Favoritism in Asymmetric Contests: Head Starts and Handicaps*

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Abstract

We examine a contest, modelled as an all-pay auction with incomplete information, in which a strong and a weak contestant compete. A contestant may suffer from a handicap or benefit from a head start. The former reduces the contestant's score by a fixed percentage; the latter is an additive bonus. The recipient of the effort is better off by giving the weak contestant a head start. However, it may or may not be profitable to handicap the strong contestant. In general, when the contestants are sufficiently heterogeneous, the weak contestant should be given a head start *and* a handicap. The effort maximizing head start and handicap may also improve the efficiency of the contest.

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1 Introduction

Writing a grant proposal can involve a significant investment of time and effort. However, this investment is not always rewarded. When several researchers compete for limited funds, even good applications are often unsuccessful.

In Canada, two criteria are used for evaluating an application for a Standard Research Grant submitted to the Social Sciences and Humanities Research Council (SSHRC). Specifically, "the score on the record of research achievement accounts for 60 per cent of the overall score, and the score on the program of research accounts for 40 per cent of the overall score".[1] The implication is that an individual with a good research record has a *head start*. That is, she will out-score a competing proposal by a less successful scholar, even if the proposed programs of research are of comparable quality.[2]

In contrast, if the applicant obtained his Ph.D. within the last five years of the application deadline he will normally be considered a "new scholar". In this case, the two components are weighted "such that either a 60/40 or 40/60 ratio will apply, depending on which will produce the more favorable overall score."

Now, compare an established scholar and a new, unproven, scholar. The established scholar has a *head start* due to her past research achievements. On the other hand, she is *handicapped* in the sense that putting more effort into the proposal impacts her score less (the program of research accounts for only 40%) than a corresponding increase in effort by the new scholar would increase his score (since the program of research accounts for up to 60% for this applicant). At first sight, it may appear that this design is self-contradictory; it is certainly unclear which scholar is favoured.

There are numerous examples of contests in which head starts and/or handicaps play important roles. Examples where they are imposed by a contest designer include sports (e.g. golf and horse racing), affirmative action, and uneven treatment of internal and external applicants for senior positions. They are sometimes exogenous and fixed components of the competition, as in R&D races between firms with different technologies and R&D procedures. Finally, they are sometimes created over time in dynamic contests, such as in a R&D race where one firm is first to make a preliminary discovery that may function as a stepping stone.

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[1] See http://www.sshrc.ca/web/apply/program_descriptions/standard_e.asp.

[2] We ignore the time element in this discussion, as well as in the model. In practice, there may be an incentive to devote the time and effort to building up the CV this year, in order to increase the chances in next year's competition. In other words, we consider the record of research achievement to be fixed when the decision to write a grant proposal is made. See Konrad (2002) for a two-stage model in which preliminary actions in the first stage affects the contest, held in stage two.

The topic of this paper is favoritism in contests in which contestants may be heterogeneous. Given the numerous instances in which contests are manipulated, it is surprising that favoritism has been subject to only limited formal study in the standard models of contests. The main objective of the paper is to challenge some common intuitions in a formal model. The following results are among the main findings: (1) a contestant who is favoured does not necessarily "slack off"; he may respond by working harder, (2) it may be profitable to handicap the *weak* contestant, and (3) it may be more profitable still to combine various instruments in ways that appear self-contradictory; specifically, to simultaneously give a contestant a head start and a handicap. The emphasis in the paper is on the second and third point, concerning the profitability of favoritism.

To begin, we posit that a statement such as "it is profitable to favour the weak contestant" is too simplistic and possibly even false, even though it is intuitively compelling. Leveling the playing field may not be profitable. First, different ways of manipulating a contest may have different consequences. This paper illustrates this point by distinguishing between two instruments: head starts and handicaps.[3] These instruments affect the contest differently, so one is no a substitute for the other. Giving a contestant a head start is not equivalent to handicapping his opponent.

Second, while it is true in the benchmark model that it is profitable to favour the weak contestant – if it is done with the right instrument – using the wrong instrument to do so may backfire. In particular, it may be profitable to handicap the weak contestant in favour of the strong contestant. As a result, we will show that the contest design implemented by SSHRC, involving a head start counteracted by a handicap, may be very effective at increasing the average quality of the program of research.[4,5] In particular, this design may be profitable if a grant is more valuable early in a scholars' career, such that new scholars are believed to value a grant substantially more than established scholars.

The contest is modelled as an all-pay auction in which bidders are privately

[3]Head starts and handicaps are identity-dependent. Alternatively, the contest can be manipulated by imposing restrictive rules that apply to all contestants. There are several papers on caps in contests, i.e., an upper bound on how much effort or money the contestants can expend. Although the cap applies to all contestant, the strategic response may be different for weak and strong contestants. See Che and Gale (1998), Gavious et al. (2002), and Sahuguet (2006). Caps are briefly discussed in Section 5.

[4]Incidentally, this work is not funded by SSHRC.

[5]There are at least two reasons for why it may be in the funding agency's interest to increase the quality of the proposed program of research. First, it is likely that there is a positive correlation between how much effort goes into the proposal and how well the money are spent. Second, impressive and visionary applications may make it easier to justify the existence of the funding agency to the government. In fact, SSHRC may make successful applications public.

informed about either their valuation of the prize or their ability. Bidders are ex ante heterogeneous, with one bidder perceived to be more likely to have high valuation or ability than another. Since we wish to emphasize the role of this heterogeneity in determining the optimal design features, we impose a few simplifying assumptions to facilitate the analysis. As in most of the existing literature on asymmetric auctions, we assume there are exactly two bidders, one "strong" and one "weak". We also assume that bidders are risk neutral and that the cost of bidding is linear in the bid. These assumptions permit the use of powerful arguments from mechanism design.[6] Extension are discussed in Section 4.

We examine two design features that imply that bidders do not necessarily compete on even terms. A bidder has a head start if he would win the auction if both bidders bid zero. In contrast, the bidder is handicapped if an increase in his bid has a smaller impact on how his bid is evaluated compared to an equal-sized increase in his competitor's bid.[7] Concretely, if bidder i bids b, his "score" is $s_i = a_i + r_i b$. The winner of the auction is the bidder with the highest score, not necessarily the bidder with the highest bid. Bidder 1 has a head start if $a_1 > a_2$, and is handicapped if $r_1 < r_2$.

A handicap has bearing on the marginal return of increasing the bid, while a head start does not. Consequently, the two instruments affect the auction in different ways. Roughly speaking, a head start influences the decision to participate in the auction. In contrast, a handicap influences the relative scores of bidders who have decided to participate.

The literature on favoritism in all-pay auctions is small. Konrad (2002) examines a two-bidder model with head starts and handicaps. However, he assumes the value of the prize is known and that it is the same for both bidders. Moreover, Konrad (2002) assumes the head start and handicap are exogenously given, whereas we allow them to be determined by a third party. See also Siegel (2009). With incomplete information, there appears to be no previous papers dealing with head starts, and only three that examine handicaps. Lien (1990) and Feess et al (2008) assume the two bidders are homogeneous.[8] Clark and Riis (2000, henceforth C&R) allow the two bidders to be heterogenous, but assume that types are drawn from uniform

[6] A few papers, including Moldovanu and Sela (2001) and Gavious, Moldovanu, and Sela (2002), allow the cost of bidding to be non-linear in the bid. However, they assume that bidders are homogeneous. Clark and Riis (2000) assume bidding costs are non-linear and that bidders are heterogeneous, with types that are drawn from (different) uniform distributions.

[7] In the SSHRC example, the "bid" is the program of research. A better program is costly to the bidder, since it requires more time and effort.

[8] Lien (1990) proves that the handicapped bidder wins less often than is efficient if types are drawn from a uniform distribution. Feess et al (2008) show that this result holds for any distribution.

distributions. In their model, it is profitable to handicap the strong bidder, and they claim the reason is that it "evens up" the contest.

The current paper is the first to examine head starts in contests with incomplete information. It is also the first to attempt to study handicaps in a more general setting than the uniform model. The former is considerably easier than the latter. Thus, the results concerning head starts are quite general. In this regard, one of the contributions of the paper is to establish a general class of models which encompasses the uniform model, and for which the analysis of the all-pay auction is particularly straightforward.[9]

The impact of handicaps on expected revenue is non-trivial. However, we identify a trade-off which makes it impossible to generalize the result of C&R that it is the strong bidder who should be handicapped. The trade-off is explained in more detail below. We then consider two special models for which it is possible to determine whether it is the weak or the strong bidder who should be handicapped. These models are special in the sense that mass points are introduced into the distribution function of at least one of the bidders.[10]

The theoretical analysis is concluded by examining the simultaneous use of head starts and handicaps. The main objective in this part is to establish that the weak bidder may be given a head start and a handicap, even when it would be optimal to handicap the strong bidder if head starts are ruled out. The uniform model has this property when the asymmetry is sufficiently large. However, a generalization is difficult. Nevertheless, we identify a set of sufficient conditions (satisfied by the uniform model) for which the result holds.

The trade-off associated with a handicap can be explained as follows. Without the handicap, the weak bidder tends to "overcompensate" for his weakness if his type is high. Consequently, he wins too often in the all-pay auction compared to the revenue maximizing mechanism when his type is high. Handicapping the strong bidder only makes this worse. On the other hand, the weak bidder wins less often in the all-pay auction compared to the revenue maximizing mechanism when his type is

[9] The weak bidder's distribution function is a "scaled down" version of the strong bidder's distribution function. This class of models does not appear to have been studied before in the auction literature. As mentioned, the uniform model is a special case. Alternatively, the uniform model is also a special case of the general class of models in which the weak bidder's distribution function is a truncation of the strong bidder's distribution function. The latter class of models is sometimes considered in the literature on asymmetric first price auctions; see e.g. Maskin and Riley (2000). However, the first class of models is more tractable in all-pay auctions.

[10] This part of the paper follows the tradition of Maskin and Riley (2000). They show that a second price auction may dominate a first price auction in terms of revenue if the weak bidder's distribution function has a mass point at the lowest end-point of the support.

low. Handicapping the strong bidder alleviates this problem. In the C&R model, it happens to be the case that handicapping the strong bidder is optimal, on balance. However, this result is not robust.

A head start affects the allocation only if bidders' types are low. Giving the weak bidder a head start allows him to win more often when his type is low. Thus, the allocation is closer to the optimal allocation. In addition, there is less reason to handicap the strong bidder when the weak bidder has a head start. If the asymmetry among bidders is sufficiently large, it is optimal to give a head start to the weak bidder, and then handicap him as well.

The remainder of the paper is organized as follows. The benchmark model is introduced in Section 2, and equilibrium strategies are derived. Optimal head starts and handicaps are considered in Section 3. Section 4 discusses the results and possible extensions. Section 5 concludes.

2 Model and Equilibrium

We model the contest as an all-pay auction. There are two bidders.[11] Each bidder is characterized by a privately known type which captures the value the bidder puts on winning the auction. Bidder i draws his type from the continuously differentiable distribution function F_i with support $[0, \overline{v}_i]$, $i = 1, 2$. Unless explicitly stated otherwise, the distribution functions have no mass points. The density, $f_i(v) = F_i'(v)$, is bounded above as well as bounded below, away from zero.

The two bidders are heterogeneous. Bidder 1 is more likely to have a low valuation than bidder 2, $f_1(0) > f_2(0)$. He is also less likely to have a high valuation, $\overline{v}_1 < \overline{v}_2$. Hence, it is helpful to think of bidder 1 as the "weak" bidder and bidder 2 as the "strong" bidder.[12]

In a standard all-pay auction bidder i must decide whether to participate in the auction, and, if so, which non-negative bid, b_i, to submit, $i = 1, 2$. The bidder with the highest bid would then be the winner. Here, however, we assume bidders receive differential treatment. It is convenient to think of bidder i as accumulating a "score", s_i. If bidder i bids b, his score is $s_i = a_i + r_i b$, where $a_i \geq 0$, $r_i > 0$. The winner is then the bidder with the highest score.[13] Hence, bidder i must decide whether to

[11] See Parreiras and Rubinchik (2006) for an analysis of aspects of all-pay auctions with many heterogenous bidders.

[12] Many of the formal results and insights depend only on how the distribution fucntions compare at the endpoints of their supports. First order stochastic dominance need not be imposed at this point.

[13] If $a_i > a_j$ and $s_1 = s_2 = a_i$, we assume that bidder j wins. This assumption ensures the

participate, and, if so, which score to aim for.

We say that bidder 1 has a head start if $a \equiv a_1 - a_2 > 0$, in which case he wins the auction if both bidders bid zero. Moreover, bidder 1 is handicapped if $r \equiv r_1/r_2 < 1$, while bidder 2 is handicapped if $r > 1$. The handicapped bidder's score responds less to an increase in the bid than is the case for the other bidder.

Turning to payoffs, we assume that bidders are risk neutral and that the true cost of a bid of b is in fact b. These assumptions are standard in the auction literature, but a more general treatment would allow for risk aversion and costs that are non-linear in the bid. The cost of obtaining a score of $s \geq a_i$ for bidder i is

$$c_i(s) = \frac{s - a_i}{r_i}. \tag{1}$$

2.1 Equilibrium allocation

We let $\varphi_i(s)$ denote bidder i's inverse strategy, i.e. bidder i scores s in equilibrium if his type is φ_i, $i = 1, 2$. Assuming that strategies are strictly increasing in type among the set of types that participate in the auction, the probability that bidder i's score is at or below s is $F_i(\varphi_i(s))$.

Given the rival's strategy, bidder i with type v maximizes his expected payoff. That is, he solves the problem

$$\max_{s \geq a_i} vF_j(\varphi_j(s)) - c_i(s),$$

or, equivalently,

$$\max_{s \geq a_i} r_i v F_j(\varphi_j(s)) - s \tag{2}$$

where $j \neq i$ is bidder i's competitor. Other things being equal (in particular $\varphi_j(s)$), the size of r_i directly affects the return of scoring higher. Consequently, it should be no surprise that r_i will influence the relative scores of the two bidders, and thus the probabilities of winning. In contrast, given the assumptions of risk neutrality and linear costs, a_i is significant only in that it determines the lowest feasible and rational score. If a_i is very high, bidder j may decide to stay out if his type is low since a strictly positive bid cannot be rationalized if it leads to a score below a_i. However, among the types that do participate, a_i does not distort the allocation. In other words, a_i will have a level effect on the scores, and may force out some of bidder j's types.

existence of an equilibrium. The tie-breaking rule is inconsequential in all other cases.

Assuming, for now, that the solution is interior, the first order condition is

$$r_i v f_j(\varphi_j(s)) \frac{d\varphi_j(s)}{ds} = 1.$$

In equilibrium, bidder i with type v obtains a score of s, meaning that $v = \varphi_i(s)$, and the first order condition can be written as

$$\frac{d\varphi_j(s)}{ds} = \frac{1}{r_i \varphi_i(s) f_j(\varphi_j(s))}. \tag{3}$$

As in Amann and Leininger (1996), dividing the two first order condition yields

$$\frac{d\varphi_1(s)}{d\varphi_2(s)} = r \frac{\varphi_1(s) f_2(\varphi_2(s))}{\varphi_2(s) f_1(\varphi_1(s))}. \tag{4}$$

To continue, define $k(v)$ as the type of bidder 1 who obtains the same score as bidder 2 with type v. In other words, bidder 2 with type v wins if bidder 1's type is below $k(v)$. Since bidder 2 of type $\varphi_2(s) = v$ achieves the same score as bidder 1 with type $\varphi_1(s) = k(v)$, (4) can be rewritten as

$$\frac{dk(v)}{dv} = r \frac{k(v) f_2(v)}{v f_1(k(v))}. \tag{5}$$

To solve the differential equation we make use of the boundary condition that $k(\overline{v}_2) = \overline{v}_1$. That is, the two bidders must share a common maximal score. Otherwise, the bidder with the highest possible score could reduce his bid without reducing the probability that he wins. With this boundary condition, (5) is then sufficient to completely determine $k(v)$. In particular, $k(v)$ must satisfy

$$\int_{k(v)}^{\overline{v}_1} \frac{f_1(x)}{x} dx = r \int_{v}^{\overline{v}_2} \frac{f_2(x)}{x} dx. \tag{6}$$

The right hand side is finite for any $v > 0$, while the left hand side is 0 if $k = \overline{v}_1$ but increases and goes to infinity as k approaches 0 from above. Hence, for any $v > 0$, there exists some k which satisfies (6), and this k is unique. Since the right hand side approaches infinity as v approaches 0, it must also be the case that $k(0) = 0$. The implication is that $k(0) = 0$ and $k(\overline{v}_2) = \overline{v}_1$ regardless of r. However, $k(v)$ depends on r for any $v \in (0, \overline{v}_2)$.

Although $k(v)$ depends or r it is independent of a_1 and a_2. This confirms that a_1 and a_2 does not affect the relative bids of the bidders (among the active types), since

k by definition reveals who bidder 2 ties with. However, we have not yet determined how the entry decision and the level of the bids is affected by changes in a_1 and a_2.[14] To do so, we assume for the sake of exposition that bidder 1 is the bidder with the head start, $a \geq 0$. As we will see in the next section, this is indeed profitable.

Since bidder 1 has a head start, bidder 2 may decide not to enter the auction at all. In particular, bidder 2 realizes that any score below a_1 will fail to win him the auction. Hence, bidder 2 either decides that it is too costly to participate and thus stays out if his type is sufficiently small, or he submits a bid of at least $c_2(a_1)$. Thus, no score below a_1 will be observed in the auction. It will never be profitable for bidder 2 to participate if $c_2(a_1) \geq \overline{v}_2$. In the following we therefore focus on the case where $c_2(a_1) \in [0, \overline{v}_2)$.

To find the critical type of bidder 2 who is indifferent between staying out of the auction and entering the auction with a score of a_1, we solve

$$vF_1(k(v)) - c_2(a_1) = 0. \tag{7}$$

Let the solution be denoted by v_2^c, and define $v_1^c \equiv k(v_2^c)$. Clearly, v_1^c and v_2^c depend directly on a and r_2. However, they also depend on r_1, since k depends on r_1.

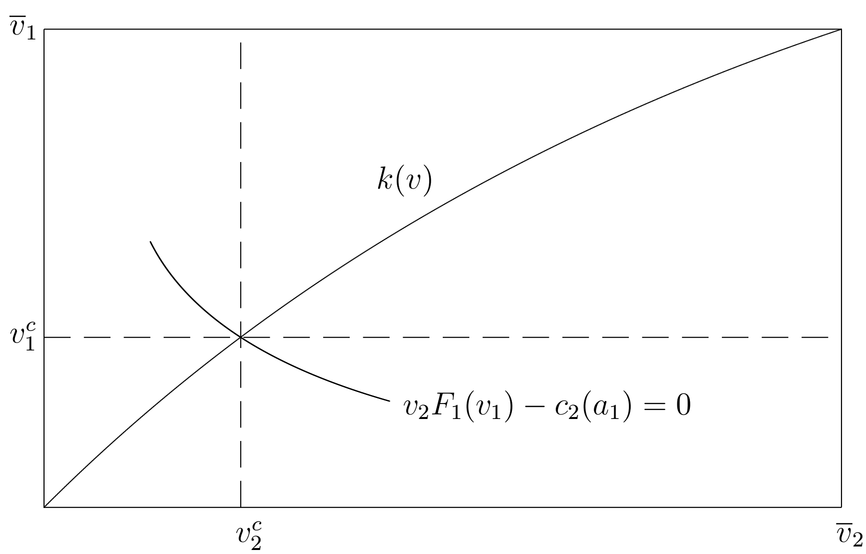

Figure 1: The equilibrium allocation. Bidder 2 wins below $k(v)$, to the right of v_2^c. Bidder 1 wins everywhere else.

[14]In particular, k was derived under the assumption that the first order conditions are satisfied, i.e. that the solution is interior. However, this is not a valid assumption for low types when $a \neq 0$.

We are now ready to outline the main properties of the equilibrium. In (v_2, v_1) space, Figure 1 depicts $v_1 = k(v_2)$ (as defined by (6)) as well as the level curve on which $v_2 F_1(v_1)$ is constant and equal to $c_2(a_1)$. Note that the former is increasing, by (5) or (6), while the latter is decreasing. The intersection of the two satisfies (7) and thus defines v_1^c and v_2^c.

In equilibrium, bidder 2 stays out of the auction if his type if strictly below v_2^c, and enters with a bid of $c_2(a_1)$ (score of a_1) if his valuation is precisely v_2^c. If his valuation is higher, he enters the auction and obtains a score equal to that obtained by bidder 1 with type $k(v)$. Bidder 1 enters the auction regardless of his type, but he submits a bid of zero, thereby obtaining a score of a_1, if his type is v_1^c, or below. If his type is higher he achieves a score to rival that of bidder 2 with type $k^{-1}(v)$.

Figure 1 also illustrates who wins the auction as a function of the bidders' types. The central point is (v_2^c, v_1^c). To the left of this point, $v_2 < v_2^c$, bidder 2 stays out of the auction, meaning that bidder 1 wins regardless of his type. To the south-east of (v_2^c, v_1^c), bidder 1 bids zero (scores a_1), while bidder 2's type is so high that he decides to be active in the auction (score above a_1). Hence, bidder 2 wins. Both bidders are active to the north-east of (v_2^c, v_1^c), but bidder 1 wins if the combination of types is above the $v_1 = k(v_2)$ curve, while bidder 2 wins below it.

The allocation in the auction can be manipulated by manipulating the two curves in Figure 1. The "level curve" can be moved to the right by increasing a or decreasing r_2, while k can be made to move down by increasing r.

In Figure 1, an increase in a corresponds to shifting the level curve defined by (7) to the north-east. Since $k(v)$ remains unchanged, v_1^c and v_2^c must increase.

On the other hand, an increase in r leads k to move downwards, in the interior (at the end-points, $k(0) = 0$ and $k(\overline{v}_2) = \overline{v}_1$ regardless of r). The right hand side of (6) is increasing in r for any $v \in (0, \overline{v}_2)$. Hence, when r increases the left hand side must increase as well, to maintain the equality. This necessitates that k declines.

However, since $k(v)$ decreases when r increases, it must also be the case that v_1^c and v_2^c change. If the increase in r comes from an increase in r_1, the level curve in Figure 1 is unchanged. As seen from Figure 1 or (7), v_2^c increases while v_1^c decreases whenever $a > 0$. Hence, bidder 2 is more likely to stay out, while bidder 1 is less likely to be satisfied with a bid of zero. The reason for the latter is that bidder 1 will exploit his increasingly advantageous position; he is more likely to be active and press his advantage.

In summary, changing a has no effect on the allocation among the types that continue submitting strictly positive bids. Hence, the head start affects the allocation only at "the bottom" (low types). In contrast, rewarding bidder 1 by increasing r_1 affects the allocation everywhere else. Thus, the two ways of favouring a bidder lead to very different outcomes. Below, we summarize the discussion thus far.

Proposition 1 *v_1^c and v_2^c are increasing in a. The former is (weakly) decreasing in r_1, while the latter is (weakly) increasing in r_1. Finally, $k(v)$ is independent of a, but decreasing in r, for all $v \in (0, \overline{v}_2)$.*

The final possibility is that r increases due to a decrease in r_2. This has two partially confounding effects. First, k shifts down. Second, the cost of obtaining a score of a_1 increases for bidder 2, meaning that the level curve in Figure 1 shifts to the right. Thus, v_2^c increases, but v_1^c may increase or decrease. To avoid this complication we will fix the values of a_2 and r_2 by normalizing

$$a_2 = 0, \ r_2 = 1.$$

In the following, when we increase a or r it should be understood that we refer to an increase in a_1 or r_1, respectively. The choice of a is a choice of where to locate the level curve, while the choice of r is a choice only of how much to manipulate k.

2.2 Equilibrium strategies

For the purposes of this paper, the equilibrium strategies themselves are of limited interest. For completeness, the following Proposition describes how a and r impact the equilibrium bids. Since this will not be used in the rest of the paper, the proof is in the Appendix.

Proposition 2 *Increases in a and r change equilibrium bids in the following way:*

1. *If a increases, bidder 1 bids less aggressively. Bidder 2 is more likely to stay out of the auction, but if he does participate he participates with a more aggressive bid.*

2. *If r increases, bidder 1 bids more aggressively if his type is low but less aggressively if his type is high. If bidder 2 continues to participate when r is increased, then he submits a less aggressive bid if his type is low, but a more aggressive bid if his type is high. However, when $a > 0$, bidder 2 participates for fewer types the higher r is.*

Proof. See the Appendix. ■

Since bidder 2 participates less often but bids more when he does participate, the effect of an increase in a on the ex ante expected payment of bidder 2 is ambiguous.

However, when a is small the latter effect can be shown to dominate and a slight increase in a will increase the expected payment of bidder 2. In contrast, the ex ante expected payment from bidder 1 unambiguously declines.[15] Thus, if a head start is to increase expected revenue, it must be because it spurs the disadvantaged bidder to bid more aggressively, and this must outweigh the declining payment from the advantaged bidder. When the problem is phrased this way, it appears the seller faces a trade-off. In the next section we rephrase the problem and show that the seller benefits from introducing a small head start to bidder 1.

An increase in r leads bidder 1 to become active for more types. That is, he submits positive bids for more types. In other words, he will start bidding more aggressively if his type is close to v_1^c. Hence, contrary to the case of a head start, a bidder will not necessarily lower his bid or effort when he is favoured more. He may be enticed to increase his bid or effort.

3 Revenue maximization

Assuming the seller's objective is to maximize expected revenue (the expected sum of payments or bids), we now discuss the optimal choice of a and r.[16]

We emphasize the significance of the fact that the two instruments change the allocation in different ways. To highlight why this is important, it is useful to follow Myerson (1981) in calculating expected revenue. He defines

$$J_i(v) \equiv v - \frac{1 - F_i(v)}{f_i(v)}, \quad v \in [0, \overline{v}_i],$$

as bidder i's *virtual valuation*. We will assume that $J_i(v)$ is strictly increasing, and we define v_i^* as the unique value of v for which $J_i(v) = 0$, $i = 1, 2$.[17] Let $\kappa(v)$ denote the strictly increasing function satisfying $J_1(\kappa) = J_2(v)$. Since $f_1(0) > f_2(0)$ implies that $J_1(0) > J_2(0)$, it must be the case that $\kappa(v) = 0$ for some $v > 0$. Moreover, since $J_2(\overline{v}_2) = \overline{v}_2 > \overline{v}_1 = J_1(\overline{v}_1)$, $\kappa(v) = \overline{v}_1$ for some $v < \overline{v}_2$.

Myerson (1981) shows that revenue in any mechanism can be written as the expected value of the virtual valuation of the winner. To calculate this, we need

[15] However, in Section 4 it is observed that this result is not robust if more than two bidders participate in the auction.

[16] In this paper the seller can use no other instruments, such as minimum bids or caps. In many of the contests mentioned in the introduction, a winner must be found. In such situations, a minimum bid is not credible. Caps are briefly discussed in Section 4.

[17] Note that $v_i^* = \arg\max_v v(1 - F_i(v))$.

bidder i's winning probability, $q_i(v)$, $i = 1, 2$,

$$q_1(v|a,r) = \begin{cases} F_2(v_2^c) & \text{if } v \in [0, v_1^c) \\ F_2(k^{-1}(v)) & \text{otherwise} \end{cases},$$

and

$$q_2(v|a,r) = \begin{cases} 0 & \text{if } v \in [0, v_2^c) \\ F_1(k(v)) & \text{otherwise} \end{cases},$$

respectively. Recall that $k(v)$, v_1^c, and v_2^c depend on a and r. Since any bidder with valuation zero has zero payoff in the mechanisms considered here, expected revenue can now be written as

$$ER(a,r) = \int_0^{\overline{v}_1} J_1(v)q_1(v)f_1(v)dv + \int_0^{\overline{v}_2} J_2(v)q_2(v)f_2(v)dv. \tag{8}$$

As a point of comparison to the all-pay auction, consider the revenue maximizing mechanism (among mechanism where the good is sold with probability one). In an optimal mechanism, the seller would maximize the expected value of the virtual valuation of the winner. Consequently, bidder 1 should win the auction if his type exceeds κ when his rival has type v, since it would then be the case that $J_1 > J_2$. Otherwise he should lose. Such a rule maximizes the expected value of the virtual valuation of the winner by ensuring that the winner is the bidder with the highest virtual valuation.

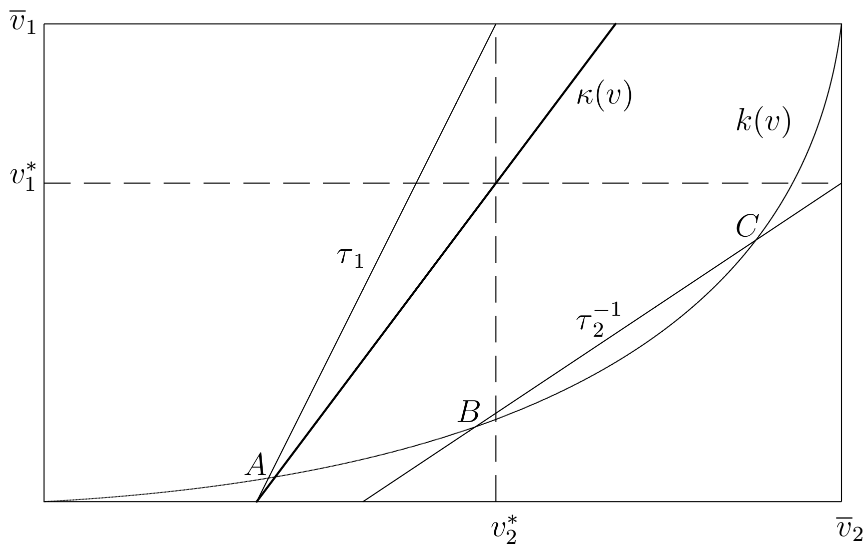

Figure 2: The optimal mechanism ($\kappa(v)$) versus the all-pay auction ($k(v)$).

Figure 2 compares an optimal mechanism an the standard all-pay auction. The standard all-pay auction is not optimal; $k(v)$ does not coincide with $\kappa(v)$. In particular, bidder 2 wins more often than is optimal "near the bottom" (when both bidders have low types), while bidder 1 wins more often than is optimal "near the top" (when both bidders have high types). Changing a and r can be seen as an exercise in manipulating the allocation to bring it as close to the optimal allocation as possible. As mentioned earlier, an increase in a corresponds to pushing the level curve to the north-east, while an increase in r is equivalent to pushing down $k(v)$.

For future reference, define $\tau_i(v)$ as the type that satisfies

$$\int_0^{\tau_i} J_i(x)\frac{f_i(x)}{F_i(\tau_i)}dx = J_j(v), \tag{9}$$

for $i, j = 1, 2$, $i \neq j$. Contingent on having a type below $\tau_1(v)$, bidder 1's expected virtual valuation is equal to bidder 2's virtual valuation when bidder 2 has type v. $\tau_1(0) = \kappa(0)$, but $\tau_1(v) > \kappa(v)$ whenever $v > 0$ and τ_1 is defined. Likewise, $\tau_2(v) > \kappa^{-1}(v)$ whenever both are defined. Finally, $\tau_i(v_j^*) = \overline{v}_i$ since the left hand side of (9) is zero when $\tau_i = \overline{v}_i$. Figure 2 illustrates κ, τ_1, and (the inverse of) τ_2.

We are now ready to examine the optimal choices of a, of r, and of a and r jointly.

3.1 Head starts

In this subsection, the handicap is assumed to be exogenous or fixed.

Compared to the optimal mechanism, one of the drawbacks of the all-pay auction is that bidder 2 wins too often near the bottom-left corner (when types are small). A head start to bidder 1 addresses this problem, as it leads bidder 1 to win when both bidders have low types. Hence, the allocation will move closer to what is optimal.

Theorem 1 *It is profitable to give bidder 1 a head start, regardless of r. The optimal head start to bidder 1 is such that $\tau_1(v_2^c) = k(v_2^c)$.*

Proof. Given (8), we observe that

$$\frac{\partial ER(a,r)}{\partial a} = f_2(v_2^c)F_1(v_1^c)\frac{\partial v_2^c}{\partial a}\left(\int_0^{v_1^c} J_1(v)\frac{f_1(v)}{F_1(v_1^c)}dv - J_2(v_2^c)\right), \tag{10}$$

the sign of which is determined by the term in parenthesis (v_2^c is increasing in a). As a approaches 0, v_1^c and v_2^c approaches 0 and this term converges to

$$-\frac{1}{f_1(0)} + \frac{1}{f_2(0)},$$

by L'Hôpital's rule. This is positive, by assumption. Hence, $ER(a, r)$ is strictly increasing in a when a is small. The first order condition is satisfied when the term in parenthesis is zero, which occurs if and only if $v_1^c = \tau_1(v_2^c)$. ■

Consider a marginal increase in bidder 1's head start. If the allocation changes, it is because bidder 2 won before, but now loses. Bidder 2's type in this event is v_2^c, while bidder 1 has a type below $k(v_2^c)$. Thus, the virtual valuation of the winner changes from $J_2(v_2^c)$ to the expected value of J_1, given bidder 1's type is below $k(v_2^c)$. The optimal head start ensures the marginal loss and gain are equated, which necessitates that $\tau_1(v_2^c) = k(v_2^c)$. In Figure 2, the intersection of τ_1 and k (the point A) thus determines v_2^c, which in turn allows the determination of a.

Although it is profitable to give a head start to the weak bidder, Theorem 1 does not claim that it is optimal or that a head start to the strong bidder is not profitable. If we are looking for the optimal head start to bidder 2, there are two points, B and C, in Figure 2 where the first order conditions are satisfied. Point B, which would require the smallest head start, is a local minimum, while point C is a local maximum. At first, expected revenue is decreasing in bidder 2's head start. Thus, if a head start to bidder 2 is profitable, it must be a large head start. In contrast, any small head start to bidder 1 is profitable. Arguably, the seller needs less information about the distribution functions to profit from a head start to the weak bidder compared to a head start to the strong bidder. See Section 3.3 for a related discussion.

The leading example in the literature on all-pay auctions (and all other auctions) is the uniform model, in which both bidders draw types from uniform distributions. C&R maintain this assumption throughout their paper. The following class of models encompasses the uniform model as a special case.

Definition 1 *F_1 is a scaled down version of F_2 if $F_i(v) = F(\frac{v}{\overline{v}_i})$, $v \in [0, \overline{v}_i]$, $i = 1, 2$, and $\overline{v}_1 < \overline{v}_2$, where F is some distribution function with support* $[0, 1]$.

If F_1 is a scaled down version of F_2, then it optimal to give the head start to the weak bidder, unless r is sufficiently greater than one. The first step in the proof of this property is to "quantify" $k(v)$. In comparison, define

$$\lambda(v) = \frac{\overline{v}_1}{\overline{v}_2} v, \ v \in [0, \overline{v}_2] \, .$$

Lemma 1 *If F_1 is a scaled down version of F_2, then*

$$r \lesseqgtr \frac{\overline{v}_2}{\overline{v}_1} \Longrightarrow k(v) \gtreqless \lambda(v) \text{ for all } v \in (0, \overline{v}_2) \, .$$

Proof. From (5),

$$k'(v) = r\frac{\overline{v}_1}{\overline{v}_2}\frac{k(v)f(\frac{v}{\overline{v}_2})}{vf(\frac{k(v)}{\overline{v}_1})}.$$

Since $k(\overline{v}_2) = \overline{v}_1$,

$$k'(\overline{v}_2) = \left(r\frac{\overline{v}_1}{\overline{v}_2}\right)\frac{\overline{v}_1}{\overline{v}_2} = \left(r\frac{\overline{v}_1}{\overline{v}_2}\right)\lambda'(\overline{v}_2). \tag{11}$$

Assume now that the term in parenthesis is less than one, in which case $k'(\overline{v}_2) < \lambda'(\overline{v}_2)$. Since $k(\overline{v}_2) = \lambda(\overline{v}_2)$, the former must therefore be above the latter immediately to the left of $\overline{v}_2$. Should $k(v)$ intersect $\lambda(v)$ as we move to the left it necessitates that the k is steeper than λ. However, if $k = \lambda$, then, by definition of λ,

$$k'(v) = r\frac{\overline{v}_1}{\overline{v}_2}\frac{\lambda(v)f(\frac{v}{\overline{v}_2})}{vf(\frac{\lambda(v)}{\overline{v}_1})} = \left(r\frac{\overline{v}_1}{\overline{v}_2}\right)\lambda'(v) < \lambda'(v),$$

which yields a contradiction. Thus, if $r < \frac{\overline{v}_2}{\overline{v}_1}$ then $k(v) > \lambda(v)$ for all $v \in (0, \overline{v}_2)$. The proof that $r > \frac{\overline{v}_2}{\overline{v}_1}$ implies $k(v) < \lambda(v)$ for all $v \in (0, \overline{v}_2)$ is analogous. ■

The second step is to "quantify" $\tau_2(v)$.

Lemma 2 *If F_1 is a scaled down version of F_2, then $\tau_2^{-1} < \lambda$ whenever τ_2^{-1} is defined.*[18]

Proof. Recall that τ_2 is an increasing function and that $\tau_2(v_1^*) = \overline{v}_2$. Hence, $\tau_2^{-1} \le v_1^*$. Likewise, κ is an increasing function and $\tau_2^{-1} < \kappa$ whenever both are defined. Moreover, since

$$J_i(v) = v - \frac{1 - F(\frac{v}{\overline{v}_i})}{\frac{1}{\overline{v}_i}f(\frac{v}{\overline{v}_i})} = \overline{v}_i\left(\frac{v}{\overline{v}_i} - \frac{1 - F(\frac{v}{\overline{v}_i})}{f(\frac{v}{\overline{v}_i})}\right)$$

and $\frac{\lambda}{\overline{v}_1} = \frac{v}{\overline{v}_2}$, it holds that $J_1(\lambda) = \frac{\overline{v}_1}{\overline{v}_2}J_2(v)$. As virtual valuations are strictly increasing and $J_1(\kappa) = J_2(v)$, $\kappa(v) = \lambda(v)$ if and only if $v = v_2^*$, in which case $\kappa(v_2^*) = v_1^*$. For any $v \in [0, v_2^*]$, $\kappa(v) < \lambda(v)$. Thus, for any $v \in [0, v_2^*]$ where τ_2^{-1} is defined, $\tau_2^{-1} < \kappa(v) < \lambda(v)$. For any $v \in (v_2^*, \overline{v}_2]$, $\tau_2^{-1} \le v_1^* < \lambda(v)$. Consequently, $\tau_2^{-1} < \lambda$. ■

Combining Lemma 1 and Lemma 2 produces the result.

Theorem 2 *Assume that F_1 is a scaled down version of F_2 and $r < \frac{\overline{v}_2}{\overline{v}_1}$. Then, any head start to the strong bidder lowers expected revenue.*

[18] In Figure 2, Lemma 2 implies that τ_2^{-1} is below the diagonal, λ.

Proof. By Lemma 1, $k(v) > \lambda(v)$ for all $v \in (0, \overline{v}_2)$, while Lemma 2 states that $\lambda(v) > \tau_2^{-1}(v)$, whenever the latter is defined. Thus, $k(v) > \tau_2^{-1}(v)$ (the two never intersect). Switching the roles of bidders 1 and 2 in (10) then proves that expected revenue is decreasing in bidder 2's head start. ■

3.2 Handicaps

In this subsection, the head start is assumed to be exogenous or fixed.

Bidder 1 wins too often when his type is high, but not often enough when his type is low in the standard all-pay auction compared to the optimal mechanism. Now, consider the effect of increasing r, i.e. handicapping bidder 2 further. $k(v)$ moves down, in the interior. Consequently, in the absence of a head start, bidder 1 wins more often, regardless of his type. This moves the allocation closer to the optimal allocation if his type is low, but farther away if his type is high. Hence, a trade-off exists, and it is not obvious that handicapping the strong bidder is optimal.

The trade-off disappears if the weak bidder has a very large head start. In this case, the weak bidder should also be handicapped.

Proposition 3 *It is optimal to handicap the weak bidder if his head start is sufficiently large.*

Proof. Assume that $a \geq \widehat{v}_2$, where $J_2(\widehat{v}_2) = \overline{v}_1$. Since $J_2(0) < 0$ and $J_2(\overline{v}_2) = \overline{v}_2 > \overline{v}_1$, such a $\widehat{v}_2$ exists. Then, regardless of r, bidder 2 will never participate if his type is below $\widehat{v}_2$. If he participates, his virtual valuation is therefore at least $\overline{v}_1$, which is higher than bidder 1's virtual valuation with probability one. Bidder 2 should be made to win more often, which occurs if the weak bidder is handicapped. If the handicap changes the allocation, it is because bidder 2, with virtual valuation above $\overline{v}_1$, now wins over bidder 1 with a virtual valuation below $\overline{v}_1$. Thus, the expected value of the virtual valuation increases. Figure 3 illustrates; the shaded area captures the combination of types for which the allocation changes in bidder 2's favor. ■

Assume now that $a = 0$. C&R have proven that it is optimal to handicap the strong bidder ($r > 1$) when types are drawn from different uniform distributions and head starts are not allowed.[19] In general, however, the trade-off makes it difficult to predict which bidder should be handicapped. In the following, we present two special models in which it is possible to determine how the optimal handicap should

[19] They argue that the reason is that handicapping the strong encourages the weak to bid more. While Clark and Riis (2000) only examine the aggregate revenue, it can in fact be shown that the seller earns more on the strong bidder (in expectation). However, the seller may or may not earn more on the weak bidder (depending on how heterogeneous the bidders are).

be implemented. Both models differ from the benchmark model studied thus far, since distributions are allowed to have mass points. The first class of models is inspired by Maskin and Riley's (2000) proof that a second price auction may be more profitable than a first price auction.

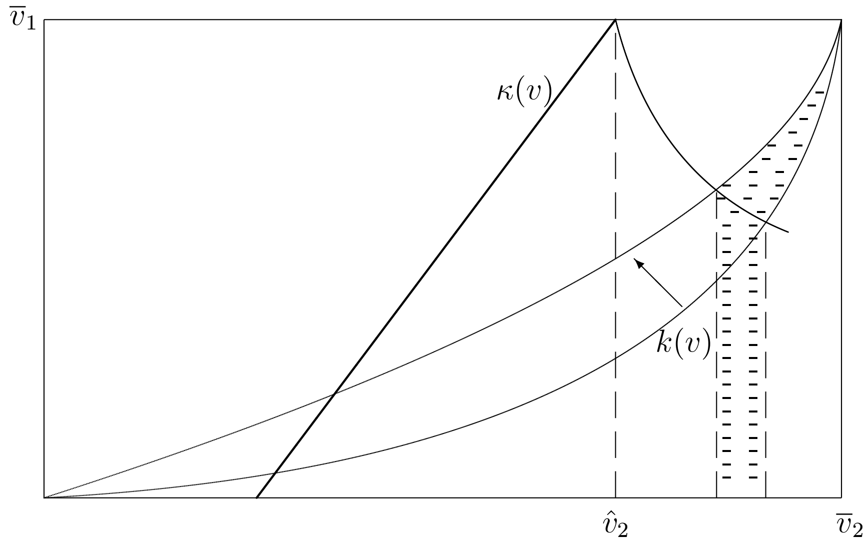

Figure 3: It is optimal to handicap the weak bidder if a is large.

3.2.1 Bidder 1 is potentially uninterested

Assume bidder 1 is *potentially uninterested* in the prize.

Definition 2 *Bidder 1 is potentially uninterested if,*

$$F_1(v) = 1 - \alpha + \alpha F_2(v),\ v \in [0, \overline{v}_2],$$

for some $\alpha \in (0,1)$.

Thus, bidder 1 is believed to be like bidder 2 with probability α, but to be uninterested in the prize with probability $1-\alpha > 0$. Although F_2 first order stochastically dominates F_1, $f_1(0) < f_2(0)$. It is readily checked that $J_1(v) = J_2(v)$, or $\kappa(v) = v$, when bidder 1 is potentially uninterested. This is the critical feature of the perturbed model.

The derivation of equilibrium in Section 2 remains valid, meaning that

$$\alpha \int_{k(v)}^{\overline{v}_2} \frac{f_2(x)}{x} dx = r \int_{v}^{\overline{v}_2} \frac{f_2(x)}{x} dx.$$

When $r = 1$, $k(v) < v$ for $v \in (0, \overline{v}_2)$. Thus, in the absence of head starts and handicaps, bidder 1 is more aggressive than bidder 2 for comparable types. However, this outcome is unequivocally negative in the current model, since it implies that $k(v) < v = \kappa(v)$, for all $v \in (0, \overline{v}_2)$. Hence, bidder 1 wins too often compared to what is optimal, regardless of his type. However, by setting $r = \alpha < 1$, we obtain $k(v) = v = \kappa(v)$. In other words, it is profitable to handicap the weak bidder. There is no trade-off by doing so.

Proposition 4 *Assume that $a = 0$. The seller profits from handicapping the weak bidder ($r < 1$) if he is potentially uninterested. The optimal value of r is $r^* = \alpha$.*

Proof. In the text. ■

In this model, the economic environment may appear to be more uneven when the weak bidder is handicapped, but the outcome actually becomes more "even", at least in the sense that the bidder with the highest valuation wins. That is, it is efficient to handicap the weak bidder.

The next example is an "approximation" of the model just discussed, but with no mass points.

Example 1: The strong bidder draws a type from the uniform distribution $F_2(v) = v$, with density $f_2(v) = 1$, $v \in [0, 1]$. In contrast, bidder 1's type is drawn from a distribution function with density

$$f_1(v) = \begin{cases} \frac{1}{2} + 100\left(\frac{1}{10} - v\right) & \text{if } v \in \left[0, \frac{1}{10}\right] \\ \frac{1}{2} & \text{if } v \in \left(\frac{1}{10}, 1\right] \end{cases}.$$

The optimal value of r is approximately 0.61 (when $a = 0$), meaning that it is the strong bidder who should be favoured.[20] □

3.2.2 Complete information

Next, assume that bidder i's type is $\overline{v}_i$ with probability one, $i = 1, 2$, with $\overline{v}_2 > \overline{v}_1 > 0$.[21] As long as bidder 2 is not handicapped too much (in particular, as long as

[20] The details of the example are omitted, but are available upon request. In this example, $J_1(v)$ is not monotonic.

[21] Konrad (2002) considers a model with exogenous head starts and handicaps in which the value of the prize is the same for both bidders. Siegel (2009) analyzes a very general model of contests that encompasses complete information, all-pay auctions with exogenous head starts and handicaps.

$rv_1 < v_2$), it is easily verified that the two bidders use mixed strategies, picking bids according to the distribution functions

$$\begin{aligned} P_1(b) &= \frac{\overline{v}_2 - r\overline{v}_1}{\overline{v}_2} + \frac{r}{\overline{v}_2} b,\ b \in [0, \overline{v}_1] \\ P_2(b) &= \frac{1}{r\overline{v}_1} b,\ b \in [0, r\overline{v}_1], \end{aligned}$$

for bidder 1 and 2, respectively. If r decreases both bid distributions become stochastically "weaker", meaning that bidders are more likely to submit low bids.[22] Thus, expected revenue unambiguously decreases if the weak bidder is handicapped. Instead, it is profitable to handicap the strong bidder.

Proposition 5 *Assume that $a = 0$. The seller profits from handicapping the strong bidder ($r > 1$) under complete information.*

Proof. Both bid distributions become stochastically "stronger" when r increases (for $rv_1 < v_2$). ■

3.3 Head starts and handicaps

Let a^* denote the optimal value of a when handicaps are ruled out ($r = 1$), and let r^* denote the optimal value of r when head starts are ruled out ($a = 0$). Let a^{**} and r^{**} denote the optimal values of a and r, respectively, when head starts and handicaps are determined jointly.

In Section 3.1 it was established that it is profitable to give the weak bidder a head start for any fixed handicap. However, when r is large, it may be even more profitable to give the strong bidder a head start. For instance, in the uniform model, it is optimal to give the strong bidder a head start if he is severely handicapped. We first show that when a and r are determined jointly, it is the weak bidder who will receive a head start in the uniform model.

Lemma 3 *Assume that $F_i(v) = \frac{v}{\overline{v}_i}$, $v \in [0, \overline{v}_i]$, $i = 1, 2$. Then, k intersects κ and τ_1 exactly once, regardless of r.*

Proof. When both distributions are uniform, it is easily verified that κ is linear. It ranges from 0 to $\overline{v}_1$ on a domain that does not include 0 or $\overline{v}_2$. k also ranges from 0

[22] In the benchmark model with incomplete information, a change in r does not lead to a stochastic deterioration or improvement in the bid distributions (Proposition 2). The comparative statics are sensitive to the assumptions regarding the information structure.

to $\overline{v}_1$ but on the larger domain $[0, \overline{v}_2]$. Thus, k and κ must intersect. Regarding the curvature of k, (5) implies

$$k''(v) = r\frac{k'(v)v - k(v)}{v^2}\frac{\overline{v}_1}{\overline{v}_2} = r\frac{r\frac{\overline{v}_1}{\overline{v}_2} - 1}{v^2}\frac{\overline{v}_1}{\overline{v}_2}k(v).$$

Thus, k is strictly concave when $r < \frac{\overline{v}_2}{\overline{v}_1}$, strictly convex when $r > \frac{\overline{v}_2}{\overline{v}_1}$, and linear when $r = \frac{\overline{v}_2}{\overline{v}_1}$. It follows that k and κ intersect only once. The proof that k intersects τ_1 exactly once is identical. ■

Proposition 6 *Assume that k and κ intersect exactly once, regardless of r. Then, $a^{**} > 0$.*

Proof. Theorem 1 implies that $a^{**} \neq 0$. It remains to show that $a^{**} \not< 0$. Following the argument in the proof of Theorem 1, any $a < 0$ that is a candidate for a maximum must produce a (v_2^c, v_1^c) pair at the intersection of τ_2^{-1} and k in Figure 2 (where k implicitly depends on r). We next show that a more profitable combination of head starts and handicaps exists. Since τ_2^{-1} is below κ, any intersection of τ_2^{-1} and k takes place in the region below κ. Since $k(0) = 0$ but $\kappa(v) = 0$ for some $v > 0$, it also follows that the unique (by assumption) intersection of k and κ must occur to the left of any intersection between τ_2^{-1} and k. In other words, k and κ do not insect to the right of the intersection between τ_2^{-1} and k, which means that k is below κ from this point on. In this region, below κ, expected revenue increases if bidder 2 wins more often. This can be achieved by lowering r (shifting k upwards) while at the same time adjusting a to keep v_1^c constant. The shaded area in Figure 4 captures the combinations of types for which the object is awarded to bidder 2 with the new mechanism but not the old mechanism. Since bidder 2's virtual valuation exceeds that of bidder 1 in this region, expected revenue has increased. ■

The advantage of handicapping the strong bidder is that it leads the weak bidder to win more often if his type is low. However, this could also be achieved by giving the weak bidder a head start. The latter option would not suffer the drawback that is associated with a handicap, namely that the weak bidder would win even more often if his type is high.

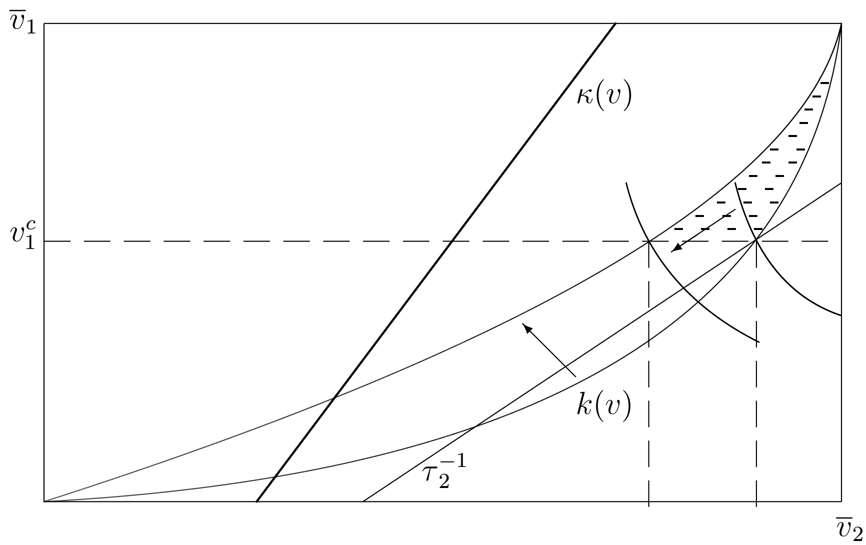

Figure 4: The weak bidder gets the head start, $a^{**} > 0$.

Once a head start is used to bring the allocation near the bottom closer to what is optimal, there is less of an incentive to handicap the strong bidder. Instead, it may be better to use the handicap as an instrument to bring the allocation closer to what is optimal near the top. To illustrate this most forcefully, consider once again the model studied by C&R. Recall that $a^* > 0$ (Theorem 2) and $r^* > 1$ (C&R). However, when a and r are chosen jointly to maximize revenue, we see that if the asymmetry is sufficiently large, the weak bidder is simultaneously given a head start ($a^{**} > 0$) and a handicap ($r^{**} < 1$).

Example 2: Assume that $F_i(v) = \frac{v}{\overline{v}_i}$, $v \in [0, \overline{v}_i]$, $i = 1, 2$, with $\overline{v}_2 > \overline{v}_1 = 1$. If $\overline{v}_2 = 2$ then $a^{**} < a^*$ and $r^{**} > 1$. However, if $\overline{v}_2 = 3$ then $a^{**} > a^* > 0$ but $r^{**} < 1$.[23]
□

Assume now that $a^* > 0$ (see Section 3.1). When a and r are determined jointly, Example 2 illustrates that the weak bidder is either given a large head start and a handicap, or the double advantage of a moderate head start and a handicapped opponent. This result holds for any combination of distributions that share the features of the uniform model identified in Lemma 3.

Proposition 7 *Assume that $a^* > 0$ and that k intersects κ and τ_1 exactly once, regardless of r. Then, either (i) $a^{**} > a^*$ and $r^{**} < 1$ or (ii) $0 < a^{**} \leq a^*$ and $r^{**} \geq 1$.*

[23] The details of the example are omitted, but are available upon request.

Proof. By Proposition 6, $a^{**} > 0$, which implies that (v_2^c, v_1^c) is determined by the intersection of k and τ_1. By assumption, this intersection is unique for any r. Hence, if $r < 1$ then k shifts up, and k must intersect τ_1 to the right of the intersection for $r = 1$. Consequently, if $r^{**} < 1$ then v_2^c increases, which necessitates that $a^{**} > a^*$. A similar argument proves that if $r^{**} \geq 1$ then v_2^c decreases, which necessitates that $a^{**} \leq a^*$. ■

To appreciate the advantages of handicapping the weak bidder while simultaneously giving him a head start, note that in the limit as $r \to 0$, the weak bidder is handicapped so much that he will not submit positive bids. Thus, he will score a. Then, from the strong bidder's point of view, a functions as a reserve price. The strong bidder would then win if his type is above a, and otherwise the weak bidder wins. If a is chosen judiciously, then this mechanism maximizes the payment that is obtainable from the strong bidder.[24] If he is very strong compared to the weak bidder, it is intuitive that it is worthwhile sacrificing revenue on the weak bidder (who pays nothing in the limiting case) to get more out of the strong bidder.

Example 3: Assume that $F_i(v) = \frac{v}{\overline{v}_i}$, $v \in [0, \overline{v}_i]$, $i = 1, 2$, with $\overline{v}_2 = 5$, $\overline{v}_1 = 1$. In this case, when a and r are chosen simultaneously, $a^{**} = 2.5$ and $r^{**} = 0$ (corner solution). Expected revenue is 1.25, all of it from the strong bidder, which exceeds what the weak bidder is willing to pay. □

Example 3 and the preceding intuition suggests that the weak bidder should be given a head start and a handicap when the asymmetry is sufficiently large or the strong bidder sufficiently strong. The next results formalizes this intuition in a special model where bidder 2 is very strong.

Proposition 8 *Assume that bidder 2's type is $\overline{v}_2 \geq \overline{v}_1$ with probability one. Then, $a^{**} = \overline{v}_2$, and $r^{**} = 0$.*

Proof. When $r^{**} = 0$ the weak bidder has no incentive to bid. Hence, he will score $a^{**} = \overline{v}_2$. It is then optimal for the strong bidder to bid $a^{**} = \overline{v}_2$ and win with probability one (given the tie-breaking rule). Social surplus is maximized, but bidders obtain zero payoff. Hence, there is no better mechanism from the seller's point of view. ■

As with Examples 2 and 3, comparing Propositions 4 and 8 produces another illustration of the potential consequences of allowing head starts and handicaps to be determined simultaneously.

[24] This occurs when $a = v_2^*$. In this case, the strong bidder wins if, and only if, his virtual valuation is non-negative.

4 Discussion

In the following we ask how head starts and handicaps impacts the efficiency of the auction. Then, we discuss the intuition in language familiar from the standard monopoly problem. Finally, we consider an alternative interpretation of the model, another way of manipulating the contest, and the consequences of allowing more bidders in the auction.

4.1 Efficiency

Thus far, the objective of the contest designer has been assumed to be revenue maximization. Depending on the context, many other objectives are possible. Here, we briefly consider the efficiency implications of head starts and handicaps.

When bidders are homogeneous, Lien (1990) and Feess et al (2008) prove that any handicap leads to a loss of efficiency. The reason is that the favoured bidder may win even when his valuation is the lowest. When bidders are heterogeneous, C&R prove that handicapping the strong bidder lowers the efficiency of the auction in the uniform model. Efficiency would be improved by handicapping the weak contestant. The intuition is that this makes it more likely that the winner is the bidder who is the most likely to have the highest valuation, namely the strong bidder.

To pursue revenue maximization, the objective is to get the allocation as close to κ as possible. To pursue efficiency, the objective is to get the allocation as close to the 45° line as possible. It can be shown that $k(v)$ generally crosses not only κ but also the 45° line (the strong bidder is more aggressive than the weak bidder when types are low). Thus, handicapping any bidder involves a trade-off in the sense that k moves closer to the 45° for some values of v but further away for others. In the C&R model, the trade-off could be quantified. However, Proposition 4 establishes that there is not necessarily a conflict between revenue maximization and efficiency when bidders are heterogeneous.

In addition, Proposition 8 proves that if the seller uses handicaps and head starts simultaneously, he may in fact cause social surplus to increase in his pursuit of higher revenue.

4.2 Monopoly pricing

Bulow and Roberts (1989) argued that the problem facing an auction designer with weak and strong bidders is similar to the problem facing a monopolist with a weak and a strong market and a random capacity. Recall that in the all-pay auction, the weak bidder wins relatively often if types are high, but less often when types are low.

This outcome roughly corresponds to the following policy by a monopolist: In the event capacity is low, sell at a discount on the weak market, but if capacity is large, sell at a discount on the strong market. Given the familiar textbook explanation of third degree price discrimination, this policy is suspect.

First, if capacity is large, it is inoptimal to give a discount to the strong market. It would be better to commit to not selling too much on the strong market, which can be achieved by dumping goods on the weak market. This is essentially what the head start to the weak bidder achieves in the context of the all-pay auction, since it rules out that bidder 2 with type below v_2^c wins.

Second, if capacity is low, it is actually not optimal to give a discount to the weak market, since this consists of consumers with low willingness-to-pay. Rather, the monopolist should sell exclusively on the strong market (remember that marginal revenue on the first unit coincides with the willingness-to-pay of the most eager consumer in the market; $J_2(\overline{v}_2) > J_1(\overline{v}_1)$ if $\overline{v}_2 > \overline{v}_1$). When the auctioneer handicaps the weak bidder, this is the direction in which he moves since it makes it less likely that the weak bidder wins if both have high types.

4.3 Head starts and participation fees

A head start to bidder 1 is isomorphic to bidder 2 having to pay a fixed fee, a, to be allowed to participate or bid.[25] In this interpretation, it is perhaps less surprising that the jointly optimal (a, r) pair may satisfy $a > 0$, $r < 1$. In particular, this would mean bidder 2 has to pay an up-front fee, but in exchange he is rewarded on the margin; additional payments are viewed more favorably.

Moreover, note that demanding a participation fee from bidder 2 eliminates the equilibrium selection problem that arises in the perturbed model when a head start is extended to bidder 1. If bidder 2 does not pay the fee, the prize is simply given to bidder 1 (even if he is uninterested).

4.4 Caps on bids

We have shown that head starts and handicaps affect the auction differently. A head start changes the allocation at the bottom, whereas a handicap's impact is global. There are other ways of manipulating an all-pay auction. For example, the possibility of imposing a cap on bids has been widely studied. This instruments changes the allocation in yet another way, since it is relevant only to bidders with high types.

[25] The bidders' scores would be $s_1 = rb_1$ and $s_2 = \max\{b_2 - a, 0\}$, where b_i is bidder i's expenditure, $i = 1, 2$. Bidder 2 wins if $b_2 > a + rb_1$, just as in the original model.

Che and Gale (1998) consider heterogenous contestants whose valuations are common knowledge (complete information). Gavious et al (2002) assume contestants are ex ante homogenous, but that valuations are private information and that costs are non-linear in the bid. In both models, caps may benefit the recipient of the effort. Sahuguet (2006) complements the two papers by extending the result to the case with ex ante heterogenous contestants, where valuations are private information. However, for tractability, it is assumed that there are exactly two contestants who draw valuations from different uniform distributions, as in the C&R model.

Maintaining the assumption of uniform distributions, it can easily be checked that imposing a cap on bids is *not* profitable in the perturbed model of Section 4 when α is high (the bidders are almost symmetric), although it may be profitable when α is low. Thus, starting from the symmetric uniform model, the profitability of a cap depends on how the asymmetry is modelled.

4.5 Many bidders

When there are two bidders, a bidder who is given a head start responds by lowering his bid. We next present an example designed to show that this prediction is not robust. That is, when there are many heterogeneous bidders it is possible that the bidder who is given a head start will in fact increase his bid. We also argue that head starts may be profitable for more reasons when there are several bidders.

To illustrate this outcome in an extreme case, the following example takes as its starting point the observation by Parreiras and Rubinchik (2006) that when there are many heterogeneous bidders some of them may never participate.

Consider the case where there are 3 bidders. The first bidder is "weak", and characterized by the distribution function $F_1(v)$, $v \in [0, \overline{v}_1]$. The remaining bidders are homogeneous, with types drawn from $F_2(v)$, $v \in [0, \overline{v}_2]$, with $\overline{v}_2 > \overline{v}_1$. For concreteness, assume $F_2(v) = v^2$, $v \in [0, 1]$. The important property is that $F_2(v)/v$ is increasing.

Intuitively, it is possible that the strong bidders compete so hard that it is not worthwhile for the weak bidder to enter the auction. If the weak bidder does not compete, the bidding strategy of the strong bidders (when $a = 0$, $r = 1$) is

$$b_2(v) = \frac{2}{3}v^3.$$

Obviously, this equilibrium hinges on the weak bidder having no incentive to enter. This condition is hardest to satisfy for type $\overline{v}_1$. If the weak bidder has type $\overline{v}_1$ his expected payoff from bidding $b_2(v)$, and thereby winning if both rivals have type below v, would be

$$\overline{v}_1 v^4 - \frac{2}{3}v^3 = v^3\left(\overline{v}_1 v - \frac{2}{3}\right).$$

We assume that $\overline{v}_1 < \frac{2}{3}$, in which case there is no incentive for the weak bidder to enter the auction, since he would earn negative payoff from doing so.

Now, if the weak bidder is given a *small* head start, he still has no incentive to submit positive bids. Nevertheless, expected revenue will increase. The reason is that the weak bidder scores a, meaning that a minimum bid is essentially imposed on the strong bidders. In this model, it is well known that minimum bids are profitable. The reason is that it excludes bidders with negative virtual valuation; $J_2(v) < 0$ when v is small, while bidder 1's expected virtual valuation is zero.

Consider next what would happen if the weak bidder is given a *large* head start. If he continues to bid zero, a still functions as a minimum bid. In this case, the strong bidders' bidding strategy is

$$b_2(v) = \frac{2}{3}v^3 + \frac{1}{3}a,$$

when $v \in \left[a^{\frac{1}{3}}, 1\right]$. To win the auction with probability one, the weak bidder with type $\overline{v}_1$ would have to bid $b_2(1)$, less a. The resulting payoff is

$$\overline{v}_1 - \frac{2}{3}(1-a),$$

while the payoff from bidding zero (scoring a) is $\overline{v}_1 a^{\frac{4}{3}}$, since he would win if both competitors have type below $a^{\frac{1}{3}}$. The former exceeds the latter if a is sufficiently high and $\overline{v}_1 > \frac{1}{2}$. Thus, if $\overline{v}_1 = .6$, for example, it is *not* an equilibrium for the weak bidder to bid zero. Athey (2001) establishes that an equilibrium exists in a large class of games encompassing the current game. Consequently, an equilibrium exists and it must involve positive bids by the weak bidder. In other words, in the current example, the weak bidder *increases* his bid when he is given a sufficiently large head start.

4.6 Favoritism by excluding rivals

Consider the previous example, but assume now that there are *many* weak bidders. In the benchmark auction, where the weak bidders stay out, expected revenue is $\frac{8}{15} \approx 0.533$. The maximal expected revenue that can be obtained from a mechanism where only the two strong bidders are active is approximately 0.585. However, if *all* the weak bidders are given very large head starts, say $a > 1$, the strong bidders will never find it profitable to enter the auction. The auction will therefore be a standard all-pay auction among a large set of weak bidders (their head starts cancel out). As the number of weak bidders grow, however, expected revenue converges to $\overline{v}_1$, which may very well exceed 0.585. In this example, the primary role of the head start is to *exclude* the strong bidders. Baye, Kovenock, and de Vries (1993) were first to prove that this may be profitable in an all-pay auction with complete information. Of course, excluding the strong bidders can be seen as a (somewhat extreme) way of favouring the weak bidders.

5 Conclusion

We considered contests or all-pay auctions with head starts and handicaps. It was pointed out that they affect the auction in different ways. Thus, one is not a substitute for the other, and it is generally profitable to use both instruments. In the benchmark model, the intuitive results holds that it is optimal to give the weak bidder a head start. However, it is not generally true that the seller profits from handicapping the strong bidder. The use of a handicap entails a trade-off, and we showed it can go both ways. Thus, it is possible that it is the weak bidder who should be handicapped. This is even more likely to be the case when head starts and handicaps can be used simultaneously. In this case, the weak bidder may be given a head start *and* a handicap. We also considered a perturbed model where there is no trade-off associated with using a handicap. In this model it is unambiguously the weak bidder who should be handicapped.

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Appendix: Proof of Proposition 2.

Preliminary step: Scores and Bids. To derive the score obtained by bidder 2 with valuation $v > v_2^c$, using the inverse function theorem on (3), for $j = 2$, yields

$$\frac{ds_2}{dv} = rk(v)f_2(v).$$

Since $s_2(v_2^c|a,r) = a$, it follows that

$$s_2(v|a,r) = a + \int_{v_2^c}^{v} rk(x)f_2(x)dx, \tag{12}$$

for $v \in [v_2^c, \overline{v}_2]$. Bidder 2's bid, $b_2(v|a,r)$, equals his score (recall the normalization $a_2 = 0$, $r_2 = 1$). If bidder 2's type is below v_2^c he stays out of the auction or scores zero.

As mentioned, in equilibrium bidder 1 with type v scores the same as bidder 2 with valuation $k^{-1}(v)$. Alternatively, we can derive $s_1(v)$ in the same manner $s_2(v)$ was derived,

$$s_1(v|a,r) = a + \int_{v_1^c}^{v} k^{-1}(x)f_1(x)dx, \tag{13}$$

for $v \in [v_1^c, \overline{v}_1]$. Given (1), bidder 1's bid is

$$b_1(v|a,r) = \begin{cases} 0 & \text{if } v \in [0, v_1^c] \\ \int_{v_1^c}^{v} \frac{1}{r}k^{-1}(x)f_1(x)dx & \text{otherwise} \end{cases}. \tag{14}$$

Proof of part 1: When a increases, v_2^c increases as well, meaning that bidder 2 stays out for more types. To see how his bid changes for the active types, note that

$$\frac{\partial b_2(v|a,r)}{\partial a} = \frac{\partial s_2(v|a,r)}{\partial a} = 1 - rv_1^c f_2(v_2^c)\frac{dv_2^c}{da}.$$

Moreover,

$$\begin{aligned} \frac{dv_2^c}{da} &= \frac{1}{F_1(v_1^c) + v_2^c f_1(v_1^c)k'(v_2^c)} \\ &= \frac{1}{F_1(v_1^c) + rv_1^c f_2(v_2^c)}, \end{aligned} \tag{15}$$

where the first equality follows from (7) and the second from (5). Hence,

$$\frac{\partial b_2(v|a,r)}{\partial a} = \frac{F_1(v_1^c)}{F_1(v_1^c) + r v_1^c f_2(v_2^c)} \geq 0 \tag{16}$$

for $v \in (v_2^c, \overline{v}_2]$. Thus, the active types responds by bidding *more aggressively*. Although the bid (or score) increases, it increases by less than a increases, as the derivative is strictly less than 1. Bidder 1 bids less aggressively, which follows directly from (14) when we recall that v_1^c increases and r and k^{-1} are unchanged.

Proof of part 2: If $a = 0$, note that implicit differentiation of (6) reveals that

$$\frac{dk}{dr} = -\frac{k}{f_1(k)} \int_v^{\overline{v}_2} \frac{f_2(x)}{x} dx = -\frac{k}{f_1(k)} \frac{1}{r} \int_k^{\overline{v}_1} \frac{f_1(x)}{x} dx,$$

where (6) was used to obtain the last equality. It follows that

$$\frac{drk}{dr} = k + r\frac{dk}{dr} = k\left[1 - \frac{1}{f_1(k)} \int_k^{\overline{v}_1} \frac{f_1(x)}{x} dx\right].$$

This is negative if k is small since the term in the bracket goes to $-\infty$ as $k \to 0$. Consequently, bidder 2's bid, $s_2(v|0,r)$, decreases in r for small types. The fact that bidder 1 bids more aggressively for small types can be proven in a similar manner. If $a > 0$, the result follows from (12) and (14) coupled with the fact that v_1^c is decreasing in r, while v_2^c is increasing in r.

Turning to high types, the highest score submitted by bidder 2 is

$$\overline{s}_2 = s_2(\overline{v}_2|a,r) = a + \int_{v_2^c}^{\overline{v}_2} r k(x) f_2(x) dx.$$

Since the bidders share the same maximal score, $\overline{s}_1 = \overline{s}_2$, we infer that bidder 1's maximal bid is

$$\overline{b}_1 = c_1(\overline{s}_2) = \frac{\overline{s}_2 - a}{r} = \int_{v_2^c}^{\overline{v}_2} k(x) f_2(x) dx.$$

This is decreasing in r since v_2^c is increasing in r and k is decreasing in r. Hence, bidder 1 bids less aggressively if his type is close to $\overline{v}_1$. In a similar manner we can calculate bidder 2's maximal bid, $\overline{b}_2$,

$$\overline{b}_2 = \overline{s}_1 = s_1(\overline{v}_1|a,r) = a + \int_{v_1^c}^{\overline{v}_1} k^{-1}(x) f_1(x) dx.$$

This increases in r since v_1^c decreases and k^{-1} increases. Thus, bidder 2 bids more aggressively if his type is high. ■