF_{-i}(v_{-i})\,dF_i(\widehat{v}_i) \\
= & \left(\int_{v_i}\int_{v_{-i}} \left(\int_{b_i,b_{-i}} \Pr\{i \text{ wins} \mid \beta_i, b_{-i}\}\,d\mu(b_i, b_{-i}|\widehat{v}_i, v_{-i}) \right) dF_{-i}(v_{-i})\,dF_i(\widehat{v}_i) \right) v_i - \beta_i \\
\geq & \left(\int_{v_i}\int_{v_{-i}} \left(\int_{b_i,b_{-i}} \mathbf{1}\left\{ \beta_i > \max_{j\neq i} b_j \right\} d\mu(b_i, b_{-i}|\widehat{v}_i, v_{-i}) \right) dF_{-i}(v_{-i})\,dF_i(\widehat{v}_i) \right) v_i - \beta_i \\
= & \; G_{-i}(\beta_i)\,v_i - \beta_i
\end{aligned}$$

The first equality makes use of the fact that bidder i bids β_i regardless of the mediator's recommendation, the inequality is implied by the rules of the auction given in (1), the last equality is by definition of G_{-i} given in (2). ■

2.3 Nash equilibria

In this section I briefly review the existing results for Nash equilibria in all-pay auctions. Nash equilibrium does not allow for pre-play communication, or more generally rules out any possibility of correlated play (unless the bidders' private valuations happen to be correlated). One may view Nash equilibria as a special class of communication equilibria where the mediator's recommendation to bidder i (b_i) is independent of the other bidders' reports ($\widehat{v}_{-i}$) as well as of the recommendations made to them (b_{-i}). Below I use a more standard approach to describing Nash equilibria in terms of the players' bidding functions.

A pure strategy of bidder i is a measurable function $b_i : [\underline{v}_i, \overline{v}_i] \rightarrow A$. In case there exists an inverse function of b_i on an interval $\left(\underline{b}, \overline{b}\right) \subset [r, \infty)$, then it is denoted by $\phi_i : \left(\underline{b}, \overline{b}\right) \rightarrow [\underline{v}_i, \overline{v}_i]$. First I state a well-known result for the case of ex-ante symmetric bidders.

Proposition 1 *Suppose $r \in [0, \overline{v})$ and $v_1, ..., v_n$ are identically distributed with a cumulative distribution F and a density f on $[\underline{v}, \overline{v}]$. Then*

(i) There exists an equilibrium such that each bidder uses the following strategy

$$b(v) = \begin{cases} r + \int_{v_r}^{v} \widetilde{v} dF^{n-1}(\widetilde{v}) & \text{if} \quad v \in (v_r, \overline{v}] \\ \emptyset & \text{if} \quad v \in [\underline{v}, v_r) \end{cases}, \text{ where } v_r \text{ is such that } F^{n-1}(v_r)\, v_r = r;$$

(ii) If $n = 2$ then this equilibrium is unique.

The existence of this type of equilibrium is a consequence of the revenue equivalence theorem (Myerson (1981)). Weber (1985) advanced a conjecture that this is a unique equilibrium. To the best of our knowledge, this conjecture was so far confirmed only for the case of two bidders: Amann and Leininger (1996) show uniqueness for the case $r = 0$; Lizzeri and Persico (2000) show uniqueness for the case $r \neq 0$.

Next I state a result for the case of two asymmetric bidders, which can be found in Lizzeri and Persico (2000).

Proposition 2 *Suppose $n = 2$ and $r \in [0, \overline{v})$. There exists a unique equilibrium. In this equilibrium each bidder i is using a bidding function which has a strictly increasing inverse $\phi_i : \left(r, \overline{b}\right] \rightarrow [\underline{v}_i, \overline{v}_i]$ such that*

$$\frac{1}{\phi_i'(b)} = g_i(\phi_i(b))\, \phi_j(b) \text{ for every } b \in \left(r, \overline{b}\right] \text{ and } i \neq j$$

Depending on the distributions F_1, F_2 and the reserve price r it may be that both, one or none of the bidders have a nondegenerate interval of types (at the low end of the support) who bid 0 *(or equivalently, $\emptyset$).*

Parreiras and Rubinchik (2008) show that the case of three or more heterogeneous bidders is considerably more complicated. For example, they prove that different bidders may have different ranges for their equilibrium bids, unlike in the case of two bidders. It is also not clear at this point whether Nash equilibrium is unique.

3 Case of two bidders and no reserve price

In this section I show that for the case of two bidders and no reserve price communication equilibria cannot be too inefficient.

Theorem 1 *Suppose $r = 0$ and $n = 2$. Then in every communication equilibrium μ the ex-ante expected surplus is bounded from below as follows:*

$$\sum_{i=1}^{2} \int_{\underline{v}_i}^{\overline{v}_i} P_i(v_i) v_i dF_i(v_i) \geq \frac{1}{2} \sum_{i=1}^{2} \int_{\underline{v}_i}^{\overline{v}_i} v_i dF_i^2(v_i)$$

Proof. Fix a communication equilibrium μ. Consider a function $\beta_i(v_i) = \int_{\underline{v}_i}^{v_i} v_i dF_i(v_i)$ defined on $[\underline{v}_i, \overline{v}_i]$. It has an inverse $\phi_i : [0, \beta_i(\overline{v}_i)] \to [\underline{v}_i, \overline{v}_i]$. Using Lemma 2 we must have

$$G_{-i}(b) \leq \frac{U_i(\phi_i(b)) + b}{\phi_i(b)} \text{ for every } b \in [0, \beta_i(\overline{v}_i)] \tag{3}$$

Note that the ex-ante expected profit is equal to

$$\Pi = \sum_{i=1}^{2} \int_{0}^{+\infty} b dG_i(b) = \sum_{i=1}^{2} \int_{0}^{+\infty} (1 - G_{-i}(b)) db$$

The profit can be bounded from below using (3) as follows:

$$\Pi \geq \sum_{i=1}^{2} \int_{0}^{\beta_i(\overline{v}_i)} (1 - G_{-i}(b)) db \geq \sum_{i=1}^{2} \int_{0}^{\beta_i(\overline{v}_i)} \left(1 - \frac{U_i(\phi_i(b_i)) + b_i}{\phi_i(b_i)}\right) db$$

Changing variables $v_i = \phi_i(b)$ and then using integration by parts we get

$$\begin{aligned} \Pi &\geq \sum_{i=1}^{2} \int_{\underline{v}_i}^{\overline{v}_i} \left(1 - \frac{U_i(v_i) + \int_{\underline{v}_i}^{v_i} \widetilde{v}_i dF_i(\widetilde{v}_i)}{v_i}\right) v_i f_i(v_i)\, dv_i \qquad (4) \\ &= \sum_{i=1}^{2} \int_{\underline{v}_i}^{\overline{v}_i} v_i F_i(v_i)\, dF_i(v_i) - \sum_{i=1}^{2} \int_{\underline{v}_i}^{\overline{v}_i} U_i(v_i)\, dF_i(v_i) \end{aligned}$$

Note that an alternative way to express the ex-ante profit is as follows:

$$\Pi = \sum_{i=1}^{2} \int_{\underline{v}_i}^{\overline{v}_i} T_i(v_i)\, dF_i(v_i) = \sum_{i=1}^{2} \int_{\underline{v}_i}^{\overline{v}_i} P_i(v_i)\, v_i dF_i(v_i) - \sum_{i=1}^{2} \int_{\underline{v}_i}^{\overline{v}_i} U_i(v_i)\, dF_i(v_i) \qquad (5)$$

Combining (4) and (5) we get the result. ■

The key observation used in the above proof comes from Lemma 2, which provides an upper bound on the cumulative distribution of equilibrium bids of any given bidder. It translates into a lower bound on the equilibrium expected payments of the bidders. As in the mechanism design literature with hidden types, higher expected payments are associated with more efficient allocation. In this way we obtain a lower bound on the expected surplus.

For the case homogeneous bidders Theorem 1 actually pins down the essentially unique communication equilibrium.

Corollary 1 *Suppose $r = 0$ and v_1, v_2 are identically distributed with a cumulative distribution F and a density f on $[\underline{v}, \overline{v}]$. Then the outcome of any communication equilibrium coincides a.s. with the outcome of the Nash equilibrium (given in Proposition 1).*

Proof. In the symmetric case the outcome of the Nash equilibrium is efficient. Thus the ex-ante expected surplus from the Nash equilibrium is the highest possible, and is equal to the expectation of the first-order statistic.

By Theorem 1 the ex-ante expected surplus from any communication equilibrium is at least $\int_{\underline{v}}^{\overline{v}} v dF^2(v)$, which is the expectation of the first-order statistic. ■

Given the uniqueness of the Nash equilibrium it is clear that, say, a simple coordination device like a publicly observed sunspot does not give rise to any new equilibria. Also it is quite intuitive that some simple minded pre-play communication scenarios where, say, one or both of the bidders announce whether the valuation is "high" or "low" would not work, since the high valuation bidders would have an incentive to lie to make the opponent bid less aggressively. However there still remains an abundance of other complicated (possibly stochastic) pre-play coordination/information revelation scenarios, and it is not clear a priori that none of them could work. Quite surprisingly, Corollary 1 shows that the possibility of pre-play communication in this case does not add any new equilibria in addition to the Nash equilibrium. In the Appendix I verify the robustness of this result by considering a discrete model with two types.

One implication of this result is that an auctioneer does not have to bother to try to prohibit pre-play communication between the bidders. Moreover, if the bidders' virtual valuation $v - ((1 - F(v)) / f(v))$ is strictly increasing on $[\underline{v}, \overline{v}]$, then the expected profit is maximized among all possible mechanisms that do not allow the auctioneer to withhold the good.[4]

The bidders on the other hand do pretty badly. If the inverse hazard rate $(1 - F(v)) / f(v)$ is strictly decreasing on $[\underline{v}, \overline{v}]$, then the bidders' ex-ante payoffs are minimized among all possible mechanisms that do not allow the auctioneer to withhold the good.[5]

The case of heterogeneous bidders is less clear cut, so to give some sense of the extent of the result in Theorem 1 I consider the following example.

Example 1 *Suppose $r = 0$, v_1 is uniformly distributed on $[0, 1]$, and v_2 is uniformly distributed on $[0, 2]$. The Nash equilibrium bidding functions (see Proposition 2) are: $b_1(v) = \frac{2}{3}v^3$, $b_2(v) = \frac{1}{3}\frac{1}{\sqrt{2}}v^{\frac{3}{2}}$. Bidder 1 wins the good whenever $2(v_1)^2 > v_2$, and thus the outcome of the Nash equilibrium is inefficient. It is straightforward to calculate that the ex-ante surplus from Nash equilibrium is $S_{ne} = \frac{21}{20}$, while the ex-ante surplus from efficient allocation is $S_{eff} = \frac{13}{12}$. By Theorem 1 the ex-ante surplus in any communication equilibrium (S_{ce}) is at least $\frac{1}{2}\left(2\int_0^1 v^2 dv + \frac{1}{2}\int_0^2 v^2 dv\right) = 1$. As another point of comparison consider a hypothetical scenario when the good is randomly allocated to bidder 1 and 2 with probabilities p and $1 - p$. The ex-ante surplus from such random allocation is $S_{\text{rand}} = 1 - \frac{1}{2}p \in \left[\frac{1}{2}, 1\right]$.*

[4] See for example Myerson (1981).

[5] See for example Section 3 in McAfee and McMillan (1992).

4 Other settings

4.1 Non-zero reserve price

In this section I restrict attention to the case of two symmetric bidders.

Lemma 3 *Suppose $r > 0$, $n = 2$ and v_1, v_2 are identically distributed with a cumulative distribution F and a density f on $[\underline{v}, \overline{v}]$. Then in every communication equilibrium μ*

$$\min_i \{\sup \{v_i : P_i(v_i) = 0\}\} \leq v_r$$

where v_r is the "cutoff type" in the Nash equilibrium (given in Proposition 1).

Proof. First note that for any given communication equilibrium $\widetilde{\mu}$ there exists a corresponding symmetric communication equilibrium μ.[6] For the remainder of the proof fix a symmetric communication equilibrium μ.

Denote by $U(v)$, $P(v)$ and $T(v)$ the equilibrium expected payoff, the probability of winning and expected payment of any bidder with valuation v. Let $G : [0, \infty) \to [0, 1]$ be the cumulative distribution function of the bids of any bidder. Denote $v^* = \sup \{v : P(v) = 0\}$, and suppose that $v^* > v_r$.

Consider the Nash equilibrium bidding function given in Proposition 1: $\beta(v) = r + \int_{v_r}^{v} \widetilde{v} dF(\widetilde{v})$ defined on $[v_r, \overline{v}]$. It has an inverse $\phi : [r, \beta(\overline{v})] \to [v_r, \overline{v}]$. Then using Lemma 2 we must have

$$G(b) \leq \frac{U(\phi(b)) + b}{\phi(b)} \text{ for every } b \in [r, b(\overline{v})]$$

Similar to the proof of Theorem 1 we can bound the profits from below as follows

$$\begin{aligned}
\Pi &= 2\int_0^{+\infty} (1 - G(b))\, db \geq 2\int_0^{\beta(\overline{v})} (1 - G(b))\, db = 2\left(b(\overline{v}) - \int_0^r G(b)\, db - \int_r^{\beta(\overline{v})} G(b)\, db\right) \\
&\geq 2\left(b(\overline{v}) - G(r)\, r - \int_r^{\beta(\overline{v})} \frac{U(\phi(b)) + b}{\phi(b)} db\right)
\end{aligned}$$

[6] If $\widetilde{\mu}$ is asymmetric, then by symmetry there exists another asymmetric communication equilibrium $\widehat{\mu}$ where the roles of the bidders are reversed. But then we can construct a symmetric communication equilibrium μ by randomizing over $\widetilde{\mu}$ and $\widehat{\mu}$ with equal probabilities. See for example Section 1 in Maskin and Riley (1984).

Changing variables $v = \phi(b)$ then using integration by parts we get

$$\begin{aligned}
\Pi &\geq 2\left(\left(r + \int_{v_r}^{\overline{v}} \widetilde{v} dF(\widetilde{v})\right) - \left(\frac{U(v_r)+r}{v_r}\right) r - \int_{v_r}^{\overline{v}} \left(\frac{U(v) + r + \int_{v_r}^{v} \widetilde{v} dF(\widetilde{v})}{v}\right) v f(v)\, dv\right) \qquad (6)\\
&= 2\left(\left(r + \int_{v_r}^{\overline{v}} \widetilde{v} dF(\widetilde{v})\right) - \left(\frac{U(v_r)+r}{v_r}\right) r - \int_{v_r}^{\overline{v}} U(v)\, dF(v) - \int_{v_r}^{\overline{v}} \left(r + \int_{v_r}^{v} \widetilde{v} dF(\widetilde{v})\right) dF(v)\right)\\
&= 2\left(-\left(\frac{U(v_r)+r}{v_r}\right) r - \int_{v_r}^{\overline{v}} U(v)\, dF(v) + rF(r) + \int_{v_r}^{\overline{v}} v f(v) F(v)\, dv\right)\\
&= \int_{v_r}^{\overline{v}} v dF^2(v) - 2\int_{v^*}^{\overline{v}} U(v)\, dF(v)
\end{aligned}$$

where the second equality uses integration by parts; the last equality uses the fact that $U(v) = 0$ for every $v \leq v^*$ and the definition of v_r $(F(v_r) v_r = r)$.

Note that an alternative way to express the ex-ante profit is as follows:

$$\Pi = 2\int_{\underline{v}}^{\overline{v}} T(v)\, dF(v) = 2\int_{v^*}^{\overline{v}} P(v)\, v dF(v) - 2\int_{v^*}^{\overline{v}} U(v)\, dF(v) \qquad (7)$$

Combining (6) and (7) we get

$$2\int_{v^*}^{\overline{v}} P(v)\, v dF(v) \geq \int_{v_r}^{\overline{v}} v dF^2(v)$$

But on the other hand we have

$$2\int_{v^*}^{\overline{v}} P(v)\, v dF(v) \leq 2\int_{v^*}^{\overline{v}} F(v)\, v dF(v) = \int_{v^*}^{\overline{v}} v dF^2(v)$$

Hence there is a contradiction since v^* is assumed to be greater than v_r. ■

The case of positive reserve price presents some difficulties for using the same argument as in Theorem 1. In particular, we can use Lemma 2 to get a bound cumulative distribution of equilibrium bids of each bidder only for the values above the reserve price r. Nonetheless, I view the partial characterization provided above as saying that the communication equilibria cannot be too inefficient, which is a similar message to the one in the previous section.

4.2 More than two bidders

In this section I assume no reserve price and continue to restrict attention to the case of symmetric bidders.

Lemma 4 *Suppose $r = 0$ and $v_1, ..., v_n$ are identically distributed with a cumulative distribution F and a density f on $[\underline{v}, \overline{v}]$. Then in every communication equilibrium μ the ex-ante expected surplus is bounded from below as follows:*

$$\sum_{i=1}^{n} \int_{\underline{v}}^{\overline{v}} P_i(v_i) v_i dF(v_i) \geq \frac{1}{2} \frac{n}{n-1} \int_{\underline{v}}^{\overline{v}} v dF^{2n-2}(v)$$

Proof. Fix a symmetric communication equilibrium μ.[7] Denote by $U(v)$, $P(v)$ and $T(v)$ the equilibrium expected payoff, the probability of winning and expected payment of any bidder with valuation v. Let $G : [0, \infty) \to [0, 1]$ be the cumulative distribution function of the bids of any bidder, $G_{\max} : [0, \infty) \to [0, 1]$ be the cumulative distribution function of the maximal bid among the opponents of any given bidder.

Consider the Nash equilibrium bidding function given in Proposition 1: $\beta(v) = \int_{\underline{v}}^{v} \widetilde{v} dF^{n-1}(\widetilde{v})$ defined on $[\underline{v}, \overline{v}]$. It has an inverse $\phi : (0, \beta(\overline{v})] \to [\underline{v}, \overline{v}]$. Using Lemma 2 we get

$$G_{\max}(b) \leq \frac{U(\phi(b)) + b}{\phi(b)} \text{for every } b \in [0, \beta(\overline{v})]$$

Using symmetry and the fact that $E\left[\sum_{j \neq i} b_j\right] \geq E[\max_{j \neq i} b_j]$ we can bound the profits from below as follows

$$\Pi = E\left[\sum_{i=1}^{n} b_i\right] \geq \frac{n}{n-1} E\left[\max_{j \neq i} b_j\right] = \frac{n}{n-1} \int_{0}^{+\infty} b dG_{\max}(b) = \frac{n}{n-1} \int_{0}^{+\infty} (1 - G_{\max}(b)) db$$

[7] As argued in Lemma 3 we can restrict attention to symmetric communication equilibria.

Similar to the proof of Theorem 1 we further bound the profits from below as follows

$$\Pi \geq \frac{n}{n-1}\int\limits_{0}^{\beta(\overline{v})}\left(1-G_{\max}\left(b\right)\right)db \geq \frac{n}{n-1}\int\limits_{0}^{\beta(\overline{v})}\left(1-\frac{U\left(\phi\left(b\right)\right)+b}{\phi\left(b\right)}\right)db$$

Changing variables $v=\phi\left(b\right)$ then using integration by parts we get

$$\begin{aligned}
\Pi &\geq \frac{n}{n-1}\int\limits_{\underline{v}}^{\overline{v}}\left(1-\frac{U\left(v\right)+\int_{\underline{v}}^{v}\widetilde{v}dF^{n-1}\left(\widetilde{v}\right)}{v}\right)vdF^{n-1}\left(v\right) \qquad (8)\\
&= \frac{n}{n-1}\left(\int\limits_{\underline{v}}^{\overline{v}}vdF^{n-1}\left(v\right)-\int\limits_{\underline{v}}^{\overline{v}}U\left(v\right)dF^{n-1}\left(v\right)-\int\limits_{\underline{v}}^{\overline{v}}\left(\int_{\underline{v}}^{v}\widetilde{v}dF^{n-1}\left(\widetilde{v}\right)\right)dF^{n-1}\left(v\right)\right)\\
&= \frac{1}{2}\frac{n}{n-1}\int\limits_{\underline{v}}^{\overline{v}}vdF^{2n-2}\left(v\right)-n\int\limits_{\underline{v}}^{\overline{v}}U\left(v\right)F^{n-2}\left(v\right)dF\left(v\right)
\end{aligned}$$

Note that an alternative way to express the ex-ante profit is as follows:

$$\Pi = n\int\limits_{\underline{v}}^{\overline{v}}T\left(v\right)dF\left(v\right) = n\int\limits_{\underline{v}}^{\overline{v}}P\left(v\right)vdF\left(v\right)-n\int\limits_{\underline{v}}^{\overline{v}}U\left(v\right)dF\left(v\right) \qquad (9)$$

Combining (8) and (9) we get

$$n\int\limits_{\underline{v}}^{\overline{v}}P\left(v\right)vdF\left(v\right) \geq \frac{1}{2}\frac{n}{n-1}\int\limits_{\underline{v}}^{\overline{v}}vdF^{2n-2}\left(v\right)+n\int\limits_{\underline{v}}^{\overline{v}}U\left(v\right)\left(1-F^{n-2}\left(v\right)\right)dF\left(v\right)$$

Since $U\left(v\right)\geq 0$ for every v we have the result. ■

When there are more than two bidders there is a new difficulty: we are interested in the distribution of each individual bid (b_i), while Lemma 2 provides us only with information about the distribution of the maximal bid among the opponents of any given bidder ($\max_{j\neq i} b_j$). So while the obtained lower bound is tight for the case of two players, it becomes less restrictive as the number of bidders increases.

Example 2 *Let $r=0$ and $v_1,...,v_n$ be uniformly distributed on $[0,1]$. The Nash equilibrium given in Proposition 1 is efficient, with the ex-ante surplus equal to $S_{eff}=\frac{n}{n+1}$. By Lemma 4 the ex-ante sur-*

plus in any communication equilibrium (S_{ce}) is at least $n\int_0^1 v^{2n-2}dv = \frac{n}{2n-1}$. *Below I present numerical values for the surplus from efficient allocation (S_{eff}), the surplus from a fair random allocation (S_{rand}), the bound on the surplus from communication equilibria (S_{ce}), and the bound on the increase in the surplus from communication equilibria over the random allocation relative to the difference between efficient surplus and the surplus from random allocation* ($(S_{ce} - S_{\text{rand}}) / (S_{eff} - S_{\text{rand}})$).

n	2	3	5	10	∞
$S_{eff} \approx$	0.667	0.75	0.833	0.909	1
$S_{\text{rand}} =$	0.5	0.5	0.5	0.5	0.5
$S_{ce} \gtrsim$	0.667	0.6	0.556	0.526	0.5
$\frac{S_{ce}-S_{\text{rand}}}{S_{eff}-S_{\text{rand}}} \gtrsim$	1	0.4	0.167	0.064	0

5 First-price auction

5.1 Preliminaries

In this section I use a similar approach to study communication equilibria in the first-price auction with a reserve price $r \geq 0$.

Each bidder has the same set of actions as in the all-pay auction: $A := [r, \infty) \cup \{\emptyset\}$. Bidders who submit the null bid $\emptyset$ do not receive the good and there is no payment. If bidder i with valuation v_i submits an active bid $b_i \geq r$, while the other bidders submit $b_{-i} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_n)$, then his payoff is

$$\begin{cases} v_i - b_i & \text{if} \quad b_i > \max\{r, \max_{j \neq i} b_j\} \\ 0 & \text{if} \quad b_i < \max\{r, \max_{j \neq i} b_j\} \\ \frac{1}{\#\{k\colon b_k = b_i\}}(v_i - b_i) & \text{if} \quad b_i = \max_{j \neq i} b_j \geq r \end{cases}$$

The definition of the communication equilibrium μ is the same as in Definition 1, with the expected payoff of bidder i with valuation v_i, when all players are truthful and obedient, given by

$$U_i(v_i) = \int_{v_{-i}} \left(\int_{b_i, b_{-i}} (\Pr\{i \text{ wins} \mid b_i, b_{-i}\}(v_i - b_i))\, d\mu(b_i, b_{-i}|v_i, v_{-i}) \right) dF_{-i}(v_{-i})$$

and the expected payoff of bidder i with valuation v_i, when he reports $\widehat{v}_i \in [\underline{v}_i, \overline{v}_i]$ and disobeys the mediator's recommendation according to $\widehat{b}_i : A \to A$ (while all players other than i are still truthful and obedient), given by

$$U_i\left(v_i; \widehat{v}_i, \widehat{b}_i(\cdot)\right) = \int_{v_{-i}} \left(\int_{b_i, b_{-i}} \left(\Pr\left\{ i \text{ wins} \mid \widehat{b}_i(b_i), b_{-i} \right\} \left(v_i - \widehat{b}_i(b_i) \right) \right) d\mu(b_i, b_{-i} | \widehat{v}_i, v_{-i}) \right) dF_{-i}(v_{-i})$$

The exact analog of Lemma 1 holds, with the definition of $T_i(v_i)$ changed to according to the rules of the first price auction. If we denote by $G_{-i} : [0, \infty) \to [0, 1]$ the ex-ante cumulative distribution function of the maximal bid among the opponents of bidder i, then we have the following analog of Lemma 2.

Lemma 5 *In every communication equilibrium* μ

$$U_i(v_i) \geq G_{-i}(\beta_i)(v_i - \beta_i) \text{ for every } v_i \in [\underline{v}_i, \overline{v}_i] \text{ and } \beta_i \in [r, \infty).$$

5.2 The case of uniform distribution

In this section I restrict attention to the case when $v_1, ..., v_n$ are independently uniformly distributed on $[0, 1]$. There is no reserve price: $r = 0$.

First, note that there exists a unique Nash equilibrium such that each bidder uses the following strategy: $b(v) = \frac{n-1}{n} v$ for every $v \in [0, 1]$.[8] This equilibrium is efficient, with the the ex-ante surplus equal to $S_{eff} = \frac{n}{n+1}$.

Lemma 6 *Suppose* $r = 0$ *and* $v_1, ..., v_n$ *are uniformly distributed on* $[0, 1]$. *Then in every communication equilibrium* μ *the ex-ante expected surplus is bounded from below as follows:*

$$\sum_{i=1}^{n} \int_0^1 P_i(v_i) v_i dv_i \geq 1 - \frac{1 + \ln n}{2n}$$

Proof. Fix a symmetric communication equilibrium μ.[9] Denote by $U(v)$, $P(v)$ and $T(v)$ the equilibrium expected payoff, the probability of winning and expected payment of any bidder with valuation v. Note that uniform distribution and symmetry implies that $U(1) = \int_0^1 P(v) dv = \frac{1}{n}$. Let $G_{\max} : [0, \infty) \to [0, 1]$ be the cumulative distribution function of the maximal bid among all

[8] See for example Lebrun (1999).

[9] As argued in Lemma 3 we can restrict attention to symmetric communication equilibria.

bidders, $G_{-1}:[0,\infty)\rightarrow[0,1]$ be the cumulative distribution function of the maximal bid among the opponents of any given bidder.

Using Lemma 5 we get

$$G_{-1}(b)\leq\frac{U(1)}{1-b}=\frac{\frac{1}{n}}{1-b}\text{for every } b\in\left[0,\frac{n-1}{n}\right] \tag{10}$$

Using symmetry and the fact that $E\left[\max_j b_j\right]\geq E\left[\max_{j\neq i} b_j\right]$ we can bound the profits from below as follows

$$\Pi=E\left[\max_j b_j\right]\geq E\left[\max_{j\neq i} b_j\right]=\int\limits_0^{+\infty} bdG_{-1}(b)=\int\limits_0^{+\infty}(1-G_{-1}(b))\,db$$

Using (10) we can further bound the profits from below as follows

$$\begin{aligned}\Pi\ &\geq\ \int\limits_0^{\frac{n-1}{n}}(1-G_{-1}(b))\,db\geq\frac{n-1}{n}-\frac{1}{n}\int\limits_0^{\frac{n-1}{n}}\frac{1}{1-b}db \\ &=\ \frac{n-1}{n}+\frac{1}{n}\left(\ln\left(\frac{1}{n}\right)-\ln(1)\right)=1-\frac{1+\ln n}{n}\end{aligned} \tag{11}$$

Note that an alternative way to express the ex-ante profit is as follows:

$$\begin{aligned}\Pi\ &=\ n\int\limits_0^1 T(v)\,dv=n\int\limits_0^1 P(v)\,vdv-n\int\limits_0^1 U(v)\,dv \\ &=\ 2n\int\limits_0^1 P(v)\,vdv-n\int\limits_0^1 P(v)\,dv=2n\int\limits_0^1 P(v)\,vdv-1\end{aligned} \tag{12}$$

where the third equality uses integration by parts; the last equality uses the fact that $\int_0^1 P(v)\,dv=\frac{1}{n}$. Combining (11) and (12) gives the result. ■

Here we face a difficulty similar to the one in Lemma 4: we are interested in the distribution of the maximal bid among all bidders ($\max_j b_j$), while Lemma 5 provides us only with information about the distribution of the maximal bid among the opponents of any given bidder ($\max_{j\neq i} b_j$). This distinction becomes less restrictive as the number of bidders increases, and so the lower bound on the surplus from communication equilibrium (S_{ce}) converges to the surplus from Nash equilibrium (S_{eff}). I believe that this lower bound can be further tightened and that it is possible to prove analogous results for distributions other than uniform.

Below I present numerical values for the surplus from efficient allocation (S_{eff}), the surplus from a fair random allocation (S_{rand}), the bound on the surplus from communication equilibria (S_{ce}), and the bound on the increase in the surplus from communication equilibria over the random allocation relative to the difference between efficient surplus and the surplus from random allocation ($\left(S_{ce} - S_{\text{rand}}\right) / \left(S_{eff} - S_{\text{rand}}\right)$).

n	2	3	5	10	∞
$S_{eff} \approx$	0.667	0.75	0.833	0.909	1
$S_{\text{rand}} =$	0.5	0.5	0.5	0.5	0.5
$S_{ce} \gtrsim$	0.577	0.65	0.739	0.835	1
$\frac{S_{ce}-S_{\text{rand}}}{S_{eff}-S_{\text{rand}}} \gtrsim$	0.46	0.601	0.717	0.819	1

6 Appendix

6.1 Discrete model with two types

Here I check whether the conclusion of Corollary 1 carries over to the all-pay auction model with discrete types. Suppose there are two bidders. Valuation of bidder i is v_i which can take two values $\underline{v}$ and $\overline{v}$, where $0 \leq \underline{v} < \overline{v}$. Valuations v_1, v_2 are distributed independently and identically: $\Pr\left(\underline{v}\right) = p$, $\Pr\left(\overline{v}\right) = 1 - p$ where $p \in (0, 1)$. There is no reserve price: $r = 0$.

First let us describe the Nash equilibrium. The result is stated without proof since the argument is standard.[10]

Lemma 7 *There exists a unique Nash equilibrium in the discrete model with two types such that each bidder uses the following strategy:*

(i) Let $\underline{v} > 0$. Type $\underline{v} = 0$ randomizes according to cumulative distribution function $\underline{H}\left(b\right) = \frac{b}{(1-p)\overline{v}}$ on $[0, p\underline{v}]$, type $\overline{v}$ randomizes according to cumulative distribution function $\overline{H}\left(b\right) = \frac{b}{(1-p)\overline{v}}$ on $\left(p\underline{v}, p\underline{v} + (1-p)\,\overline{v}\right]$.

(ii) Let $\underline{v} = 0$. Type $\underline{v} = 0$ bids 0, type $\overline{v}$ randomizes according to cumulative distribution function $\overline{H}\left(b\right) = \frac{b}{(1-p)\overline{v}}$ on $\left(0, (1-p)\,\overline{v}\right]$.

[10] See for example Section 6 in Fudenberg and Tirole (1991).

For future reference note that the outcome of the Nash equilibrium is efficient, and the ex-ante expected surplus is equal to $p^2\underline{v}+\left(1-p^2\right)\overline{v}$.

Lemma 8 *The outcome of any communication equilibrium in the discrete model with two types coincides a.s. with the outcome of the Nash equilibrium.*

Proof. I prove the result only for the case when $\underline{v}>0$. The argument for the other case is identical.

Fix a symmetric communication equilibrium μ.[11] Denote by $U(v)$, $P(v)$ and $T(v)$ the equilibrium expected payoff, the probability of winning and expected payment of any bidder with valuation v. Let $G:[0,\infty)\to[0,1]$ be the cumulative distribution function of the bids of any bidder.

Using Lemma 2 we get

$$\begin{aligned} G(b) &\le \frac{U(\underline{v})+b}{\underline{v}} \text{ for every } b\in[0,p\underline{v}]\text{, and} \\ G(b) &\le \frac{U(\overline{v})+b}{\overline{v}} \text{ for every } b\in(p\underline{v},p\underline{v}+(1-p)\overline{v}]\,. \end{aligned} \tag{13}$$

Similar to the proof of Theorem 1 we can bound the profit from below using (13)

$$\begin{aligned} \Pi &= 2\int_0^{+\infty}(1-G(b))\,db \ge 2\int_0^{p\underline{v}+(1-p)\overline{v}}(1-G(b))\,db \\ &\ge 2\left((p\underline{v}+(1-p)\overline{v})-\int_0^{p\underline{v}}\frac{U(\underline{v})+b}{\underline{v}}db-\int_{p\underline{v}}^{p\underline{v}+(1-p)\overline{v}}\frac{U(\overline{v})+b}{\overline{v}}db\right) \\ &= \left(p^2\underline{v}+\left(1-p^2\right)\overline{v}\right)-2\left(pU(\underline{v})+(1-p)U(\overline{v})\right) \end{aligned} \tag{14}$$

Note that an alternative way to express the profit is as follows:

$$\Pi = 2\left(pT(\underline{v})+(1-p)T(\overline{v})\right)=2\left(pP(\underline{v})\,\underline{v}+(1-p)P(\overline{v})\,\overline{v}\right)-2\left(pU(\underline{v})+(1-p)U(\overline{v})\right) \tag{15}$$

Combining (14) and (15) we get

$$2\left(pP(\underline{v})\,\underline{v}+(1-p)P(\overline{v})\,\overline{v}\right)\ge p^2\underline{v}+\left(1-p^2\right)\overline{v}$$

[11] As argued in Lemma 3 we can restrict attention to symmetric communication equilibria.

The expression on the right is the ex-ante expected surplus from the Nash equilibrium, which is the highest possible, since the outcome of the Nash equilibrium is efficient. ■

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