Game-theoretic Foundations of Competitive Search Equilibrium*

Manolis Galenianos†, Philipp Kircher‡

March 23, 2009

Abstract

We prove the existence of Nash equilibria in pure firm strategies for a finite directed search economy with heterogeneous firms, homogeneous workers and a general production and matching technology. We exploit the existence result to provide a new characterization of the equilibrium set for the finite economy. We also show that, as the number of agents increases, the equilibria of the finite economy converge to the equilibria of competitive search economies with a continuum of agents; therefore, competitive search models have solid micro-foundations.

*This work supersedes the related results that we distributed in the working paper "Heterogeneous Firms in a Finite Directed Search Economy". We thank Daron Acemoğlu, Ken Burdett, Jan Eeckhout, Mike Peters, Philip Reny, Shouyong Shi, Neil Wallace and Randy Wright for helpful comments. Kircher thanks the National Science Foundation for financial support (grant SES-0752076).

†Pennsylvania State University. E-mail: manolis@psu.edu.

‡University of Pennsylvania. Corresponding author. E-mail: kircher@econ.upenn.edu.

1 Introduction

Models of *competitive search* combine frictions, which are seen as an important feature of labor markets, with a significant role for pricing, which is mostly absent in models of random search. This is achieved by postulating a positive relationship between the speed of filling a vacancy and the payoff that a firm offers to workers.[1] The positive relationship between hiring and wages is based on the idea that a firm can attract more applicants by advertising more desirable payoffs for its prospective employees. However, this type of information diffusion runs counter to the most common interpretation of search frictions: workers lack information about the location and/or pay of job opportunities.

Models of *directed search* provide an alternative interpretation of frictions which is consistent with common knowledge of firms' locations and wage offers: it is inherently difficult for workers to coordinate their job search. This idea is formalized as a game where every firm announces the payoffs that it offers and each worker sends one job application after observing all announcements.[2] Lack of coordination is captured by restricting attention to equilibria where workers follow symmetric strategies. In such equilibria some firms receive too many workers (i.e. there are more applicants than available vacancies) while others too few: workers' lack of coordination provides micro-foundations for the matching function.

To apply these insights to large (continuum) economies used in competitive search, the standard approach is to derive the equilibrium of a finite economy, where strategies and off-equilibrium payoffs are well-defined, and examine the limit as the number of agents grows. Such analysis has so far been performed in relatively simple environments with risk-neutral agents, fixed productivity on the job and no informational or incentive problems beyond matching frictions (see Peters (1991) and Burdett, Shi and Wright (2001) for the case of homogeneous firms; see Peters (2000) for the case of heterogeneous firms). Additionally, existence has been proved in mixed firm strategies, except for the special case of identical firms. As we discuss below, mixed firm strategies present some difficulties in further analyzing these games.

We consider a finite directed search economy with heterogeneous firms and general matching and production technologies that encompasses a variety of preference and informational structures including most of the environments to which competitive search has been applied. Our main result is to prove that equilibria in pure firm strategies exist if the production function satisfies a simple condition (essentially, concavity and regularity). As in the earlier literature, we assume that workers are homogeneous and frictions arise from lack of coordination.

[1] One prominent implication of this market structure is that constrained efficiency obtains because firms can "price" their hiring rates (see Moen (1997), Shi (2001), Shimer (2005)).

[2] Multiple applications are considered in Albrecht, Gautier and Vroman (2006), Galenianos and Kircher (2009) and Kircher (2008). In models of finite economies, multiple applications lead to severe technical complications as shown in Albrecht, Gautier, Tan and Vroman (2005). See Julien, Kennes and King (2000) and Camera and Selcuk (2008) for models where prices are (potentially) renegotiated after matching.

The existence of equilibrium in pure firm strategies is are result that is both non-trivial and useful in a number of ways. The technical difficulty arises from the strategic interaction among the agents in a finite environment. Specifically, the action of a single firm affects the payoffs of all market participants, which means that we need to keep track of the full distribution of announcements when deriving the equilibrium conditions. The difficulties of establishing existence in pure firm strategies are discussed in detail in Peters (1997). Furthermore, it is not a priori obvious that such equilibria exist. For instance, Acemoğlu and Ozdaglar (2006) show that equilibria in pure strategies may not exist in a related environment where pricing and congestion interact non-trivially.[3]

Our existence result yields two immediate additional benefits that enable a deeper analysis of the interaction of competitive price setting and matching frictions in finite economies. First, we currently lack characterization results for finite economies.[4] Our existence result means that we need only evaluate a firm's strategy against its competitor's *pure* strategies which significantly reduces the complexity of characterizing equilibria. We prove that, under an additional assumption on the production function, there is positive relationship between the productivity of a firm and compensation it offers to its workers. Building on this result we also show that the pure strategy equilibrium is unique when firms are homogeneous. Proving these results is still non-trivial due to the finite nature of the market, as we discuss in detail in Section 4, but pure strategies mean that such results are feasible. An additional application of our existence result can be found in Galenianos, Kircher and Virag (2009) where it is shown that constrained efficiency does not obtain in finite economies, unlike in the continuum ones, at least for certain production specifications. We expect additional comparative statics and characterization results to be within reach.

Second, we show that, as the number of agents grows, the finite economy directed search equilibria converge to the competitive search equilibria. This is particularly relevant because the competitive search framework has been applied to increasingly complicated environments with match-specific private information (Guerrieri (2008)), risk-averse workers (Acemoğlu and Shimer (1999)), endogenous choice of the intensive margin (hours) of work (Faig and Jerez (2004), Rocheteau and Wright (2005), Berentsen, Menzio and Wright (2008)) and moral hazard (Moen and Rozen (2007)). Our setup encompasses those environments and the limit result shows that these environment have solid micro-foundations. The existence of equilibrium in pure strategies allows us to extend to the entire game the analysis that Peters (1997) performs for the subgame among workers, thus side-stepping the much more involved construction of convergence in mixed strategies of Peters (2000).

[3] In their model, prices and congestion interact additively while in directed search the congestion (probability of trade) interacts with the price multiplicatively. Existence obtains in our setting for a large class of functional forms for the trading probability.

[4] An exception is Burdett, Shi and Wright (2001) for the case where firms and workers are homogeneous.

2 The General Environment and Some Applications

We start with a brief description of our environment and show how various models from the literature can be mapped into our setting. A more detailed description of outcomes and payoffs is given in Section 3.

2.1 The Environment

The economy is populated with a finite number of firms and workers, denoted by $M = \{1, ..., m\}$ and $N = \{1, ..., n\}$ respectively, where $m \geq 2$ and $n \geq 2$. For production to take place, a firm needs to hire a worker. All workers are ex ante identical and each of the (potentially heterogeneous) firms can hire at most one worker. The game starts with the hiring process. Then production takes place and payoffs are realized. The split of the surplus between worker and firm is determined during the hiring process according to the posting game described below. The payoff of being unmatched is normalized to zero for both firms and workers. Firms maximize their expected profits and workers maximize their expected utility.

The hiring process has three stages. First, each firm simultaneously makes a public announcement: It commits to the utility v that it will provide to the worker that it hires.[5] Second, workers observe the announcements of all firms and each worker simultaneously applies to one firm. Last, a firm that receives one or more applications hires one of its applicants at random and the other applicants remain unmatched; a firm without applications remains idle. We follow the directed search literature in focusing our attention on subgame perfect equilibria where workers follow symmetric strategies. Such equilibria are intended to capture the frictions of labor markets.

The surplus generated when firm j fills its vacancy and pays v to its worker is denoted by $S_j(v)$.[6] The firm's ex-post profits (i.e. conditional on a hire) are denoted by $\pi_j(v)$ so that $S_j(v) = \pi_j(v) + v$. We impose the following restrictions on the firms' profit functions:[7]

Assumption 1 *We consider environments where for all $j \in M$:*

i. $\pi_j(v)$ *weakly concave,*

ii. $\pi_j(v)$ *is twice continuously differentiable,*

iii. there are unique $\overline{v}_j$ *and* $\underline{v}_j$ *such that* $\pi_j(\overline{v}_j) = 0$ *and* $\pi_j(\underline{v}_j) = \max_{v \geq 0} \pi_j(v)$.[8]

[5] In some environments, the worker's payoff within a match is stochastic. In that case, v represents the worker's expected utility conditional on getting the job. See Section 2.2 for illustrations.

[6] The surplus may depend non-trivially on the worker's payoff when, for instance, the worker has to exert costly effort. See Section 2.2.

[7] One can rewrite these conditions in terms of $S_j(\cdot)$. It turns out to be more convenient to work with $\pi_j(\cdot)$.

[8] Workers' individual rationality means that $v_j \geq 0$ is a necessary condition for a hire to occur.

The Pareto frontier between a worker and a firm is linear (strictly concave) when $\pi_j(v)$ is linear (strictly concave). In the case of strict concavity, utility is imperfectly transferable between workers and firms.

2.2 Examples

This section shows how to map a number of production environments that have been analyzed in the competitive search literature into our framework and demonstrates that they satisfy Assumption 1. These environments differ with respect to workers' preferences, the production technology and the informational structure within a match.

Example 1: The canonical example in the literature is the case where workers are risk-neutral and production at firm j is deterministic at x_j. In this environment each firm posts a wage w, the value to the worker who obtains this wage is $v = w$, the surplus is a fixed number $S_j(v) = x_j$, and profits are given by $\pi_j(v) = x_j - v$. This model is examined in Burdett, Shi and Wright (2001), Moen (1997), Montgomery (1991) and Peters (2000).

Example 2: Workers are risk averse, there is no insurance and production is deterministic. Firms post wages and cannot insure workers against unemployment. Letting $\vartheta(w)$ denote the workers' utility function we have $S_j(v) = x_j - w + \vartheta(w)$, $v = \vartheta(w)$ and $\pi_j(v) = x_j - w$. Therefore we can rewrite $\pi_j(v) = x_j - \vartheta^{-1}(v)$ and note that $\vartheta^{-1}(\cdot)$ is convex due to risk aversion. Together with the requirement that $x_j > \vartheta^{-1}(0)$, this environment satisfied Assumption 1. A competitive search model with this structure is analyzed in Acemoğlu and Shimer (1999).

Example 3: Workers are risk neutral and they have private information concerning their match-specific disutility of work. Firms post wages. After a worker and a firm match, the worker draws his disutility of work ϕ from some distribution Φ and privately observes it. When the wage is w and the disutility is ϕ, the worker's net utility is $w - \phi$. Individual rationality means that the worker will refuse the job if $\phi > w$. Therefore, $S_j(v) = \int_{\phi \leq w_j} [x_j - \phi] d\Phi(\phi)$. The worker's ex ante utility is $v = \int_{\phi \leq w_j} [w_j - \phi] d\Phi(\phi)$ which implicitly defines a unique function $w_j(v)$ under the standard monotone hazard rate condition for Φ. Profits are given by $\pi_j(v) = \int_{\phi \leq w_j(v)} [x_j - w_j(v)] d\Phi(\phi)$. It is not hard to show that this function is concave in v under the monotone hazard rate condition.[9] Guerrieri (2008) analyzes competitive search in such an environment.

Example 4: Production is linear and disutility of work is convex in hours of work, each firm post an hourly wage w and each hired worker decides how many hours to work. The surplus

[9] Profit $\pi_j(v)$ is concave if $w(v)$ is convex, which is equivalent with v being concave in w. Since $v'(w) = -\Phi(w)$ we have $v''(w) = -\Phi'(w) \leq 0$, since the density $\Phi'(w)$ of Φ is positive.

is given by $S_j(v) = x_j t - k(t)$ where t is the time spent working and $k(t)$ is a strictly convex function representing the disutility of work. The worker's net utility is given by $v = wt - k(t)$ which leads to $w(v) = [v + k(t)]/t$. Optimal time allocation requires that $w = k'(t)$ which implicitly defines $t(w)$, or $t(v)$. The firm's profits are $\pi_j(v) = x_j t - wt = x_j t(v) - v - k(t(v))$. A sufficient condition for this profit function to be concave is $k'''(t) \geq 0$.[10] This environment is very similar to the product market model of Rocheteau and Wright (2005) and Berentsen, Menzio and Wright (2008).[11]

Example 5: Production is linear and disutility of work is convex in hours of work; firms post a (possibly non-linear) wage schedule $w(t)$ that determines payments as a function of hours; workers decide how many hours to work after privately observing a disutility shock ϕ drawn from some distribution Φ that satisfies monotone hazard rate. The worker's net utility after observing ϕ is $w(t) - \phi k(t)$ where $k(t)$ is strictly convex. Given the realization of ϕ the worker chooses $t(\phi)$ that maximizes $\max_t w(t) - \phi k(t)$ and his expected utility before observing ϕ is $v = \int [w(t(\phi)) - \phi k(t(\phi))] d\Phi(\phi)$. Profits are given by $\pi_j(v) = \int [x_j t(\phi) - w(t(\phi))] d\Phi(\phi)$ and the surplus function is $S_j(v) = \int [x_j t(\phi) - \phi k(t(\phi))] d\Phi(\phi)$. For each v left to the worker there is a contract that yields the highest profit $\pi(v)$ to the firm. $\pi(v)$ is concave if $k'''(t) \geq 0$. Faig and Jerez (2004) examine this environment in a product market setting where a worker is a buyer and ϕ corresponds to his marginal valuation for the seller's (in our setting, firm's) good.

Example 6: The firm does not observe the worker's effort t (moral hazard), output within the match y is stochastic and the firm posts an output-contingent wage schedule $w(y)$. The output is given by $y = xt + \phi$ where ϕ is a shock which is only observable to the worker and is drawn from some distribution Φ with an increasing hazard rate. The worker observes ϕ and chooses $t(\phi)$ to maximize his net utility $w(y) - k(t)$, where $k(t)$ is a convex cost of effort. His expected utility from a schedule $w(y)$ is $v = \int [w(xt(\phi) + \phi) - k(t(\phi))] d\Phi(\phi)$. Again, for each v left to the worker there is a contract that yields the highest profit $\pi(v)$ to the firm. Also, $k'''(t) > 0$ is a sufficient condition for $\pi(\cdot)$ to be concave. Moen and Rozen (2007) analyze this setting in a competitive search framework.

---

[10]Since $v - k(t(v))t(v) + k(t(v)) = 0$ defines $t(v)$, we have $t'(v) = [k''(t(v))t(v)]^{-1} \geq 0$ and $t''(v) = -[k''(t(v))t(v)]^{-3}[k'''(t(v))t(v) + k''(t(v))] \leq 0$. Then $\pi_j''(v) = [x - k'(t(v))]t''(v) - k''(t(v))t'(v)t(v)$, which is negative when $x - k'(t(v)) \geq 0$. This is the case everywhere on $[\underline{v}_j, \overline{v}_j]$. To see this, note that $k'(t(v))$ is equal to the wage that implements this utility, but only for $x_j - w \geq 0$ the firm makes weakly positive profits, which defined the range of possible offers $[\underline{v}_j, \overline{v}_j]$.

[11]Their product market considers buyers (in our framework the workers) and sellers (in our framework the firms) where the former can buy a continuous quantity q $(= \bar{t} - t)$ of a good at a unit price p $(= 1 - w)$. Our framework does not address the holding cost of money which is a feature in these papers.

3 Analysis of the Finite Model

In this section we state our main result regarding the existence of an equilibrium in pure strategies by the firms. The following subsections are devoted to the proof.

Theorem 1 *A directed search equilibrium exists when Assumption 1 holds.*

We start by providing a formal definition of equilibrium and then prove the theorem in three parts. First, we examine the subgame that follows an arbitrary announcement by the firms and show that the workers' probability of applying to some firm j is quasi-concave in that firm's announcement. Then, we show that a firm's expected profits are quasi-concave in its announcement, v_j. Finally, we prove existence by using a fixed point argument which is extended to deal with the discontinuity in profits that arises in this type of model.

3.1 Formal Definitions

We now formally present the agents' strategies and the equilibrium concept. We examine in turn the matching structure, the subgame that follows an arbitrary announcement by the firms and the first stage strategies of firms.

The strategy of worker i specifies the probability with which he applies to each firm after observing some announcement $\mathbf{v} = (v_1, v_2, ..., v_m)$. Under Assumption 1, no firm has an incentive to make an offer lower than $\underline{v}_j$ or higher than $\overline{v}_j$ and therefore the space of announcements is $\mathcal{V} \equiv \times_{j=1}^{m} [\underline{v}_j, \overline{v}_j]$. Let $p_j^i(\mathbf{v})$ denote the probability that worker i applies to firm j after observing $\mathbf{v}$. We focus our attention on equilibria where workers follow symmetric strategies: $p_j^i(\mathbf{v}) = p_j^l(\mathbf{v}) = p_j(\mathbf{v})$, for all $i,\ l \in N$. We denote the strategy of workers by the vector $\mathbf{p}(\mathbf{v}) = (p_1(\mathbf{v}), ..., p_m(\mathbf{v}))$. When there is no possibility for confusion, we suppress the argument $\mathbf{v}$ to keep notation simple.

We now specify the mapping from the application strategies to the probabilities of filling a vacancy (for firms) and finding a job (for workers). The probability that a firm fills its vacancy when workers apply there with probability p is denoted by $H(p)$. The probability that a worker is hired by the firm where he applied when the other workers apply there with probability p is denoted by $G(p)$. We allow for general functional forms for $H(p)$ and $G(p)$ that encompass several specifications.[12] This flexible setup This highlights that our results are not specific to the

[12]Most directed search models (e.g. Peters (1991) or Burdett, Shi and Wright (2001)) consider urn-ball matching: the number of applications received by a firm is distributed according to a Binomial distribution with parameters n and p, where p is the probability that an individual worker applies there. This specification leads to $H(p) = 1 - (1-p)^n$ and trade in pairs implies $G(p) = H(p)/(np)$. Changes in the physical environment give rise to different matching functions. For example, one can consider the case that an application is lost with some probability ρ and, as a result, $H(p) = 1 - (1 - (1-\rho)p)^n$.

exact form of the matching frictions. Additionally, it allows for for different matching functions in the limit large game (see Section 5). The next assumption summarizes the structure of the matching function.

Assumption 2 *$H(p)$ and $G(p)$ satisfy the following conditions for $p \in [0,1]$:*

i. *$H(p)$ is twice continuously differentiable, strictly increasing, concave and $H(p) \in [0,1]$.*

ii. *$G(p)$ is twice continuously differentiable, strictly decreasing, convex and $G(p) \in [0,1]$.*

iii. *$\frac{1}{G(p)}$ is convex.*

iv. *$H(p) = npG(p)$.*

Furthermore, define $h(p) \equiv H'(p)$ and $g(p) \equiv G'(p)$. Parts *i* and *ii* ensure that $H(p)$ and $G(p)$ are probabilities and that they behave nicely.[13] Part *iii* implies that the probability that a firm fills its vacancy is concave in the probability with which a worker becomes employed in that firm. This assumption is commonly used in models of directed search (see e.g. Menzio (2007)). Part *iv* is a consistency condition (in expectation terms): it states that the probability that a firm fills its vacancy is equal to the probability that a worker is hired by that firm times the average number of workers. It also implies that a firm without applicants cannot hire: $H(0) = 0$. Most matching functions that are used in the search literature satisfy Assumption 2, including the urn-ball ($H(p) = 1-(1-p)^n$), telephone line ($H(p) = np/(np+1)$) and CES matching function ($H(p) = [(np)^{-\sigma}+1]^{-1/\sigma}$ with $\sigma \in (0,1)$).

A worker's expected utility from applying to firm j is given by $G(p_j)v_j$. Utility maximization leads to the following definition of the equilibrium in a subgame.

Definition 1 (Symmetric Subgame Equilibrium) *A symmetric equilibrium in the subgame that follows announcements $\mathbf{v}$ is a vector $\mathbf{p}(\mathbf{v}) = (p_1(\mathbf{v}), ..., p_n(\mathbf{v}))$ such that $\sum_j p_j(\mathbf{v}) = 1$ and for all $j \in M$*

$$p_j(\mathbf{v}) > 0 \Rightarrow G(p_j(\mathbf{v}))\ v_j = \max_{k \in M} G(p_k(\mathbf{v}))\ v_k. \quad (1)$$

In words, for a worker to apply to firm j ($p_j > 0$), he needs to receive a level of expected utility that is at least as high as what he can get at any other firm.

Each announcement $\mathbf{v}$ leads to a unique vector of application strategies if at least one firm offers strictly positive utility. That is, the subgame equilibrium $\mathbf{p}(\mathbf{v})$ is *unique* given any $\mathbf{v}$ with $v_j > 0$ for some $j \in M$ (Peters (1984), Proposition 1).[14] When $\mathbf{v} = \mathbf{0}$ the workers' strategy is

[13] Some readers may prefer to interpret p as search intensity rather than the probability of applying to a firm since we allow for $H(p) > 1-(1-p)^n$.

[14] Peters (1984) proves this result for urn-ball matching but his proof can be extended in a straightforward way to our setting.

arbitrary. From now on we assume that $p_j(\mathbf{0}) = 1/m$ for all $j \in M$ but our results hold for any specification of $\mathbf{p}(\mathbf{0})$. We define *market utility* to be the expected utility that workers obtain in the subgame and denote it by $U(\mathbf{v})$.

We say that firm j is *active* when $p_j > 0$ and it is *inactive* when $p_j = 0$. In the former case the probability that the firm hires a worker is strictly positive; in the latter case it is zero. Let $A(\mathbf{v}) \equiv \{j \in M | p_j(\mathbf{v}) > 0\}$ denote the set of active firms for a given $\mathbf{v}$ and note that it is non-empty. The set of inactive firms is denoted by $A^C(\mathbf{v})$. Following announcement $\mathbf{v}$ we can without loss of generality reshuffle the firms' index so that $A(\mathbf{v}) = \{1, ..., l\}$ and $A^C(\mathbf{v}) = \{l+1, ..., m\}$ if $l < m$, or $A^C(\mathbf{v}) = \emptyset$ if $l = m$.

We now turn to the firms' problem in the first stage of the hiring process. Firm j takes as given the announcements of the other firms, $\mathbf{v}_{-j}$, and the response of workers in the subgame, $\mathbf{p}(\mathbf{v})$. The expected profits of firm j are denoted by

$$\Pi_j(\mathbf{v}) \equiv H(p_j(\mathbf{v}))\ \pi_j(v_j), \tag{2}$$

where $p_j(\mathbf{v})$ solves (1). Profits are uniquely determined given $\mathbf{v}$ since each announcement leads to a unique set of application probabilities in the subgame.

We are ready to define the equilibrium of this game. A directed search equilibrium is a pure strategy Nash equilibrium in the game among firms with payoffs $\Pi_j(\mathbf{v})$. Formally:

Definition 2 (Directed Search Equilibrium) *A directed search equilibrium is a vector of announcements $\mathbf{v} \in \mathcal{V}$ such that $\Pi_j(\mathbf{v}) \geq \Pi_j(v_j', \mathbf{v}_{-j})$ for all $v_j' \in [\underline{v}_j, \overline{v}_j]$ and all $j \in M$ where the workers' strategies are given by the symmetric subgame equilibrium.*

3.2 Analysis of the Subgame

In this section we characterize the workers' response to an arbitrary announcement by the firms, $\mathbf{v}$, and we determine how that response changes when some v_j changes.

Characterization of Subgame: We characterize $\mathbf{p}(\mathbf{v})$ in two steps. First, we determine the set of active firms. Then we determine the exact probabilities with which workers visit the active firms.

Recalling that $U(\mathbf{v}) = \max_j G(p_j(\mathbf{v}))v_j$, we rewrite equation (1) as

$$\begin{aligned} G(p_j(\mathbf{v}))v_j &= U(\mathbf{v}), \quad \forall j \in A(\mathbf{v}), \\ G(p_j(\mathbf{v}))v_j &\leq U(\mathbf{v}), \quad \forall j \in A^C(\mathbf{v}). \end{aligned}$$

To determine whether firm j is active or inactive, compare v_j with $U(\mathbf{v})$. If $v_j > U(v_j, \mathbf{v}_{-j})$, then $p_j > 0$. Equivalently, $v_j < U(v_j, \mathbf{v}_{-j})$ implies that $p_j = 0$. Last, if the announcement of some firm j is exactly on the boundary ($v_j = U(v_j, \mathbf{v}_{-j})$) then that firm is inactive ($p_j = 0$): if it were active then $G(p_j) < 1$ leading to $G(p_j)\ v_j < U(\mathbf{v})$ which contradicts subgame equilibrium.[15]

To summarize these results, note that the market utility only depends on active firms: if $p_j = 0$ then $U(v_j, \mathbf{v}_{-j}) = U(0, \mathbf{v}_{-j})$. The following condition determines whether a firm is (in)active:

$$j \in A^C(\mathbf{v}) \Leftrightarrow v_j \leq \hat{v}_j(\mathbf{v}_{-j}) \equiv U(0, \mathbf{v}_{-j}). \tag{3}$$

We now focus on the active firms. In equilibrium, the exact probability with which a worker applies to each of the firms in $A(\mathbf{v})$ is determined by the requirement that he is indifferent across them:

$$G(p_k)\ v_k - G(p_l)\ v_l = 0,\ \forall\ k \in A(\mathbf{v})/\{l\}, \tag{4}$$

$$\sum_{k \in A(\mathbf{v})} p_k - 1 = 0. \tag{5}$$

Equations (4) and (5) define a system $\mathbf{F}$ of l equations with l exogenous and l endogenous variables. The announcements $\hat{\mathbf{v}} \equiv (v_1, ..., v_l)$ of the active firms are the exogenous variables and the probabilities $\hat{\mathbf{p}} \equiv (p_1, ..., p_l)$ are the endogenous variables.

Equations (3), (4) and (5) fully describe the equilibrium of the subgame. As noted in Section 3.1, $\mathbf{p}(\mathbf{v})$ is uniquely defined when $v_j > 0$ for some $j \in M$ and we assume that $p_j(\mathbf{0}) = 1/m$.

Workers' reaction to a change in a firm's announcement: We now examine how the equilibrium of the subgame changes in response to a perturbation in some v_j. Let $\mathbf{v}$ denote the initial announcement and suppose that $v_k > 0$ for some $k \in M$. The case of $\mathbf{v} = \mathbf{0}$ is treated separately below. We will use the implicit function theorem on equations (4) and (5) but we first need to determine whether the set of active firms, $A(\mathbf{v})$, changes.

Take some v_j' "near" v_j and compare $A(v_j', \mathbf{v}_{-j})$ to $A(v_j, \mathbf{v}_{-j})$. Note that $U(\mathbf{v})$ is continuous

[15] In other words, the correspondence $A(\mathbf{v})$ is lower hemi-continuous in $\mathbf{v}$.

in its argument and $U(\mathbf{v}') > U(\mathbf{v}) \Leftrightarrow v_j' > v_j$. Given any $k \neq j$: [16]

$$
\begin{aligned}
v_k > U(\mathbf{v}) &\Rightarrow k \in A(\mathbf{v}) \text{ and } k \in A(v_j', \mathbf{v}_{-j}) \\
v_k < U(\mathbf{v}) &\Rightarrow k \in A^C(\mathbf{v}) \text{ and } k \in A^C(v_j', \mathbf{v}_{-j}) \\
v_k = U(\mathbf{v}) \text{ and } v_j' > v_j &\Rightarrow k \in A^C(\mathbf{v}) \text{ and } k \in A^C(v_j', \mathbf{v}_{-j}) \\
v_k = U(\mathbf{v}) \text{ and } v_j' < v_j &\Rightarrow k \in A^C(\mathbf{v}) \text{ and } k \in A(v_j', \mathbf{v}_{-j})
\end{aligned}
$$

Essentially, $A(\mathbf{v})$ is constant in v_j unless some firm is exactly on the boundary for being active. For a given $\mathbf{v}_{-j}$, this argument implies that there are at most m critical points for $v_j \in [\underline{v}_j, \overline{v}_j]$ where some firm (possibly including j) is exactly on the boundary. Let $\Psi_j(\mathbf{v}_{-j})$ denote the set of announcements by firm j where some firm is on the boundary, given $\mathbf{v}_{-j}$; similarly, let $\Omega_j(\mathbf{v}_{-j})$ denote the set of announcements where $v_k \neq U(\mathbf{v})$ for all $k \in M$ (we occasionally omit the argument $\mathbf{v}_{-j}$ for notational simplicity). The lemma summarizes our results.

Lemma 1 *Given $\mathbf{v}_{-j}$ the set $\Psi_j(\mathbf{v}_{-j})$ contains a finite number of points.*

Proof. See above. ■

We now characterize how $\mathbf{p}$ changes in response to a change in v_j. We will show that $p_j(v_j, \mathbf{v}_{-j})$ is quasi-concave in v_j. We first focus on announcements in Ω_j and then generalize our results to the full domain $\Omega_j \cup \Psi_j$.

Consider an announcement $(v_j, \mathbf{v}_{-j})$ where $v_j \in \Omega_j(\mathbf{v}_{-j})$ and some perturbation $\mathbf{v}' = (v_j', \mathbf{v}_{-j})$. When v_j' is close enough to v_j the set of active firms does not change: $A(\mathbf{v}) = A(\mathbf{v}')$. If $v_j < U(\mathbf{v})$ then firm j is inactive both under $\mathbf{v}$ and under $\mathbf{v}'$ and therefore $\mathbf{p}$ is not affected by a small change in v_j, i.e. $\partial p_k / \partial v_j = 0 \ \forall \ k$. If $v_j > U(\mathbf{v})$, we shall apply the implicit function theorem around $\mathbf{F}(\hat{\mathbf{p}}, \hat{\mathbf{v}}) = \mathbf{0}$. The Jacobian of $\mathbf{F}$ with respect to $(p_1, ..., p_l)$ is given by

$$
D_{\mathbf{p}}F = \begin{pmatrix}
\xi_1(\mathbf{v}) & 0 & 0 & ... & 0 & -\xi_l(\mathbf{v}) \\
0 & \xi_2(\mathbf{v}) & 0 & ... & 0 & -\xi_l(\mathbf{v}) \\
... & ... & ... & ... & ... & ... \\
0 & 0 & 0 & ... & \xi_{l-1}(\mathbf{v}) & -\xi_l(\mathbf{v}) \\
1 & 1 & 1 & ... & 1 & 1
\end{pmatrix},
$$

where $\xi_k(\mathbf{v}) \equiv g(p_k(\mathbf{v}))\ v_k$ denotes the change in the expected utility offered by firm k due to an increase in p_k. The rank of this matrix is l: the expected utility of applying to firm k decreases in p_k and therefore $\xi_k \neq 0$ for all $k \in A(\mathbf{v})$. As a result we can apply the implicit

[16] A similar argument holds for the case where $k = j$. The only difference is that when $v_j = U(\mathbf{v})$ then $j \in A(v_j', \mathbf{v}_{-j})$ if $v_{j'} > v_j$ and $j \in A^C(v_j', \mathbf{v}_{-j})$ if $v_{j'} < v_j$.

function theorem to show that $\partial p_j(\mathbf{v})/\partial v_j$ exists locally around $\mathbf{v}$ and that the matrix of partial derivatives is defined by $D_{\mathbf{v}}\mathbf{p} = -(D_{\mathbf{p}}\mathbf{F})^{-1}D_{\mathbf{v}}\mathbf{F}$. The following lemma describes our result:

Lemma 2 (Workers' response to a perturbation in the announcements) *When $v_j \in \Omega_j(\mathbf{v}_{-j})$ and $j \in A(\mathbf{v})$ a change in v_j leads to*

$$\frac{\partial p_j(\mathbf{v})}{\partial v_j} \;=\; T_j(\mathbf{v})^{-1}\; G(p_j(\mathbf{v})), \tag{6}$$

where $T_j(\mathbf{v}) = -\xi_j(\mathbf{v}) - [\sum_{k \in A(\mathbf{v})\setminus\{j\}} \xi_k(\mathbf{v})^{-1}]^{-1}$.

Proof. See the appendix. ■

The result establishes that the change of the application probability depends on the probability of getting a job at firm j, as well as on a term which is a function of the ξ_ks. When the marginal utilities (ξ_k) are large in absolute terms, a change in a firm's announced utility leads to a small change in $\mathbf{p}$ because a small change is sufficient to equalize utilities across all firms.

Finally, when $\mathbf{v}_{-j} = \mathbf{0}_{-j}$ we have $p_j(v_j, \mathbf{0}_{-j}) = 1/m$ for $v_j = 0$ and $p_j(v_j, \mathbf{0}_{-j}) = 1$ for $v_j > 0$. Similarly, for all $k \neq j$ we have $p_k(\mathbf{0}) = 1/m$ when $v_j = 0$ and $p_k(\mathbf{v}) = 0$ when $v_j > 0$. In other words, $p_j(v_j, \mathbf{0}_{-j})$ is discontinuous at $v_j = 0$ and $0 \in \Psi_j(\mathbf{0}_{-j})$.

We use Lemma 2 to prove p_j is quasi-concave on the *full* domain of announcements. In particular when $\mathbf{v}_{-j} \neq \mathbf{0}_{-j}$ the application probability $p_j(v_j, \mathbf{v}_{-j})$ is equal to zero for $v_j \leq \hat{v}_j(\mathbf{v}_{-j})$ and it is strictly concave for $v_j \geq \hat{v}_j(\mathbf{v}_{-j})$. When $\mathbf{v}_{-j} = \mathbf{0}_{-j}$ the application probability is discontinuous at $v_j = 0$ with $p_j(0, \mathbf{0}_{-j}) = 1/m$ and $p_j(v_j, \mathbf{0}_{-j}) = 1$ for $v_j > 0$.

Lemma 3 *The application probability $p_j(v_j, \mathbf{v}_{-j})$ is quasi-concave in v_j for given $\mathbf{v}_{-j}$.*

Proof. See the appendix. ■

3.3 Analysis of firms' strategies

We now analyze how profits change when a firm's announcement is perturbed. The goal is to prove the quasi-concavity of expected profits.

Consider firm j and fix the other firms' announcement $\mathbf{v}_{-j}$. We first focus on $v_j \in \Omega_j(\mathbf{v}_{-j})$ and we describe how to extend our results to $v_j \in \Psi_j(\mathbf{v}_{-j})$ below (the case of $\mathbf{v}_{-j} = \mathbf{0}$ is considered separately). If $v_j < \hat{v}_j(\mathbf{v}_{-j})$ then firm j is inactive, its expected profits are zero and $\partial \Pi_j(\mathbf{v})/\partial v_j = 0$. If $v_j > \hat{v}_j(\mathbf{v}_{-j})$ then firm j is active and the first derivative of its expected profits with respect to its own announcement is

$$\frac{\partial \Pi_j(v_j, \mathbf{v}_{-j})}{\partial v_j} = H(p_j(v_j, \mathbf{v}_{-j}))\frac{d\pi_j(v_j)}{dv_j} + h(p_j(v_j, \mathbf{v}_{-j}))\pi_j(v_j)\frac{\partial p_j(v_j, \mathbf{v}_{-j})}{\partial v_j}. \tag{7}$$

The second derivative is

$$
\begin{aligned}
\frac{\partial^2 \Pi_j(v_j, \mathbf{v}_{-j})}{\partial v_j^2} &= H(p_j(v_j, \mathbf{v}_{-j}))\frac{d^2\pi_j(v_j)}{dv_j^2} + 2\ h(p_j(v_j, \mathbf{v}_{-j}))\frac{d\pi_j(v_j)}{dv_j}\frac{\partial p_j(v_j, \mathbf{v}_{-j})}{\partial v_j} \\
&+ h'(p_j(v_j, \mathbf{v}_{-j}))\Big(\frac{\partial p_j(v_j, \mathbf{v}_{-j})}{\partial v_j}\Big)^2 \pi_j(v_j) + h(p_j(v_j, \mathbf{v}_{-j}))\pi_j(v_j)\frac{\partial^2 p_j(v_j, \mathbf{v}_{-j})}{\partial v_j^2} \quad (8)
\end{aligned}
$$

It is not hard to see that equation (8) is negative. The first term is weakly negative since π_j is weakly concave. The second term is weakly negative since π_j is weakly decreasing in $[\underline{v}_j, \overline{v}_j]$, $h(p_j) > 0$ and $\partial p_i/\partial v_i > 0$. The third term is negative since $h'(p_i) \leq 0$, and the fourth term is strictly negative because of $\partial^2 p_i/\partial v_i^2 < 0$. Therefore, expected profits Π_j are strictly concave on $(\hat{v}_j(\mathbf{v}_{-j}), \overline{v}_j) \cap \Omega_j(\mathbf{v}_{-j})$. This result can be extended to the elements in $\Psi_j(\mathbf{v}_{-j})$ using the same arguments as in the proof of Lemma 3. When $\mathbf{v}_{-j} = \mathbf{0}$, the expected profits of firm j are discontinuous at $v_j = 0$ due to the discontinuity of p_j at $\mathbf{v} = \mathbf{0}$. More specifically, $\Pi_j(v_j, \mathbf{0}_{-j}) = \pi_j(0)/m$ when $v_j = 0$ and $\Pi_j(v_j, \mathbf{0}_{-j}) = \pi_j(v_j)$ when $v_j > 0$.

We have established that profits are quasi-concave. In particular we have shown that when $\mathbf{v}_{-j} \neq \mathbf{0}_{-j}$ the expected profits pf firm j are equal to zero for $v_j \in [\underline{v}_j, \hat{v}_j(\mathbf{v}_{-j})]$ and are strictly concave for $v_j \in [\hat{v}_j(\mathbf{v}_{-j}), \overline{v}_j]$. When $\mathbf{v}_{-j} = \mathbf{0}_{-j}$, the expected profits are discontinuous at $v_j = 0$ with $\Pi_j(0, \mathbf{0}_{-j}) = \pi_j(0)/m$ and $\Pi_j(v_j, \mathbf{0}_{-j}) = \pi_j(v_j)$ for $v_j \in (0, \overline{v}_j]$.

Lemma 4 *Expected profits* $\Pi_j(v_j, \mathbf{v}_{-j})$ *are quasi-concave in* v_j *for given* $\mathbf{v}_{-j}$.

Proof. See above. ■

It is worth remarking that despite this lemma we cannot rule out mixed strategy equilibria because quasi-concavity is too weak a concept. Global strict concavity is sufficient to establish that an equilibrium has to be in pure strategies since the sum of strictly convex functions remains strictly convex and therefore best replies are a singleton. However, since we only have quasi-concavity and the sum of quasi-concave functions does not necessarily remain quasi-concave this argument does not go through.

3.4 Finding a Fixed Point

The final step to prove the existence of a directed search equilibrium is to find a fixed point in firms' strategies. The strategy space, $\mathcal{V}$, is compact and the expected profit function is quasi-concave. However, as show above, profits are discontinuous at $\mathbf{v} = \mathbf{0}$.

When $\mathcal{V}$ does not include $\mathbf{0}$, i.e. if $\underline{v}_j > 0$ for some j, then existence follows by standard fixed point arguments: the expected profit function is continuous and therefore the best response correspondence of the firms is upper hemi-continuous by Berge's Theorem. Quasi-concavity of profits leads to a convex-valued best-response correspondence and Kakutani's fixed point theorem ensures the existence of an equilibrium.

However, when $\mathbf{0} \in \mathcal{V}$ we have to deal with the resulting discontinuity. To prove existence we use the concept of Better-Reply Security of Reny (1999). In our environment Better-Reply Security means the following. Consider any announcements $\mathbf{v} \in \mathcal{V}$ that is not an equilibrium and any sequence $\mathbf{v}_h \in \mathcal{V}$ such that $\mathbf{v}_h \to \mathbf{v}$ as $h \to \infty$ with limit payoff vector $(\overline{\Pi}_1, \overline{\Pi}_2, .., \overline{\Pi}_m) = \lim_{h\to\infty}(\Pi_1(\mathbf{v}_h), \Pi_2(\mathbf{v}_h), .., \Pi_m(\mathbf{v}_h))$. The game among firms is Better-Reply Secure if there exists a player j and an action $\tilde{v}_j$ such that $\Pi_j(\tilde{v}_j, \tilde{\mathbf{v}}_{-j}) > \overline{\Pi}_j$ for all $\tilde{\mathbf{v}}_{-j}$ in the neighborhood of $\mathbf{v}_{-j}$. That is, if the original announcement is not an equilibrium then there exists a firm that can always do strictly better even if the other firms slightly deviate from the profile. When profits are continuous around $\mathbf{v}$, this is trivially the case.

We only have to check the condition for the case when all firms offer zero, i.e. at $\mathbf{v} = \mathbf{0}$. For any sequence of $\mathbf{v}_h$ converging to zero there is some firm j that in the limit has an application probability below the average, i.e. $p_j \leq 1/m$ and its payoffs are $\overline{\Pi}_j \leq H(1/m)\pi_j(0)$. If firm j offers $\tilde{v}_j = \varepsilon$, then all workers apply to firm j as long as $v_k < \varepsilon/n$ for all $k \neq j$. So for every ε there is a neighborhood around the strategy of the other firms such that firm j hires with probability one, and therefore for ε small by continuity of the ex post profit function it can ensure itself a payoff close to $\pi_j(0)$, which is strictly higher than $\overline{\Pi}_j$ because the firm can now hire for sure. Therefore, the game is Better-Reply Secure and an equilibrium exists by the fixed point Theorem 3.1 in Reny (1999).

This concludes the proof of Theorem 1.

4 Characterization of the Equilibrium Set

This section characterizes the equilibrium set. We show that more productive firms will in equilibrium offer higher utility to workers, under an additional assumption on the production technology. As a corollary we show that the directed search equilibrium is unique when firms are homogeneous. We first define how we rank firms according to productivity, then discuss the difficulties involved in obtaining these results, and finally prove our theorems.

We first need to rank firms by their productivity. We will use the following definition and only consider environments where the firms can be ranked accordingly.

Definition 3 *We say that firm j is more productive than firm k if*

$$\pi_j(0) \geq \pi_k(0) \text{ and} \tag{9}$$

$$d\pi_j(v)/dv \geq d\pi_k(v)/dv \ \forall\ v. \tag{10}$$

If one of the inequalities is strict, we say that firm j is strictly more productive than firm k. If both (9) *and* (10) *hold with equality, then we say that firms j and k are equally productive.*

Equation (9) states that when workers receive zero utility the profits of firm j are weakly higher than the profits of firm k. Equation (10) states that the profits of firm j increase faster (or drop more slowly) than k's when workers' utility increases. It immediately follows that for a given level of worker utility, firm j makes higher profits than k.[17] For example, in the linear profit functions $\pi_j(v) = x_j - v$ of Montgomery (1991) and Burdett, Shi and Wright (2001), Definition 3 translates into our usual notion of being more productive ($x_j \geq x_k$) because the slopes of the profit functions are identical.

Proving that more productive firms offer higher utility to prospective employees is straightforward in the context of a continuum economy. One only needs to establish the following simple single-crossing condition between the probability of hiring, H, and the utility that is offered to workers, v: for a given increase in H, a more productive firm is always willing to raise v by a larger amount than a less productive firm. In a continuum economy, this argument is sufficient to show that more productive firms offer higher utility to workers.

However, this logic does not apply in a finite economy because a single firm's action affects market outcomes and, in particular, the probability of hiring when making a given offer. Consider two firms (say 1 and 2) that currently offer different levels of utility (v_1 and v_2) and are both contemplating a deviation to some $\hat{v}$. The hiring probability that firm 1 faces if it offers $\hat{v}$ is different from the one that firm 2 faces because the overall distribution of offers will be different: if firm 1 deviates to $\hat{v}$ then the distribution includes $\hat{v}$ and v_2 but not v_1; if firm 2 deviates, the distribution includes $\hat{v}$ and v_1 but not v_2. Therefore the hiring probability when offering $\hat{v}$ depends on which firm is making that offer. As a result, single-crossing in terms of preferences is not enough because the "technology" by which a firm can convert the utility that it offers into the probability of hiring differs for the different firms.

To prove our result, we will directly compare the first order conditions of the firms. However, it is not necessary for the equilibrium to be characterized by the first order conditions and we provide an additional condition which guarantees that this first order approach is valid. The reason why the first order conditions need not hold in equilibrium is that a firm's expected profits may contain kinks. To see this, consider a firm (say, firm 1) that offers v_1 and is active and suppose that some other firm (say, firm 2) offers v_2 and is on the boundary for being active. Think of how the expected profits of firm 1 are affected by a change in v_1: If firm 1 reduces its announcement the market utility will fall and firm 2 will become active, adding a competitor for workers' services; this makes the supply of workers more elastic with respect to

[17]Note, however, that Definition 3 is a strictly stronger requirement than $\pi_j(v) \geq \pi_j(v)$ for all v.

the announcement. If firm 1 increases its offer the market utility will increase, firm 2 will remain inactive and the supply of workers will be less elastic with respect to v_1. This creates a kink in the expected profits of firm 1 and the first order conditions of firm 1 need not be satisfied if v_1 and v_2 turn out to be profit maximizing.

The following assumption is sufficient to rule out the scenario described above by guaranteeing that all firms are active. More precisely, it states that every firm is active in equilibrium, even when all of its competitors offer the maximum individually rational utility.

Assumption 3 $p_j(\overline{\mathbf{v}}) > 0$ *for all* j *where* $\overline{\mathbf{v}} = (\overline{v}_1, ..., \overline{v}_m)$.

It is easy to show that Assumption 3 holds as long as the maximum utilities that firms are willing to offer are not too far apart, i.e. there exists parameter $\gamma < 1$ such that Assumption 3 holds whenever $\min_j \overline{v}_j > \gamma \max_j \overline{v}_j$. Note that we only rely on Assumption 3 for the characterization proof of Section 4 and we do not needed for our other results.

We now prove that if a low productivity firm's first order conditions hold and it offers higher utility than a high productivity firm then the high productivity firm's first order conditions are not satisfied. This shows that in equilibrium a more productive firm offers higher utility to workers.

Theorem 2 *If Assumption 3 holds, then in any directed search equilibrium* $v_j > v_k$ *if firm* j *is strictly more productive than firm* k *and* $v_j = v_k$ *if firm* j *is equally productive to firm* k.

Proof. See Appendix. ■

Theorem 2 has immediate consequences for the homogeneous firm environment. First, it trivially establishes that all firms post identical offers. Second, it provides a straightforward proof for the uniqueness of equilibrium.

Theorem 3 *When all firms are equally productive, the directed search equilibrium is unique.*

Proof. When all firms are equally productive Assumption 3 holds and in equilibrium all firms offer the same level of utility by Theorem 2. As a result, $p_j = 1/m$ for all $j \in M$ in all possible equilibria. Suppose there are two candidate equilibria A and B where firms offer v_A and $v_B > v_A$, respectively, and consider the firms' first order conditions. The terms $H(p)$ and $h(p)$ are the same in both candidate equilibria. The concavity of the profit function implies that $d\pi(v_A)/dv_A \geq d\pi(v_B)/dv_B$. Profits are a decreasing function of offered utility in $\mathcal{V}$ which implies that $\pi(v_A) > \pi(v_B)$. Finally, $\partial p_j/\partial v_A > \partial p_j/\partial v_B$ follows from equation (15): $G(p)$ and $g(p)$ are the same in both equilibria and $T_j(\mathbf{v}_A) < T_j(\mathbf{v}_B)$. ■

5 Competitive Search as a Limit

In this section we present the standard one-shot version of a competitive search economy with a continuum of agents and show that it is the limit of the finite game as the number of agents becomes large. This setup encompasses the models described in Section 2.2. Our exposition is most closely related to Peters (1997).

Consider an economy with measure one of firms and measure b of workers. The workers are homogeneous and firms are potentially heterogeneous with types distributed on $\Theta = [0, 1]$ according to probability measure P. When a firm of type $\theta \in \Theta$ fills its vacancy and pays v to its worker it makes profits $\pi_\theta(v)$, where π_θ satisfies Assumption 1 and $\bar{v} \equiv \sup_{\theta\in\Theta} \bar{v}_\theta < \infty$.

The timing of the model is the same as in the finite case: firms post announcements, workers decide where to apply for a job, matching occurs and payoffs are realized. The workers' strategies result in an expected *queue length* λ which represents the ratio of the expected number of applications per firm at each announcement level v and corresponds to np_j in the finite case. The probability that a firm facing queue length λ hires a worker is given by $r_f(\lambda)$ and the probability that a worker who applies to such a firm finds a job is $r_w(\lambda)$, where $r_w(\lambda) = r_f(\lambda)/\lambda$. Additionally, r_f is strictly increasing and concave, r_w is strictly decreasing and convex and they are both twice continuously differentiable.

The queue length across different announcements is determined by an indifference condition, similar to equation (1), which states that a worker receives at least the market utility U when applying to a firm. An important additional assumption is that this relation holds both on and *off* the equilibrium path, i.e. it determines a firm's hiring probability from offering some v that is not posted by anyone else:

$$\text{If } v > U \text{ then } \lambda \text{ is s.t. } r_w(\lambda)v = U \text{ , otherwise } \lambda = 0. \tag{11}$$

As in the finite case, an announcement that is too low ($v \leq U$) receives no applicants ($\lambda = 0$) and a firm is active only if $v > U$. Let $\lambda(v, U)$ be the queue length defined by (11). Each firm anticipates this relation between the queue length and its announcement, and solves the problem

$$\max_v r_f(\lambda(v, U))\pi_\theta(v) \tag{12}$$

Definition 4 (Competitive Search Equilibrium) *A competitive search equilibrium is the workers' market utility U^* and and a cumulative distribution of announcements Y^* defined on any $\mathcal{V} \subset [0, \overline{v}]$ such that*

$$\int_{\mathcal{V}} dY^*(v) \leq P\{\theta \in \Theta : \textit{some } v \in \mathcal{V} \textit{ solves } (12)\} \tag{13}$$

and

$$\int \lambda\left(v, U^*\right) dY^*(v) = b. \tag{14}$$

The left hand side of equation (13) gives the measure of announcements in $\mathcal{V}$. Since the firms choose the announcements, there has to be at least an equally large measure of firms that find it optimal to offer these announcements, which is given by the right hand side of (13).[18] Equation (14) ensures that the worker-firm ratio integrated across all firms actually adds up to the measure of workers in the economy. It ensures that the utility that the workers obtain indeed reflects their scarcity.

For some of the convergence results it is more useful to talk about a firms' rank in the distribution. We define a firm as being of rank $x \in [0,1]$ if a fraction x of other firms has a weakly lower type. We can back out the actual type of the firm that has rank x as $\tau(x) = \sup\{\theta \in \theta | P([0,\theta]) \leq x\}$). Let Π^*_x denote the expected profit of a firm of rank x in the competitive equilibrium.

We will now explore the connection of this limit game to games of the finite economy that we analyzed in Section 3. Consider a finite economy with m firms and $n = bm$ identical workers. In what follows, we index the variables that refer to the finite economy by m. We label firms in the finite economy by their rank in the productivity distribution, so that firm j is of rank j/m. Furthermore, we assume that the rank remain unchanged as the economy grow in that it coincides with that of firm of type $\tau(j/m)$ in the limit economy. Therefore, by construction the distribution of types in the finite economy converges weakly to the type distribution in the limit economy. Theorem 1 proves that the finite economy has a pure strategy equilibrium. Let Y_m denote the distribution of announcements for that equilibrium, U_m the market utility of the workers and $\Pi_{m,x}$ the expected profit of firm $j = mx$.

In the finite game we have some trading probabilities given by $H(p)$ and $G(p)$ when workers apply with probability p to a firm, where H and G fulfill Assumption 2. The matching probabilities change when we increase the number of workers n, and to make this dependence obvious we can write $H(n,p)$ and $G(n,p)$.[19] Intuitively, np reflects the expected number of workers at this firm. We will consider matching functions r_w and r_f that can be approached as the limits of H and G as $n \to \infty$ keeping $np = \lambda$. Since Assumption 2 is quite general, this includes most

[18] The inequality can be strict when firms are indifferent about several different v's. If firms have a unique optimum, then (13) has to hold with equality. To see this, note that Y is a CDF and therefore integrates to one over the entire space - meaning that the unit measure of firms trades at some announcement. Since P is a probability measure the only way to have them add up is to have equality everywhere.

[19] It is more convenient to index these probability by n. Of course, this is identical to indexing them by m since $n = bm$.

matching functions that have been used in the literature. For example:

$$
\begin{aligned}
1-e^{-\lambda} &= \lim_{n\to\infty|np=\lambda} 1-(1-p)^n \\
\frac{\lambda}{1+\lambda} &= \lim_{n\to\infty|np=\lambda} \frac{np}{1+np} \\
(1+\lambda^{-\sigma})^{-1/\sigma} &= \lim_{n\to\infty|np=\lambda} (1+(np)^{-\sigma})^{-1/\sigma}
\end{aligned}
$$

We will show that an allocation that can be supported for the limit of finite games constitutes a competitive search equilibrium, and vice versa. The following result shows the payoffs of workers and firms converge for large m to those in the limit economy, which implicitly means that the equilibrium matching probabilities converge.

Theorem 4 *For any convergent subsequence of equilibria such that $Y_m \to Y^*$ there exists U^* such that $\{U^*, Y^*\}$ constitutes a competitive search equilibrium, and expected utilities converge ($U_m \to U^*$) as well as expected profits ($\Pi_{m,x} \to \Pi_x^*$). Conversely, for any competitive search equilibrium $\{U^*, Y^*\}$ there exists a subsequence of equilibria such that $Y_m \to Y^*$, $U_m \to U^*$, and $\Pi_{m,x} \to \Pi_x^*$.*

Proof. The analysis for the subgame against a convergent distribution $Y_m \to Y^*$ of (*possibly non-equilibrium*) offers follows directly from Peters (1997), Theorem 3 and Theorem 4.[20] He characterizes the payoffs for the firms that offer any of the wages in Y_m. Peters (1997, p. 256) lays out that his equivalence theorems extend directly to convergence of finite *equilibria* if the finite equilibria exist in pure posting strategies (because in this case the equilibrium can be represented as a step function Y_m). Our Theorem 1 establishes such existence in pure posting strategies. ■

Existence in pure posting strategies is crucial for this otherwise straightforward extension of Peters' result.

6 Conclusions

In this paper we prove the existence of Nash equilibria in pure firm strategies for finite directed search economies with heterogeneous firms, homogeneous workers and general production and matching structures. In addition to being interesting in its own right, this result is useful in a number of ways. A more complete characterization of the equilibrium set is feasible (see Section 4) and examining the efficiency properties of the finite economy becomes easier (see Galenianos, Kircher and Virag (2009)). Furthermore, proving the convergence of finite equilibria

[20] The proofs in Peters (1997) work with the function $H(n,p) = 1-(1-p)^n$, but straightforward replacement by the general functional form $H(n,p)$ shows convergence for more general matching functions.

to the continuum economies becomes relatively straightforward (see Section 5), showing that the competitive search models that have been considered in the literature have solid micro-foundations.

A number of questions remain open for this class of models. The cardinality of the pure strategy equilibrium set has not been characterized (especially as concerns uniqueness) while the existence of non-degenerate mixed strategy equilibria has not been proved or disproved. A different research direction would be to introduce heterogeneity on the worker side. With two-sided heterogeneity one can address questions regarding the sorting patterns between workers and firms. This question has been examined in continuum models by Shi (2001), Shimer (2005) and Eeckhout and Kircher (2009) but, to our knowledge, only Peters (2009) has made progress in analyzing a finite economy.[21]

[21] Peters (2009) considers the game among heterogeneous workers for given wage offers by firms, while strategic decisions of the firms are not analyzed for finite numbers. He does integrate firms' decisions in a limit game.

7 Appendix

Lemma 2.

Proof. We show that the partial derivatives translate into equation (6). (See Korn and Korn (1968) for the relevant matrix algebra). $D_{\mathbf{p}}\mathbf{F}$ is a matrix with elements $\alpha_{ss} = \xi_s(\mathbf{v})$ and $\alpha_{sl} = -\xi_l(\mathbf{v})$ for $s \in \{1,...,l-1\}$, $\alpha_{ls} = 1$ for $s \in \{1,...,l\}$ and $\alpha_{sk} = 0$ otherwise. To calculate the determinant $|D_{\mathbf{p}}\mathbf{F}|$ we use Laplace's development to expand the last row and obtain $|D_{\mathbf{p}}\mathbf{F}| = \sum_{s=1}^{l} \Lambda_{ls}$, where Λ_{ls} is the cofactor to element α_{ls}. That is, $\Lambda_{ls} = (-1)^{l+s}|Q_{ls}|$, where Q_{ls} is the matrix resulting from $D_{\mathbf{p}}\mathbf{F}$ by elimination of the lth row and the sth column. Since Q_{ll} is a diagonal matrix we have $|Q_{ll}| = \prod_{k\in L(\mathbf{v})\backslash\{l\}} \xi_k(\mathbf{v})$. For $s < l$ we expand the s th row of $|Q_{ls}|$ which yields $|Q_{ls}| = (-1)^{l-1+s}(-\xi_l(\mathbf{v}))|B_{ls}|$, where B_{ls} is a $(l-2)^2$ -dimensional diagonal matrix with diagonal elements $\xi_k(w)$ for all $k \in A(\mathbf{v})\backslash\{s,l\}$. We therefore have $|Q_{ls}| = (-1)^{l+s} \prod_{k\in A(\mathbf{v})\backslash\{s\}} \xi_k(\mathbf{v})$, which yields that $|D_{\mathbf{p}}\mathbf{F}| = \sum_{s=1}^{l} \prod_{k\in A(\mathbf{v})\backslash\{s\}} \xi_k(\mathbf{v})$.

Next, consider the matrix $D_{\mathbf{v}}\mathbf{p} = -(D_{\mathbf{p}}\mathbf{F})^{-1}D_{\mathbf{v}}\mathbf{F}$ of partial derivatives. As an implication of Cramer's Rule $(D_{\mathbf{p}}\mathbf{F})^{-1} = |D_{\mathbf{p}}\mathbf{F}|^{-1}C$, where C is the matrix with elements $\gamma_{js} = \Lambda_{sj}$. The Jacobian with respect to the exogenous variables $D_{\mathbf{v}}\mathbf{F}$ evaluated at $(\mathbf{p}(\mathbf{v}), \mathbf{v})$ is simply a diagonal matrix except for the last column, with elements $\beta_{ss} = G(p_s(\mathbf{v}))$ and $\beta_{sl} = -G(p_l(\mathbf{v}))$ for $s \in \{1,..,l-1\}$ and zeros elsewhere. We therefore have $\partial p_j(\mathbf{v})/\partial v_j = -\Lambda_{jj}|D_{\mathbf{p}}\mathbf{F}|^{-1}G(p_j(\mathbf{v}))$. This follows immediately for $j \in \{1,..,l-1\}$, and holds for $j = l$ by symmetry which is cumbersome but straightforward to verify analytically. Since the cofactor Λ_{jj} has a similar structure as the determinant $|D_{\mathbf{p}}\mathbf{F}|$ only with row and column j missing, we have $\Lambda_{jj} = \sum_{s\in A(\mathbf{v})\backslash\{j\}} \prod_{k\in A(\mathbf{v})\backslash\{j,s\}} \xi_k(\mathbf{v})$, and we obtain

$$\frac{\partial p_j(\mathbf{v})}{\partial v_j} = -\frac{\sum_{s\in A(\mathbf{v})\backslash\{j\}} \prod_{k\in A(\mathbf{v})\backslash\{j,s\}} \xi_k(\mathbf{v})}{\sum_{s\in A(\mathbf{v})} \prod_{k\in A(\mathbf{v})\backslash\{s\}} \xi_k(\mathbf{v})} G(p_j(\mathbf{v})). \tag{15}$$

Equation (6) follows then from simple algebraic manipulations. ■

Lemma 3.

Proof. Fix $\mathbf{v}_{-j}$. We first consider $\hat{v}_j \in \Psi_j(\mathbf{v}_{-j})$, i.e. points where the workers reaction is not differentiable. We have already established there is only a finite number of such points. At these points the concavity of $p_j(v_j, \mathbf{v}_{-j})$ follows trivially because a decrease in the announcement by firm j increases other firms' expected number of applicants, while an increase does not. That is, by continuity of $p_j(\cdot)$, $\xi_j(\cdot)$ and $G(\cdot)$ equation (15) implies that $\lim_{v_j \nearrow \hat{v}_j} \partial p_j(v_j, \mathbf{v}_{-j})/\partial v_j < \lim_{v_j \searrow \hat{v}_j} \partial p_j(v_j, \mathbf{v}_{-j})/\partial v_j$.

The remaining task is to show that $p_j(v_j, \mathbf{v}_{-j})$ is strictly concave for $\hat{v}_j \in \Omega_j(\mathbf{v}_{-j})$. Recall that $T_j(\mathbf{v}) = -\xi_j(\mathbf{v}) - X_j(\mathbf{v})$ where $X_j(\mathbf{v}) = 1/\sum_{k\in A(\mathbf{v})\backslash\{j\}} \frac{1}{\xi_k(\mathbf{v})}$. We differentiate (15) with respect to v_j to obtain the following:

$$\frac{\partial^2 p_j}{\partial v_j^2} = -\frac{1}{T_j^2}\left(g(p_j)\frac{\partial p_j}{\partial v_j}[X_j + v_j] - G(p_j)\Big[g'(p_j)\frac{\partial p_j}{\partial v_j}v_j + g(p_j) + \frac{\partial X_j}{\partial v_j}\Big]\right), \tag{16}$$

where $\mathbf{v}$ is omitted for brevity. We now show that (16) is strictly negative. We split the term in the round bracket into three parts, B_1, B_2 and B_3, and show that each is non-negative.

The first part is given by $B_1 = g(p_j)\ [\partial p_j/\partial v_j]\ X_j$ and it is strictly positive because $g(p_j)$ and X_j are strictly negative. Part B_2 is given by

$$B_2 = g(p_j)\ \frac{\partial p_j}{\partial v_j}\ v_j -\ G(p_j)\ \Big[g'(p_j)\ \frac{\partial p_j}{\partial v_j}\ v_j + g(p_j)\Big].$$

Rearranging the above and using (15) yields

$$B_2 = G(p_j)v_j[2g(p_j)^2 - g'(p_j)\ G(p_j)] + X_j g(p_j) G(p_j).$$

The last term is positive so we only need to show that term in the square bracket is positive, which holds exactly when $1/G(p)$ is convex.

Finally, consider $B_3 = -G(p_j)[\partial X_j/\partial v_j]$. Note that

$$\frac{\partial X_j}{\partial v_j} = X_j^2\Big[\sum_{k\in A(\mathbf{v})\setminus\{j\}} \frac{g'(p_k)}{g(p_k)^2 v_k}\frac{\partial p_k}{\partial v_j}\Big],$$

Since $\partial p_k/\partial v_j \leq 0$ for $k \neq j$ and $g'(p_k) \geq 0$, due to the convexity of $G(p)$, we have shown that B_3 is non-negative. ■

Theorem 2.
Proof. Under Assumption 3, $A(\mathbf{v}) = M$ and the announcement of every firm is characterized by its first order condition:

$$\frac{\partial \Pi_j}{\partial v_j} = H(p_j)\frac{d\pi_j(v_j)}{dv_j} + h(p_j)\pi_j(v_j)\frac{\partial p_j}{\partial v_j} = 0 \ \ \forall\ j \in M. \tag{17}$$

From now on we focus on firms 1 and 2 without loss of generality. Let firm 1 be strictly more productive than firm 2. The proof proceeds by contradiction. Assume $v_1 \leq v_2$ (the proof for equal productivities and $v_1 < v_2$ is analogous). Under this assumption we will show that $\partial\Pi_2/\partial v_2 = 0$ and then $\partial\Pi_1/\partial v_1 > 0$, which contradicts profit maximization for firm 1 and proves that $v_1 > v_2$ is a necessary condition for equilibrium.

We proceed by assuming $v_1 \leq v_2$. To compare the first order conditions of firms 1 and 2 we can work with the following two sets of inequalities:

$$\frac{d\pi_1(v_1)}{dv_1} \geq \frac{d\pi_2(v_1)}{dv_1} \geq \frac{d\pi_2(v_2)}{dv_2}, \quad (18)$$

$$\pi_1(v_1) \geq \pi_2(v_1) \geq \pi_2(v_2). \quad (19)$$

The first inequality of equations (18) and (19) is due to firm 1 being more productive and at least one of them has to hold strictly (according to Definition 3). The second inequality of equation (18) is due to the (weak) concavity of $\pi_j(\cdot)$. The second inequality of equation (19) is due to the fact that $\pi_j(v_j)$ is decreasing in v_j in the relevant range.

Rearranging equation (17) yields

$$\frac{d\pi_j(v_j)}{dv_j} + \frac{h(p_j)}{H(p_j)}\frac{\partial p_j}{\partial v_j}\pi_j(v_j) = 0. \quad (20)$$

If the term multiplying $\pi_j(v_j)$ is higher for firm 1 than for firm 2, then the first derivative of firm 1 is strictly positive when $v_1 \leq v_2$ which proves our result. Using equation (6) we can rewrite:

$$\frac{h(p_j)}{H(p_j)}\frac{\partial p_j}{\partial v_j} = -\frac{h(p_j)}{H(p_j)}G(p_j)\frac{\sum_{s\in M\setminus\{j\}}\prod_{k\in M\setminus\{j,s\}}\xi_k}{\sum_{s\in M}\prod_{k\in M\setminus\{s\}}\xi_k}. \quad (21)$$

Note that the last term has the same denominator for all j. Therefore we need only show that

$$\frac{h(p_1)G(p_1)}{H(p_1)}\sum_{s\neq 1}\prod_{k\notin\{1,s\}}|g(p_k)|v_k \geq \frac{h(p_2)G(p_2)}{H(p_2)}\sum_{s\neq 2}\prod_{k\notin\{2,s\}}|g(p_k)|v_k \quad (22)$$

recalling that $\xi_k \equiv g(p_k)v_k$ and $g(p_k) < 0$. The assumption that $v_1 \leq v_2$ implies $p_1 \leq p_2$ and hence $h(p_1 \geq h(p_2)$, $H(p_1) \leq H(p_2)$ and $G(p_1) \geq G(p_2)$. The term $\prod_{k\notin\{1,2\}}|g(p_k))|v_k$ is contained inside the summation in both sides of inequality (22). It is therefore sufficient to show:

$$\frac{h(p_1)G(p_1)}{H(p_1)}|g(p_2)|v_2 \geq \frac{h(p_2)G(p_2)}{H(p_2)}|g(p_1)|v_1. \quad (23)$$

Subgame equilibrium implies that $v_2/v_1 = G(p_1)/G(p_2))$. Together with $G(p_j) = H(p_j)/(np_j)$ and $|g(p_j)| = [G(p_j) + h(p_j)/n]/p$, inequality (23) reduces to

$$\frac{G(p_2) + h(p_2)/n}{G(p_2)h(p_2)/n} \geq \frac{G(p_1) + h(p_1)/n}{G(p_1)h(p_1)/n}.$$

If $R(p) \equiv G(p)^{-1} + nh(p)^{-1}$ is strictly increasing in p we have our result. Differentiation yields $R'(p) = -G(p)^{-2}g(p) - nh(p)^{-2}h'(p)$ which is strictly positive for any $p \in (0,1)$ because $h'(p) \leq 0$ and $g(p) < 0$. ■

References

[1] Acemoğlu, Daron and Asuman Ozdaglar (2007): "Competition and Efficiency in Congested Markets," *Mathematics of Operations Research*," 32, 1-31.

[2] Acemoğlu, Daron and Robert Shimer (1999): "Efficient Unemployment Insurance," *Journal of Political Economy*, 107, 893-928.

[3] Albrecht, James, Pieter A. Gautier, and Susan Vroman (2006): "Equilibrium Directed Search with Multiple Applications," *Review of Economic Studies*, 73, 869-891.

[4] Albrecht, James, Pieter A. Gautier, Serene Tan, and Susan Vroman (2004): "Matching with Multiple Applications Revisited." *Economics Letters*, 84, 311-314.

[5] Berentsen, Alexander, Guido Menzio and Randall Wright (2008): "Inflation and Unemployment in the Long Run," mimeo.

[6] Burdett, Kenneth, Shouyong Shi and Randall Wright (2001): "Pricing and Matching with Frictions," *Journal of Political Economy*, 109, 1060-1085.

[7] Camera, Gabriele and Cemil Selcuk (2008): "Price Dispersion with Directed Search," *Journal of the European Economic Association*, forthcoming.

[8] Faig, Miguel and Belen Jerez (2006): "Inflation, Prices, and Information in Competitive Search," *Advances in Macroeconomics* 6 (1),

[9] Galenianos, Manolis and Philipp Kircher (2009): "Directed Search with Multiple Job Applications." *Journal of Economic Theory*, 114(2), 445-471.

[10] Galenianos, Manolis, Philipp Kircher and Gabor Virag (2009): "Market Power and Efficiency in a Search Model," mimeo.

[11] Eeckhout, Jan and Philipp Kircher (2009): "Sorting and Decentralized Price Competition," mimeo.

[12] Julien, Benoit, John Kennes, and Ian King: "Bidding for Labor." *Review of Economic Dynamics*, 3 (2000), 619-629.

[13] Kircher, Philipp (2008): "Efficiency of Simultaneous Search," mimeo.

[14] Korn, Granino A. and Theresa M. Korn (1968): *Mathematical Handbook for Scientists and Engineers: Definitions, Theorems and Formulas for Reference and Review*, McGraw-Hill, N.Y.

[15] Moen, Espen R. (1997): "Competitive Search Equilibrium," *Journal of Political Economy*, 105, 385-411.

[16] Moen, Espen R. and Asa Rosen (2007): "Incentives in Competitive Search Equilibrium," mimeo.

[17] Montgomery, James D. (1991): "Equilibrium Wage Dispersion and Interindustry Wage Differentials," *Quarterly Journal of Economics*, 106, 163-179.

[18] Peters, Michael (1984): "Bertrand Competition with Capacity Constraints and Mobility Constraints," *Econometrica*, 52, 1117-1128.

[19] Peters, Michael (1991): "Ex Ante Price Offers in Matching Games: Non-Steady States," *Econometrica*, 59 1425-1454.

[20] Peters, Michael (1997): "On the Equivalence of Walrasian and Non-Walrasian Equilibria in Contract Markets: The Case of Complete Contracts," *Review of Economic Studies*, 64, 241-264.

[21] Peters, Michael (2000): "Limits of Exact Equilibria for Capacity Constrained Sellers with Costly Search," *Journal of Economic Theory*, 95, 139-168.

[22] Peters, Michael (2009): "Unobservable Heterogeneity in Directed Search,"mimeo.

[23] Reny, Philip J. (1999): "On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games," *Econometrica*, 67, 1029-1056.

[24] Rocheteau, Guillaume, and Randall Wright (2005): "Money in Search Equilibrium, in Competitive Equilibrium, and in Competitive Search Equilibrium," *Econometrica*, 73, 175-202.

[25] Shi, Shouyong (2001): "Frictional Assignment I: Efficiency," *Journal of Economic Theory*, 98, 232-260.

[26] Shimer, Robert (2005): "The Assignment of Workers to jobs in an Economy with Coordination Frictions," *Journal of Political Economy*, 113/5, 996-1025.