

Reputational Incentives and Dynamics*

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Abstract

We study a dynamic moral hazard model, where a firm can invest into the quality of its product which, in turn, is imperfectly observed by consumers. We analyse how investment incentives depend on the firm's reputation and the information structure of consumer learning, and solve for the resulting reputational dynamics.

When consumers learn through good news, investment incentives are increasing in reputation, leading to a unique *work-shirk* equilibrium and convergent dynamics. When consumers learn through bad news, investment incentives are decreasing in reputation, leading to a continuum of *shirk-work* equilibria and divergent dynamics. Finally, when consumers learn through Brownian motion and the cost of investment is low, there is a work-shirk equilibrium, but no shirk-work equilibrium.

1 Introduction

In most industries firms can affect the quality of their products or services through research and development, human capital investment and organisational change. However, if the firm's customers can only imperfectly assess the investments and the resulting product quality, the value created by these investments does not immediately increase their willingness-to-pay. One way in which firms can reap some of the value created is through building a reputation for high quality, justifying premium prices. This paper analyses the incentives for investment in such a market, characterising how these incentives depend on the current reputation of the firm and solving for the long-term reputational dynamics.

The model captures key features of many important industries. In labour markets such as those for academics, artists and advertising executives, agents spend much of their time investing

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in skills and perfecting their trade. Their reputation and future compensation, however, depends heavily on their best paper, performance or campaign. In the computer industry, component manufacturers invest heavily into research and development while customers are only able to observe the performance of the entire computer. Customers therefore often learn about the quality of the product through newsworthy incidences, such as Dell's 2006 recall of 4 million Sony lithium-ion batteries.¹ In the car industry, firms devote considerable resources to improving quality standards through organisational change and new production processes. Since these actions are not observable, customers only learn about the true quality slowly, through consumer reports and the media.²

In the model, illustrated in Figure 1, one long-lived firm sells a product of high or low quality to a continuum of identical short-lived consumers. Product quality is a function of the firm's past investments. The quality then determines future prices through imperfect market learning: a high quality product generally leads to a higher consumer utility than a low quality product, but learning is obstructed by noise. At each point in time, consumers' willingness to pay is determined by the market belief that the quality is high, x_t , which we call the *reputation* of the firm. This reputation changes over time as a function of (a) the equilibrium beliefs of the firm's investments, and (b) market learning about the product quality. This market learning depends on the specifics of the signal structure. Our model nests three types of signal structures that have received attention in the literature on imperfect monitoring:

1. In the *good news* case the product usually generates constant utility. However, at random times a high-quality product enjoys a breakthrough, revealing its high quality. Such good news may occur in academia when a paper becomes famous, in the bio-tech industry when a trial succeeds, and for actors when they win an Oscar.
2. In the *bad news* case the product usually generates constant utility. However, at random times the low quality product suffers a breakdown, revealing its low quality. Such bad news may occur in the computer industry when batteries explode, in the airline industry when a plane crashes, and for doctors when they are sued for medical malpractice.
3. In the *Brownian news* case a high-quality product generates a higher mean utility than a low-quality product, but a normally distributed random error means that customers only learn slowly. As a result, beliefs changes continuously over time. Such continuous updating occurs as drivers learn about the build-quality of a car, as clients learn about the skills of a consultancy, and as callers learn about the customer service of a telephone service provider.

¹See "Dell to Recall 4m Laptop Batteries", Financial Times, 15th August 2006.

²See "Detroit Carmakers on a Journey to Recover Reputation", Financial Times, 24th December 2008.

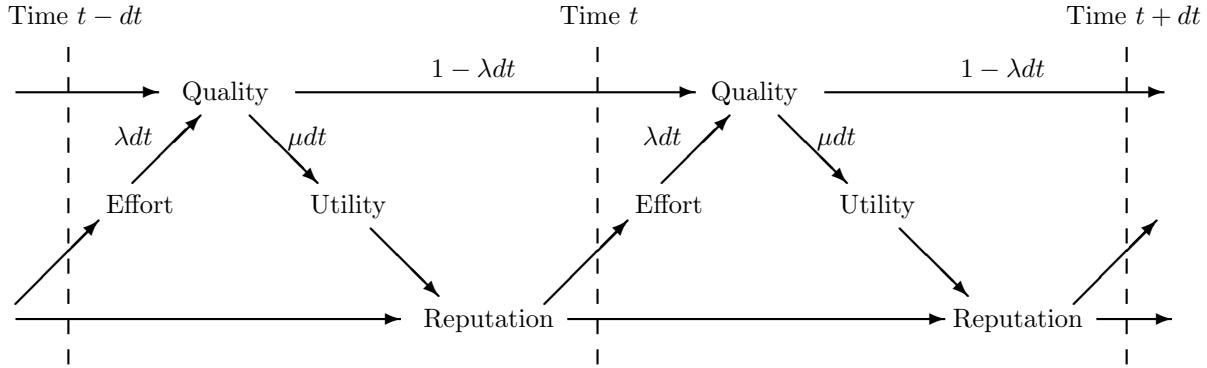


Figure 1: **Timeline.**

In a Markovian equilibrium the firm's value is a function of its product quality and its reputation. As illustrated in figure 1, both quality and reputation move slowly and therefore can be interpreted as assets, which the firm sometimes builds up and other times depletes. Reputation is valuable because it determines the firm's revenue. Quality in turn is valuable because it generates a *reputational dividend*: a high quality product yields higher expected utility to customers, increasing the firm's future reputation. Proposition 1 formalizes this idea by writing the asset value of quality, i.e. the difference in the value between a high quality firm and a low quality firm, as the net present value of its future reputational dividends. This formula is important because it is precisely this value of quality which determines the investment incentives of the firm. To analyse these incentives further, we consider the three information structures mentioned above.

In the *good news case* (Section 4) equilibria are *work-shirk*: The firm works if and only if its reputation lies below a cutoff x^* . Intuitively, the reputational dividend is an increase in the probability of a product breakthrough, revealing the firm's high quality and boosting its reputation to 1. Since the benefit of such a reputational boost decreases in the firm's reputation, so do investment incentives. The form of the equilibrium implies that reputational dynamics converge to a cycle: A firm with low reputation works, eventually jumps to reputation 1 where it starts shirking, and drifts down until it hits the cutoff and starts working again. We also show that an increase in believed effort lowers the firm's incentives to exert effort, implying that the equilibrium is unique.

The *bad news case* (Section 5) is in many ways the opposite to the good news case. Equilibria are *shirk-work*: The firm works if and only if its reputation lies above a cutoff x^* . Intuitively, the reputational dividend is insurance against a product breakdown, revealing the firm's low quality and driving its reputation to 0. Since the benefit of such insurance increases in the firm's reputation, so do investment incentives. The form of the equilibrium implies that reputational dynamics diverge: A firm with reputation below the cutoff shirks forever, causing its reputation to fall to 0; a firm with reputation above the cutoff works forever, causing its reputation to approach 1, unless it suffers a product breakdown while its product quality is still low. At the cutoff, the firm's

incentives depend discontinuously on market beliefs, leading to a continuum of equilibria.

Our analysis of the *Brownian news case* (Section 6) indicates that the good news results are more robust than those for bad news. When effort is sufficiently cheap, there is a unique work-shirk equilibrium. In contrast, there is no shirk-work equilibrium. This asymmetry hinges on the reputational drift due to equilibrium beliefs. When $x \approx 0$ and $x \approx 1$, market learning is slow and the reputational dividend is small. At the top work is not sustainable: if the firm is believed to work, the firm's reputation stays high and the reputational dividend stays small, undermining the incentive to invest. At the bottom work may be sustainable: if the firm is believed to work, the firm's reputation drifts up and reputational dividends increase, sustaining the incentive to invest. Crucially, a firm exerts effort at $x \approx 0$ not because of current reputational dividends, but because of those in the future. This argument is self-fulfilling: the firm works at low reputations because it is believed to work. This suggests another, *shirk-work-shirk*, type of equilibrium where a firm with a low reputation is trapped in a shirk-hole in which market learning is too slow to incentivise effort. Simulations show that such an equilibrium may exist, but both formal arguments and simulations indicate that it disappears for small costs.

We can link our results to models in which quality is chosen in every period (e.g. Klein and Leffler (1981), Mailath and Samuelson (2001)) by taking the obsolescence rate of quality λ to infinity. With complete information, an increase in λ frontloads the returns to investment and increases investment incentives. With incomplete information, there is a countervailing effect: For large values of λ , equilibrium beliefs dominate market learning in determining reputational dynamics. In the good news and Brownian news cases the distribution of reputations degenerates to a peak at the work-shirk cutoff, expected reputational dividends vanish, and we show (not quite yet in the Brownian case) that pure shirking is the only equilibrium. In the bad news case, to the contrary, investment incentives increase in the obsolescence rate and any shirk-work profile is sustainable as an equilibrium.

1.1 Theoretical Literature

Our paper forms a bridge between classical models of reputation with exogenous types and models of repeated games. In contrast to reputation models of Holmström (1999) and Mailath and Samuelson (2001), we suppose the state variable is the quality of the firm's product rather than some exogenous ability type of the firm (see figure 2). As a consequence, reputational dynamics are entirely driven by the reputation itself, rather than by exogenous type changes. In comparison to the repeated games literature, we suppose there is a state variable which links the periods.

To be more specific, Mailath and Samuelson (2001) consider a firm that sells a good of unknown quality. There are two types of firms: a competent firm who can choose high or low effort, and an inept firm who can only choose low effort. The actual product quality is then a noisy function of the firm's effort. From the consumer's perspective, utility is determined by the probability the

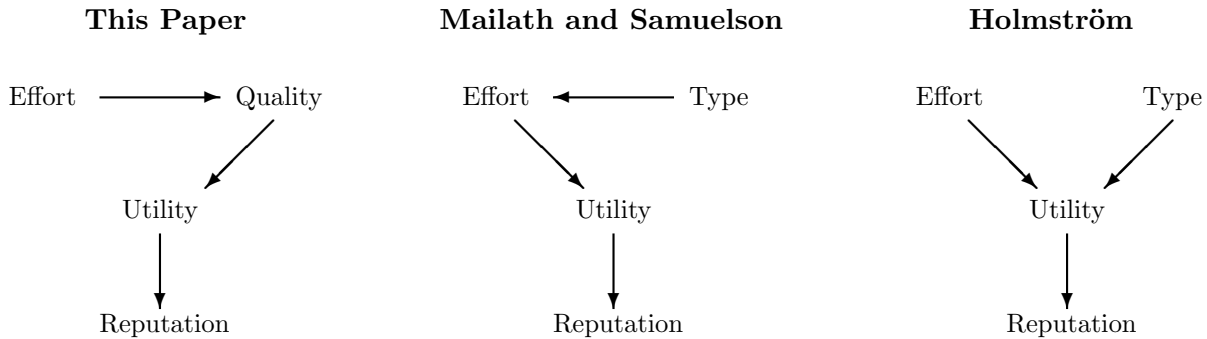


Figure 2: **Relation to Literature.** This figure shows the relationships between this paper, Mailath and Samuelson (2001) and Holmström (1999).

firm is competent (the firm’s reputation) multiplied by the probability a competent firm exerts effort.

Mailath and Samuelson derive a striking result: there is a unique Markov perfect equilibrium in pure strategies in which the competent firm always chooses low effort. When the reputation is close to 1, it is impossible to sustain high effort for the same reason as in our paper. Effort then unravels from the top: If the firm is known to be shirking when its reputation passes some cutoff, it has no incentive to exert effort just below this cutoff since success would mean an increase in reputation and an immediate collapse in the price. In contrast, in our paper, product quality is persistent. Thus, the price drifts down continuously when the firm starts to shirk and unravelling is prevented.

Holmström (1999) examines a signal-jamming model where an agent of unknown ability can exert effort to confuse the learning of her employer. When the agent’s type is constant, the employer gradually learns the agent’s ability, and effort declines over time. When the agent’s type exogenously changes over time, some effort level is sustained in the stationary equilibrium.³

When compared to the repeated games literature (e.g. Fudenberg, Kreps, and Maskin (1990)), our model has an evolving state variable. Nevertheless, the way that incentives for effort are determined by the signal structure in our paper echoes a similar theme in the repeated games literature. Abreu, Milgrom, and Pearce (1991) and Sannikov and Skrzypacz (2007a) consider a repeated prisoners’ dilemma game with imperfect public monitoring. They find that first-best is attainable as players become patient if the public signal indicates defection (\sim bad news), but is not attainable if the public signal indicates cooperation (\sim good news) or is generated by Brownian motion.⁴

³There is an important, wider literature on reputation models with moral hazard, where types are fixed (e.g. Kreps, Milgrom, Roberts, and Wilson (1982), Kreps and Wilson (1982), Diamond (1989) Fudenberg and Levine (1992)). A number of these models have been applied to experience goods. For example, Tadelis (1999, 2002) analyses the market for reputations, Hörner (2002) looks at the competition between firms, while Bar-Isaac (2003) considers the exit decision of firms.

⁴Also see Faingold and Sannikov (2007) on games with long-run and short-run players, and Sannikov and Skrzy-

Finally, our paper is related to contract design with persistent effort. Fernandes and Phelan (2000) suppose that an agent’s output depends today on effort both today and yesterday, and derive a recursive formulation to solve for the principal’s optimal contract. Jarque (2008) shows that the problem is much simpler when output depends on the geometric sum of past efforts and the cost of effort is linear. Unlike these papers, our consumers simply react to the firm’s actions, rather than designing contingent contracts.

1.2 Empirical Literature

There are a number of empirical papers examining the importance of reputation in internet auctions (eBay). Resnick, Zeckhauser, Swanson, and Lockwood (2006) find that a new seller obtains significantly lower prices than a seller with a good feedback score. Cabral and Hortaçsu (2009) similarly find that a seller with negative feedback obtains significantly lower prices. More interesting, Cabral and Hortaçsu (2009) look at the seller’s reactions, showing that a seller who receives one negative feedback is more likely to obtain a second negative feedback, and is more likely to exit. This suggests that either underlying quality is correlated over time, or a seller who receives a negative feedback exerts less effort, as in our bad news case.

Studies have also examined the role of reputation in other markets. In the airline industry, a crash reduces the stock market value of the airline and manufacturer in question, reduces demand for all aviation, but increases the value of firms who compete directly with the crashed airline (Chalk (1987), Borenstein and Zimmerman (1988), Bosch, Eckard, and Singal (1998)). In the restaurant market, the introduction of grade cards increased investments in hygiene, and had the biggest effect on non-chain restaurants (Jin and Leslie (2003, 2009)). In the vehicle emission testing market, garages with higher pass rates can demand higher prices (Hubbard (1998, 2002)). In all of these cases, firms make investments that affect the quality of the product, and hence their reputation. While these studies demonstrate the importance in maintaining a reputation, there is little evidence on the effect of reputation on the firm’s investment incentives, as examined in this paper.

2 Model

Basics: Time $t \in [0, \infty)$ is continuous and infinite. The common interest rate is $r \in (0, \infty)$.

Firm and Consumers: There is one firm and a continuum of consumers. At any point in time t the firm’s product can have high or low quality, $\theta_t \in \{L = 0, H = \mu\}$, where $\mu > 0$. The expected instantaneous value of the product to the consumer equals the increment of a stochastic

pacz (2007b) on simultaneous move games with Brownian and Poisson news.

process $dZ_t = dZ_t(\theta_t, \varepsilon_t)$ with expected value $E[dZ_t] = \theta_t dt$ and stochastic component ε_t that is independent over time. We will often focus on three special cases:

- (a) *Good news*: $dZ_t = 0$ almost always but a good product has a *breakthrough* with arrival rate μ generating consumer utility of $dZ_t = 1$
- (b) *Bad news*: $dZ_t = \mu dt$ almost always but a bad product has a *breakdown* with arrival rate μ generating consumer disutility of $dZ_t = -1$
- (c) *Brownian news*: $dZ_t = \theta_t dt + dW_t$

Strategies: At time t the firm chooses effort $\eta_t \in [0, 1]$ at cost $c\eta_t dt$. Product quality θ_t is a function of past effort $\eta_{s \leq t}$ via a Poisson process with arrival rate λ (independent of dZ_t) that models quality obsolescence. Absent a shock quality is constant: $\theta_{t+dt} = \theta_t$, while at a shock previous quality becomes obsolescent and quality is determined by the level of investment: $Pr(\theta_{t+dt} = \mu) = \eta_t$. This implies $Pr(\theta_t = H) = \int_0^t \lambda e^{\lambda(s-t)} \eta_s ds + e^{-\lambda t} Pr(\theta_0 = H)$.

Information: Realized consumer utility dZ_t is public information, while actual product quality θ_t is observed only by the firm.⁵ The market belief about product quality $x_t = Pr(\theta_t = \mu)$ at time t is called the firm's reputation.

Reputation Updating: The reputation increment $dx_t = x_{t+dt} - x_t$ is governed by realized consumer utility dZ_t and believed effort $\tilde{\eta}_t$. As these are independent, dx_t can be decomposed additively:

$$dx_t = \lambda(\tilde{\eta}_t - x_t)dt + x_t(1 - x_t) \frac{\Pr(dZ_t|H) - \Pr(dZ_t|L)}{x_t \Pr(dZ_t|H) + (1 - x_t) \Pr(dZ_t|L)}. \quad (2.1)$$

$d_\theta x_t$ denotes the increments conditional on quality θ .⁶

Profit and Consumer Surplus: At time t the firm sets price equal to the expected value μx_t . While consumers get 0 in expectation, the firm's instantaneous profit is $(\mu x_t - c\eta_t)dt$ and its discounted present value is thus given by⁷

$$V_\theta(x; \eta, \tilde{\eta}) := r \int_{t=0}^{\infty} e^{-rt} \mathbb{E}_{\theta_0=\theta, x_0=x, \eta, \tilde{\eta}} [\mu x_t - c\eta_t] dt. \quad (2.2)$$

⁵The assumption that the firm knows the quality of its product is irrelevant as long as cost functions are symmetric.

⁶The reason for modeling time as continuous is purely pragmatic. If time was measured in discrete periods, the updating equation (2.1) would be complicated by a dt^2 term because in every period market learning would already take the equilibrium effort decision into account.

⁷That the firm's value $V_\theta(x; \eta, \tilde{\eta})$ is independent of history depends on the focus on Markovian effort and beliefs, justified in Lemma 1.

Markov-Perfect-Equilibrium: We assume Markovian beliefs $\tilde{\eta} = \tilde{\eta}(x)$ and show below that optimal effort $\eta = \eta(x)$ is independent of history and current product quality θ . A Markov-Perfect-Equilibrium $\langle \eta, \tilde{\eta} \rangle$ consists of a Markovian effort function $\eta : [0, 1] \rightarrow [0, 1]$ for the firm and Markovian market beliefs $\tilde{\eta} : [0, 1] \rightarrow [0, 1]$ such that 1) $\eta \in \eta^*(\tilde{\eta}) := \arg \max_{\eta} \{V_{\theta}(x; \eta, \tilde{\eta})\}$ maximizes firm value $V_{\theta}(x; \eta, \tilde{\eta})$, given x and $\tilde{\eta}$, and 2) market beliefs are correct: $\tilde{\eta} = \eta$. In a Markovian equilibrium η we will write the firm's value as a function of its quality and its reputation: $V_{\theta}(x)$.

2.1 Optimal Investment Choice

In principle, the firm's effort choice η as well as market beliefs $\tilde{\eta}$ could depend on the entire public history $dZ^t = (dZ_s)_{0 \leq s < t}$, as well as the private history $\theta^t = (\theta_s)_{0 \leq s < t}$ and time t . We assume that market beliefs are Markovian $\tilde{\eta}$ because we think of the continuum of consumers as sharing their experience in an imperfect way, e.g. through consumer reports. For Markovian beliefs $\tilde{\eta}$ all payoff relevant parameters at time t depend on the history only via the current product quality θ_t and the firm's reputation x_t . Thus, the optimal effort choice of the firm only needs to depend on these two parameters.

The benefit of effort in $[t; t + dt]$ is the probability of a technology shock hitting, λdt , times the difference in value functions $\Delta(x) := V_H(x) - V_L(x)$, which we will refer to as the *asset value of quality*. The marginal cost of investment is rc , and thus optimal effort $\eta(x)$ is given by

$$\eta(x) = \begin{cases} 1 & \text{if } rc < \lambda \Delta(x) \\ 0 & \text{if } rc > \lambda \Delta(x) \end{cases}, \quad (2.3)$$

independently of the current quality of the firm θ_t !

Lemma 1 summarizes this discussion:

Lemma 1 *For Markovian beliefs $\tilde{\eta}(x)$ there is an optimal Markovian effort function $\eta(x)$ that depends solely on the firm's reputation but not on its product quality. Additionally, $\eta(x)$ satisfies equation 2.3.*

Equation (2.3) makes the model tractable and is the reason that we assume the costs of effort to be independent of product quality and past effort. An implication of equation (2.3) is that our results are not at all driven by the asymmetric information about product quality θ , but solely by the unobserved investment η into future quality.

2.2 Cutoff Equilibria and Reputational Dynamics

We call an equilibrium *work-shirk*, if there exists a cutoff x^* such that a firm with low reputation $x < x^*$ is exerting effort $\eta(x) = 1$, whereas a firm with a high reputation $x > x^*$ does not $\eta(x) = 0$. The opposite case, where low reputations shirk and high reputations work, is called a *shirk-work* equilibrium.

Reputational dynamics of work-shirk equilibria are fundamentally different from those of shirk-work equilibria. Net of market learning, the dynamics $dx = \lambda(\tilde{\eta}_t - x_t)dt$ are convergent in a work-shirk equilibrium, i.e. $dx > 0$ for $x < x^*$ and $dx < 0$ for $x > x^*$, but divergent in a shirk-work equilibrium. Accordingly we will find that in the good news case, where all equilibria are work-shirk, a firm with any initial reputation x_0 , will eventually find itself circling around the cutoff reputation x^* . In contrast, in the bad news case where all equilibria are shirk-work, reputations will diverge. Either the firm starts with a low reputation, shirks and its reputation converges to 0, or it starts with a high reputation, works and unless a breakdown hits while its quality is low its reputation converges to 1.

A set of reputations $S \subseteq [0, 1]$ is called a *shirk-hole* if the firm shirks $\eta(x) = 0$ for all $x \in S$ and S is closed under reputational dynamics in that $\Pr(x_t \in S) = 1$ if $x_0 \in S$. For future use we define $\Delta_{x^*}(x)$ to be the asset value of quality for a firm with reputation x when both actual effort η and believed effort $\tilde{\eta}$ are work-shirk with cutoff x^* .

3 General Results

To truly understand the reputational incentives and dynamics of the firm, we will need to take a stance on the market information structure. We do this in Sections 4, 5, and 6. In this section we prepare the ground-work that is common among good, bad and Brownian news, most importantly the concept of reputational dividends and Proposition 1.

3.1 Welfare

Suppose product quality is publicly observed. Then the benefit of exerting effort equals the obsolescence rate λ times the price differential μ divided by the effective discount rate $r + \lambda$. Thus the first-best effort choice is given by:

$$\eta = \begin{cases} 1 & \text{if } c < \frac{\lambda}{r+\lambda}\mu \\ 0 & \text{if } c > \frac{\lambda}{r+\lambda}\mu \end{cases} . \quad (3.1)$$

There is no equilibrium with positive effort if $c > \frac{\lambda}{r+\lambda}\mu$. In this case, welfare is negative and, as consumers receive zero utility in equilibrium, the firm makes negative profits. The firm therefore prefers to shirk at all levels of reputation, thereby guaranteeing itself a non-negative payoff.

We thus restrict attention in the paper to the case $c < \frac{\lambda}{r+\lambda}\mu$.

3.2 Value of Quality and Reputation

We noted above that the firm's value in a MPE is a function of its quality and its reputation: $V = V_\theta(x)$. The value of the firm's current reputation x derives from its effect on current price μx and future prices $\mu x_t|_{x_0=x}$, as reputational dynamics are in expectation inert. Quality θ on the other hand derives its value indirectly, through its effect on future reputation. In this subsection we will confirm that in equilibrium $V_\theta(x)$ is increasing in reputation x and that $V_H(x) \geq V_L(x)$.⁸ In doing so, we will also derive a useful formula that interprets $\Delta(x)$ as the NPV of reputational dividends.

Lemma 2 *Given an optimal response to market beliefs $\eta^*(\tilde{\eta})$, the value function of the firm $V_\theta(x; \eta^*(\tilde{\eta}), \tilde{\eta})$ is strictly increasing in its reputation x and increasing in market beliefs $\tilde{\eta}$.*

Proof. Fix $(x', \tilde{\eta}')$ and $(x', \tilde{\eta}') \geq (x'', \tilde{\eta}'')$, i.e. $x'' \geq x'$ and $\tilde{\eta}''(x) \geq \tilde{\eta}'(x)$ for all x . We can rewrite the best response $\eta^*(\tilde{\eta})$ to the Markovian beliefs $\tilde{\eta}'$ in a non-Markovian way as a function of the public history $\eta^*(\tilde{\eta}) = \eta_t(dZ^t)$.

Now consider the firm with the higher reputation $(x'', \tilde{\eta}'')$. If it exerts effort according to the non-Markovian strategy $\eta_t(dZ^t)$ ⁹ it will by construction have the same quality θ_t and exert the same effort η_t for every time t and every realization of the random processes. On the other hand, the reputation x_t will always be weakly higher when it starts at $x'' > x'$ because the updating equation (2.1) implies that the future reputation x_{t+dt} is increasing in the reputation x_t and the Markovian beliefs $\tilde{\eta}$. Thus, by mimicking the effort of the firm with the lower initial reputation x' the firm with the higher initial reputation can secure itself a strictly higher value. By Lemma 1 there must be a Markov strategy that is at least as good as this mimicking strategy. \square

This lemma implies that across equilibria η, η' , with $\eta'(x) \geq \eta(x)$ for all x , the firm's value is increasing in effort $V_\theta(x; \eta', \eta') \geq V_\theta(x; \eta, \eta)$. Corollary 1 complements this lemma by showing that $V_H(x; \eta^*(\tilde{\eta}), \tilde{\eta}) \geq V_L(x; \eta^*(\tilde{\eta}), \tilde{\eta})$.

As shown in Section 2.1, investment incentives are driven by the asset value of quality $\Delta(x)$. To analyse this value, we can decompose it into (a) the immediate benefit of a positive market signal, called the *reputational dividend*, and (b) the continuation benefit of a high quality product:

$$\begin{aligned} \Delta(x) &= (1 - rdt)(1 - \lambda dt)\mathbb{E}[V_H(x + d_Hx) - V_L(x + d_Lx)] \\ &= (1 - rdt - \lambda dt)\mathbb{E}[(V_H(x + d_Hx) - V_H(x + d_Lx)) + \Delta(x + d_Lx)]. \end{aligned} \tag{3.2}$$

⁸While it is unclear whether these monotonicity results hold for any η and $\tilde{\eta}$, Lemma 6 in Appendix C.1 extends them to work-shirk effort functions where the cutoff type x^* is indifferent between working and shirking.

⁹This strategy is Markovian when the dynamics start at x' but it will in general be non-Markovian when it starts at $x'' \neq x'$.

The first line uses the principle of dynamic programming, while the second adds and subtracts $V_H(x + d_Lx)$. Integrating up yields equation (3.3) in Proposition 1, which expresses the asset value of quality as the discounted sum of future reputational dividends. Equation (3.4) follows from the alternative decomposition of (3.2) when we add and subtract $V_L(x + d_Hx)$ instead of $V_H(x + d_Lx)$. These expressions serve as a work-horses throughout the paper.

Proposition 1 *Fix any Markovian beliefs $\tilde{\eta}$ and a Markovian best response $\eta^*(\tilde{\eta})$. Then two closed-form expressions for the value of quality $\Delta(x)$ are given by*

$$\Delta(x) = \int_0^\infty e^{-(\lambda+r)t} \mathbb{E}_{x_0=x, \theta_{s \leq t}=L} [D_H(x_t)] dt \quad (3.3)$$

$$= \int_0^\infty e^{-(\lambda+r)t} \mathbb{E}_{x_0=x, \theta_{s \leq t}=H} [D_L(x)] dt \quad (3.4)$$

where the reputational dividend $D_\theta(x)$ is defined by

$$D_\theta(x) := \mathbb{E}[V_\theta(x + d_Hx) - V_\theta(x + d_Lx)].$$

Proof. To integrate up (3.2), fix x and set $\psi(t) := \mathbb{E}_{x_0=x, \theta_{s \leq t}=L} [\Delta(x_t)]$. Up to terms of order $o(dt)$ we have

$$\begin{aligned} -d \left(\psi(t) e^{-(r+\lambda)t} \right) &= -e^{-(r+\lambda)t} (\psi(t+dt) - \psi(t) - (r+\lambda) dt \psi(t)) \\ &= e^{-(r+\lambda)t} \mathbb{E}_{x_0=x, \theta_{s \leq t}=L} [D_H(x_t)] \end{aligned}$$

and (3.3) follows. \square

Corollary 1 *Fix any Markovian beliefs $\tilde{\eta}$ and a Markovian best response $\eta^*(\tilde{\eta})$. For a given reputation x , a high-quality firm has a higher value than a low-quality firm, i.e. $V_H(x) \geq V_L(x)$.*

Proof. By the updating equation (2.1) we have $d_Hx \geq d_Lx$, by Lemma 2 we get $D_\theta(x) = V_\theta(x_t + d_Hx_t) - V_\theta(x_t + d_Lx_t) \geq 0$, and finally by Proposition 1 we get $\Delta(x) = V_H(x) - V_L(x) \geq 0$.

\square

3.3 Investment Levels

Lemma 3 (Some Effort Somewhere) *For sufficiently low costs c , pure shirking, i.e. $\eta(x) = 0$ for all x , is not an equilibrium.*

Proof. Fixing $\eta(x) = \tilde{\eta}(x) = 0$, value functions and Δ do not depend on c and $\Delta(x) > 0$ for some x . Thus, as the cost of effort vanish $c \rightarrow 0$, its benefits $\lambda\Delta(x)$ are constant and thus bounded away from 0, contradicting the equilibrium condition. \square

This result is in contrast to reputational models with inept firms where shirking is always an equilibrium, even if costs of working are 0. As discussed in the introduction the critical difference in this model is that expected quality and price still depend on the firm's reputation - and in particular are greater than 0 - even when the firm is believed to be shirking.

Lemma 4 (No Effort at the Top) *If no market signal dZ_t perfectly reveals low quality, i.e. if $\frac{\Pr(dZ_t|L)}{\Pr(dZ_t|H)} < \infty$ for all dZ_t , then a firm with a perfect reputation $x = 1$ must be shirking with positive probability $\eta(1) < 1$.*

Proof. Assume to the contrary that $\eta(1) = 1$ in equilibrium. By the assumption that $\frac{\Pr(dZ_t|L)}{\Pr(dZ_t|H)} < \infty$ a firm with perfect reputation $x = 1$ can never lose its reputation, irrespectively of the consumer utility dZ_t it generates. Thus $d_H x_t = d_L x_t = 0$ and by Proposition 1 the asset value of quality $\Delta(x)$ equals 0. Thus, full effort $\eta(1) = 1$ can not be sustained in equilibrium. \square

The proof actually shows a slightly stronger statement: In no equilibrium the firm can be working for all reputation levels x in an interval of arbitrary high reputations $(1 - \varepsilon, 1)$.

4 Good News

Assume that consumers learn about quality θ_t from infrequent product *breakthroughs* that reveal $\theta = H$ with arrival rate μ . Absent a breakthrough, updating evolves deterministically according to

$$\frac{dx}{dt} = \lambda(\eta(x) - x) - \mu x(1 - x). \quad (4.1)$$

We can define x_t as the deterministic solution of the ODE (4.1) with initial value x_0 .

The reputational dividend, which is the value of having a high quality in the next instant, equals the value of increasing the reputation from x to 1 times the probability of a breakthrough. That is,

$$D_H(x) = V_H(x + d_H x) - V_H(x + d_L x) = \mu(V_H(1) - V_H(x))dt$$

Conditioning on the firm always being low quality, in order to prevent a breakthrough, the time path of reputation is given by (4.1). Using equation (3.3), the asset value of quality is

$$\Delta(x_0) = \int_0^\infty e^{-(\lambda+r)t} \mu[V_H(1) - V_H(x_t)]dt \quad (4.2)$$

The reputational dividend $V_H(1) - V_H(x_t)$ is decreasing in x_t , so that $\Delta(x)$ is decreasing in x . It follows that any equilibrium is work-shirk. Intuitively, when a breakthrough occurs the firm's reputation immediately jumps to 1. Since this jump is larger for a low-reputation firm, incentives to invest decrease in reputation and the equilibrium is *work-shirk*.

The form of the equilibrium implies that the reputational dynamics converge to a cycle. Absent a breakthrough, a firm's reputation converges to a stationary point $\hat{x} = \min\{\lambda/\mu, x^*\}$ where the firm works with positive probability. When a breakthrough occurs a firm's reputation jumps to 1. The firm is then believed to be shirking, so its reputation drifts down to \hat{x} , absent another breakthrough. In the long-run, the firm's reputation therefore cycles over the range $[\hat{x}, 1]$.

Proposition 2 *In the good news case*

- (a) *Every equilibrium is work-shirk.*
- (b) *Reputational dynamics converge to a non-trivial cycle.*
- (c) *If $\lambda \geq \mu$ the equilibrium is unique.*
- (d) *For sufficiently high λ , zero-effort is the only equilibrium.*

Proof. Part (a). Reputation x_t follows (4.1), so an increase in x_0 raises x_t at each point in time. Lemma 2 says that $V_H(x)$ is strictly increasing in x , so equation (4.2) implies that $\Delta(x_0)$ is decreasing in x_0 . Part (b) follows from (a).

Part (c). Suppose $\lambda \geq \mu$ and fix a cutoff x^* . Since $\lambda/\mu > x^*$, it is a best response for the firm to shirk for all reputation levels $x \geq x^*$. Consider the dynamic process (4.1), denoted by x_t , starting at $x_0 = 1$, and let $\tau(x^*)$ be the time it hits x^* . In Appendix A.1 we show that the asset value of quality at reputation x^* is given by

$$\Delta_{x^*}(x^*) = \frac{\mu}{r + \lambda} \int_{t=0}^{\tau(x^*)} e^{-rt} r \mu (x_t - x^*) \left[\frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\mu + \lambda)t} \right] dt \quad (4.3)$$

Intuitively, $\Delta_{x^*}(x^*)$ equals the discounted probability that a breakthrough arrives before a technology shock (the term before the integral), multiplied by the value of such a breakthrough (the integral). Differentiating,

$$\frac{d}{dx^*} \Delta_{x^*}(x^*) = -\frac{\mu}{r + \lambda} \int_{t=0}^{\tau(x^*)} e^{-rt} r \mu x^* \left[\frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\mu + \lambda)t} \right] dt$$

which is strictly negative. Hence there is at most one x^* at which $\lambda \Delta_{x^*}(x^*) = rc$.

Part (d). Let $x^* = 0$, and observe that $\tau(0) = \infty$. Using (4.3) and the fact that $x_t \leq e^{-\lambda t}$,

$$\lambda \Delta_0(0) \leq \frac{\lambda}{r + \lambda} \mu \int_{t=0}^{\infty} e^{-(r + \lambda)t} r \mu dt = \frac{\lambda}{r + \lambda} \frac{r \mu^2}{r + \lambda}$$

When λ is sufficiently high $\lambda \Delta_0(0) < rc$, and the unique equilibrium exhibits zero effort. \square

Suppose that $\lambda \geq \mu$, as shown in the left panel of Figure 3. In this case, the firm's reputation drifts up whenever it is known to be working, which means that the stationary point will be $\hat{x} = x^*$.

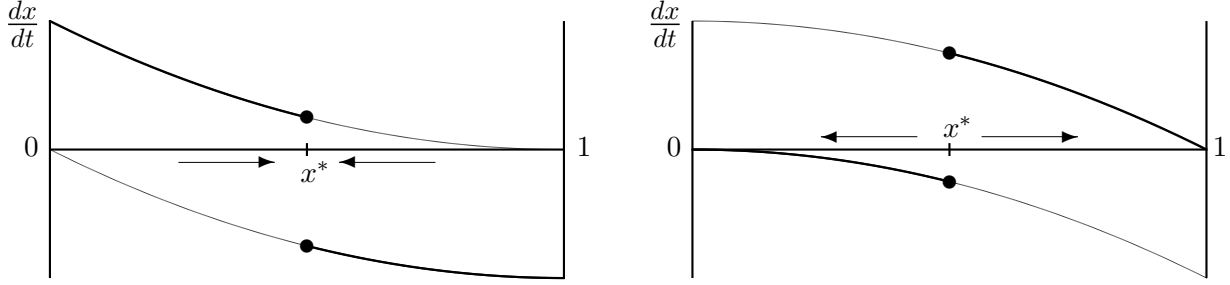


Figure 3: **Dynamics in Good News (left) and Bad News (right)**. This picture illustrates how the reputational drift dx/dt , absent a breakthrough, changes with the reputation of the firm, x . These pictures assume $\lambda = \mu$ and $x^* = 1/2$. The dark line shows the drift under the optimal effort.

At this cutoff the firm randomises so as to keep its reputation constant, choosing to invest with probability $\eta(x^*) = x^* \left(1 + \frac{\mu}{\lambda} (1 - x^*)\right)$. Proposition 2(c) shows that the equilibrium is unique. To understand this result, suppose the market believes the cutoff is \tilde{x} , and denote the firm's best response by $x^*(\tilde{x})$. An increase in \tilde{x} means the firm's reputation will not drift down as far, absent a breakthrough. This change benefits low-quality firms more than high-quality firms, reducing $\Delta(x)$. As a result, $x^*(\tilde{x})$ is decreasing in \tilde{x} and there is a unique fixed point where $x^*(\tilde{x}) = \tilde{x}$.

Proposition 2(d) says that, as technology shocks become more frequent the incentives to exert effort disappear. Intuitively, when λ is high, equilibrium beliefs move the reputation towards x^* very quickly. Hence a breakthrough, which raises the reputation to 1, is quickly depreciated. This means there is little value to being a high-quality firm and little value to investing. As discussed in the Introduction, Proposition 2(d) implies that investment is non-monotone in the frequency of technology shocks. When shocks are too frequent, market beliefs drive reputation and so there is no incentive to invest in actual quality. When shocks are too infrequent, investment takes too long to pay off and, again, there is little incentive to invest in quality. For intermediate frequencies, however, effort can be supported.

5 Bad News

Assume that x_t is generated by *breakdowns* that reveal $\theta = L$ with arrival rate μ . Absent a breakdown, updating evolves deterministically according to

$$\frac{dx}{dt} = \lambda(\eta(x) - x) + \mu x(1 - x). \quad (5.1)$$

Thus, we can define x_t as the solution of the ODE (5.1) with initial value x_0 .

The reputational dividend, which is the value of having a high quality in the next instant,

equals the value of not losing one's reputation times the probability of a breakdown. That is,

$$D_L(x) = V_L(x_t + d_H x_t) - V_L(x_t + d_L x_t) = \mu(V_L(x_t) - V_L(0))dt$$

Conditioning on the firm always being high quality, in order to prevent a breakdown, the time path of reputation is given by (5.1). Using equation (3.4), the asset value of quality is

$$\Delta(x_0) = \int_0^\infty e^{-(\lambda+r)t} \mu[V_L(x_t) - V_L(0)]dt. \quad (5.2)$$

The jump size $V_H(x_t) - V_H(0)$ is increasing in x_t , so that $\Delta(x)$ is increasing in x . It follows that any equilibrium is shirk-work. Intuitively, when a breakdown occurs a firm immediately jumps to reputation $x = 0$. Since this jump is larger for a high-reputation firm, incentives to invest increase in reputation and the equilibrium is shirk-work.¹⁰

The form of the equilibrium implies that the reputational dynamics diverge. Suppose we are in an equilibrium with $x^* \in (0, 1)$, so firms below x^* shirk while firms above x^* work. A firm that starts with reputation above x^* converges to reputation $x = 1$, absent a breakdown. If such a breakdown does occur before the firm become high-quality, then it becomes stuck in a shirk-hole with reputation $x = 0$. A firm with reputation below x^* initially shirks and may have either rising or falling reputation, depending upon parameters. In either case, its reputation will either end up at $x = 0$ or $x = 1$.

Proposition 3 uses the following non-triviality assumption:

$$\frac{\lambda}{r + \lambda + \mu} \mu(\mu - c) > rc \quad (\text{PE})$$

Assumption (PE) is necessary and sufficient to have an equilibrium with positive effort.

Proposition 3 *Assume (PE) holds. In the bad news case*

- (a) *Every equilibrium is shirk-work.*
- (b) *If $x^* \in (0, 1)$ then reputational dynamics diverge to 0 or 1.*
- (c) *If $\lambda \geq \mu$ there is an interval of equilibria with cutoffs $x^* \in [a, b]$ and $b > a$.*
- (d) *If λ is sufficiently high, every $x^* \in (0, 1]$ defines an equilibrium.*

¹⁰For example, when writing about the explosion of Sony's batteries in Dell's laptops the Financial Times wrote that "The withdrawal comes at a sensitive time for [Dell], which has been fighting broader perceptions of poor customer service and slowing sales growth. [However] it could have a deeper impact on Sony, given the Japanese company's reputation for quality in the consumer electronics industry" (15th August 2006). This illustrates the point that the jump in reputation is larger for higher reputation firms.

Proof. Part (a). Reputation x_t follows (5.1), so an increase in x_0 raises x_t at each point in time. Lemma 2 says that $V_L(x)$ is strictly increasing in x , so equation (5.2) implies that $\Delta(x_0)$ is increasing in x_0 . Part (b) follows from (a).

Part (c). If $\lambda \geq \mu$ the dynamics are divergent at x^* : if $x_0 = x_t - \epsilon$, then x_t drifts down to $x = 0$; if $x_0 = x_t + \epsilon$, then x_t drifts up to $x = 1$, absent a breakdown. Lemma 2 says that $V_L(x_t)$ is strictly increasing in x_t , so (5.2) implies that for $x^* \in (0, 1)$,

$$\Delta_{x^*}^+(x^*) := \lim_{x \nearrow x^*} \lambda \Delta_{x^*}(x) > \lim_{x \searrow x^*} \lambda \Delta_{x^*}(x) =: \Delta_{x^*}^-(x^*). \quad (5.3)$$

A cutoff x^* thus defines a shirk-work equilibrium if

$$\lambda \Delta_{x^*}^+(x^*) \geq rc \geq \lambda \Delta_{x^*}^-(x^*) \quad (5.4)$$

If $\lambda \geq \mu$ then $V_L(0) = 0$. Equation (5.2) then implies that $\Delta_{x^*}^+(x^*)$ and $\Delta_{x^*}^-(x^*)$ are increasing in x^* and equation (5.4) defines an interval of cutoffs, $[a, b]$.

We wish to show that $b > a$. First, observe that $\lambda \Delta_0^-(0) = 0$ so $b > 0$. Second, observe that for any cutoff x^* , $V_L(1) = \frac{\lambda+r}{\lambda+r+\mu}(\mu - c)$ and $V_L(0) = 0$. Hence equation (5.2) yields $\lambda \Delta_1(1) = \frac{\lambda}{r+\lambda+\mu} \mu(\mu - c)$, and (PE) implies that $a < 1$. Since either a or b are interior, (5.3) implies that $b > a$.

Part (d). Pick any $x^* > 0$. First suppose $x > x^*$ observe that $x_t \geq 1 - e^{-\lambda t}$. In Appendix B.1 we derive the closed form for the asset value of value of quality. Taking limits,

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \lambda \Delta_{x^*}(x) &= \lim_{\lambda \rightarrow \infty} \frac{\lambda \mu}{\lambda + \mu} \int_{t=0}^{\infty} e^{-rt} r(\mu x_t - c)(1 - e^{-(\lambda+\mu)t}) dt \\ &\geq \lim_{\lambda \rightarrow \infty} \frac{\lambda \mu}{\lambda + \mu} r \int_{t=0}^{\infty} e^{-rt} (\mu(1 - e^{-\lambda t}) - c)(1 - e^{-(\lambda+\mu)t}) dt = \mu(\mu - c) \end{aligned} \quad (5.5)$$

where the final equality uses the fact that the integral converges to $(\mu - c)/r$. Assumption (PE) implies that $\mu(\mu - c) > rc$. Hence for sufficiently large λ , working is optimal for all $x > x^*$ and any x^* .

Next suppose $x < x^*$ and observe that $x_t \leq e^{-(\lambda-\mu)t}$. In Appendix B.1 we derive the closed form for the asset value of value of quality. Taking limits,

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \lambda \Delta_{x^*}(x) &= \lim_{\lambda \rightarrow \infty} \frac{\lambda \mu}{\lambda - \mu} \int_{t=0}^{\infty} e^{-rt} r \mu x_t (e^{-\mu t} - e^{-\lambda t}) dt \\ &\leq \lim_{\lambda \rightarrow \infty} \frac{\lambda \mu}{\lambda - \mu} \int_{t=0}^{\infty} e^{-rt} r \mu e^{-(\lambda-\mu)t} dt = 0 \end{aligned} \quad (5.6)$$

Hence for sufficiently large λ , shirking is optimal for all $x < x^*$ and any x^* . \square

Suppose $\lambda \geq \mu$, so that whenever the firm is known to be shirking its reputation drifts down

(see figure 3). In this case, the region below x^* is a shirk-hole: when a firm's reputation is below the cutoff, it is certain to see its reputation decrease because of the unfavorable market beliefs. In equilibrium it always shirks, eventually giving rise to a low quality product and a product breakdown destroying whatever is left of its reputation. When a firm's reputation is above the cutoff, favorable market beliefs contribute to an increasing reputation and the firm invests to insure itself against a product breakdown. At the cutoff, the firm works when it is believed to be working and shirks whenever it is believed to be shirking.

Proposition 3(c) shows that there are an interval of equilibrium cutoffs satisfying (5.4). The multiplicity is driven by a discontinuity in the value function at x^* , caused by the divergent reputational dynamics. Intuitively, the market's beliefs become self-fulfilling. If the market believes the firm is shirking, its reputation falls, undermining any incentive to invest. Conversely, if the market believes the firm is working, its reputation rises, causing the firm to invest in order to avoid a breakdown.

When $\lambda < \mu$ the dynamics are somewhat different, although multiplicity may remain. Define $\tilde{x} = 1 - \frac{\lambda}{\mu}$ to be the stationary point in the dynamics when the firm is believed to be shirking. There are two types of equilibria, which may coexist.

1. *Trapped equilibria.* When $\tilde{x} < x^*$, a firm with reputation $x \in (0, x^*)$ finds its reputation converging to \tilde{x} , and remains stuck in a shirk-hole. At some point it suffers a breakdown and remains at $x = 0$ thereafter. Since the dynamics are divergent at x^* the value function is discontinuous, and there is an interval of such equilibria.
2. *Permeable equilibria.* When $\tilde{x} > x^*$, a firm with reputation $x \in (0, x^*)$ finds its reputation increasing. If it does not suffer a breakdown, it moves into the work-region and its reputation converges to one. Since the dynamics are convergent at x^* , there is at most one permeable equilibrium.

Finally, Proposition 3(d) shows that as technology shocks become more frequent, then any cutoff can be an equilibrium. Intuitively, a firm that starts below the cutoff finds its reputation falling to zero instantly and gives up, while a firm above the cutoff finds its reputation rising to one instantly and works hard. While outside the model, this multiplicity creates an incentive for firms to invest in marketing in order to shape consumers expectations.

6 Brownian News

Assume that consumers gradually learn about quality θ_t through the evolution of a Brownian motion with state-dependent drift, $dZ_t = \theta_t dt + dW_t$. Updating then evolves according to

$$\begin{aligned} d_H x &= \lambda(\eta(x) - x)dt + \mu^2 x(1-x)^2 dt + \mu x(1-x)dW \\ d_L x &= \lambda(\eta(x) - x)dt - \mu^2 x^2(1-x)dt + \mu x(1-x)dW. \end{aligned}$$

To calculate the value of quality we apply Itô's formula to get:

$$\begin{aligned} \mathbb{E}_x[V_\theta(x + d_H x)] &= V_\theta(x) + \mu^2 x(1-x)^2 V'_\theta(x) + \frac{(\mu x(1-x))^2}{2} V''_\theta(x) \\ \mathbb{E}_x[V_\theta(x + d_L x)] &= V_\theta(x) - \mu^2 x^2(1-x)^2 V'_\theta(x) + \frac{(\mu x(1-x))^2}{2} V''_\theta(x). \end{aligned}$$

The dividend is thus

$$D_H(x) = \mathbb{E}_x[V_H(x + d_H x) - V_H(x + d_L x)] = \mu^2 x(1-x)V'_H(x). \quad (6.1)$$

$D_H(x)$ declines to zero in either tail, as reputational updating becomes very slow. Using equation (3.3), the value of quality reduces to

$$\Delta(x) = \int_0^\infty e^{-(\lambda+r)t} \mathbb{E}_{x_0=x, \theta^t=L}[\mu^2 x_t(1-x_t)V'_H(x_t)]dt. \quad (6.2)$$

In Section 6.1, we show that when costs are small there exists a work-shirk equilibrium, and that the cutoff is uniquely determined. In Section 6.2 we consider other forms of equilibria. Finally, in Section 6.3 we show that as the obsolescence rate increases ($\lambda \rightarrow \infty$) then, in a work-shirk equilibrium, effort will decrease to zero.

6.1 Work-Shirk Equilibria

Proposition 4 shows that, for small costs, there exists a work-shirk equilibrium whereby the firm works when its reputation is below some cutoff, x^* . In such an equilibrium, the reputation of a firm below the cutoff tends to rise, whereas the reputation of a firm above the cutoff tends to fall, leading to cyclical dynamics.

Proposition 4 *There is c^* such that for all $c \in (0, c^*)$:*

- (a) *There exists $x^* \in (0, 1)$ such that work-shirk with cutoff x^* is an equilibrium.*
- (b) *The cutoff is uniquely determined.*
- (c) *Reputational dynamics converge to a non-trivial cycle.*

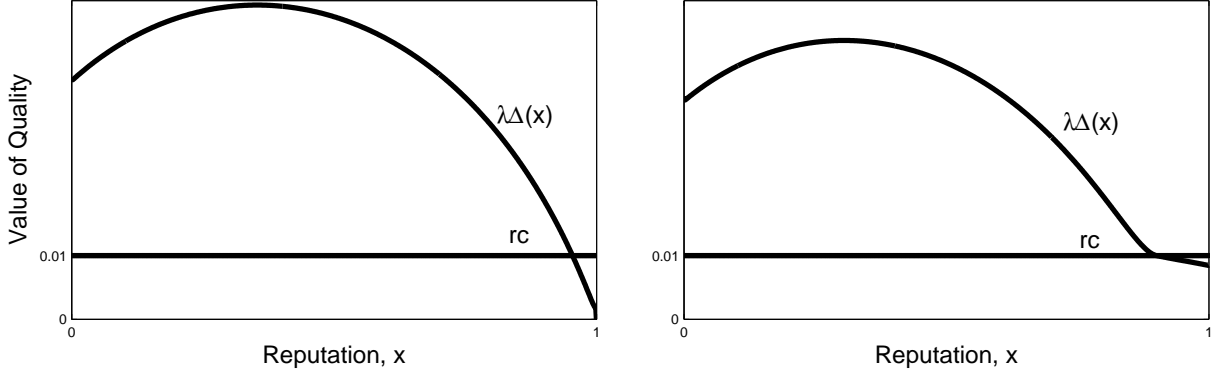


Figure 4: **Asset Value of Quality under Full Effort (left) and in Work-Shirk Equilibrium (right).** This figure assumes that $\mu = 1$, $\lambda = 1$, $r = 1$ and $c = 0.01$. In the work-shirk equilibrium, the resulting cutoff is $x^* = 0.900$.

Proof. See Appendix C. \square

When costs are low, there is a work-shirk equilibrium but no shirk-work equilibrium (Lemma 4). This asymmetry is illustrated in figure 4: When the firm is believed to be working, the asset value of quality is zero at $x = 1$ but strictly positive at $x = 0$. Intuitively, when $x = 1$ the firm's reputation is stuck at $x = 1$ and the firm collects low dividends thereafter. In contrast, when $x = 0$ the firm's reputation drifts into the interior, and the firm eventually collects high dividends. In other words, the firm wishes to invest at $x = 0$ not because a high quality is valuable immediately, but because high quality is valuable once its reputation is in the interior, where its reputation is sensitive to its true quality.

Figure 4 is almost a proof of part (a) of the proposition. Elementary calculations show that:

$$\Delta_1(x) \begin{cases} > 0 & \text{for } x < 1 \\ = 0 & \text{for } x = 1 \end{cases} \quad \text{and } \Delta'_1(1) < 0.$$

and thus for small c there exists x^* such that

$$\lambda\Delta_1(x) \begin{cases} > rc & \text{for } x < x^* & \text{(Low types work)} \\ = rc & \text{for } x = x^* & \text{(} x^* \text{ indifferent)} \\ < rc & \text{for } x > x^* & \text{(High types shirk).} \end{cases} \quad (6.3)$$

To prove part (a) we just need to replace Δ_1 on the LHS with Δ_{x^*} . The problem with this simple argument is that it implicitly assumes continuity of Δ'_{x^*} as $x^* \rightarrow 1$. However, it is straightforward to show that $\lim_{x^* \rightarrow 1} \Delta'_{x^*}(1) = 0 > \Delta'_1(1)$. As a result, it could be that $\Delta_{x^*}(x)$ is increasing in x for $x > x^*$, contradicting the last condition in (6.3).

To understand this complication, consider the marginal value of reputation $V'_\theta(x)$ to a firm with reputation $x \in [x^*, 1]$ where $x^* \approx 1$. A reputational increment dx is valuable to the firm only as long as $x_t|_{x_0=x+dx} > x^*$: As soon as $x_t|_{x_0=x+dx} = x^*$ the increment $x_t|_{x_0=x+dx} - x_t|_{x_0=x}$ vanishes because of the difference of drift $\mathbb{E}[dx]$ to the left of x^* and to the right of x^* . As a consequence, $V'_\theta(x)$ and also $D_\theta(x)$ may be minimized at x when x^* is sufficiently high and simulations confirm that this is actually the case for certain parameter values. So one may be concerned that $\Delta_{x^*}(x)$ is also minimised at x^* .

To overcome this complication and show that $\lambda\Delta_{x^*}(x^*) > \lambda\Delta_{x^*}(x)$ for $x \in (x^*, 1]$ we need a better understanding of the reputational dynamics dx and the marginal values $V'_{\theta,x^*}(x)$ for $x, x^* \approx 1$. Fortunately, the dynamics of $(1-x)$ approximate a geometric Brownian motion which is reflected at $(1-x^*)$ by the huge relative difference in the drift terms. For the high type,

$$\begin{aligned} d_H(1-x) &= -\lambda(\eta-x)dt - \mu^2x(1-x)^2dt + \mu x(1-x)dW \\ &\approx \begin{cases} -\lambda(1-x)dt - \mu(1-x)dW & \text{for } x < x^* \\ \lambda x dt & \text{for } x > x^* \end{cases} \end{aligned}$$

and similar for $d_L(1-x)$.

This has two implications. First, while the dividend may be minimised at x^* , the value of quality at the cutoff $\Delta_{x^*}(x^*)$ is largely determined by the dividends at $x < x^*$. Second, the marginal value of reputation and the dividend at $x > x^*$ are small in relation to those at $x < x^*$, because a reputational increment essentially disappears when $x_t = x^*$ and this happens much sooner for initial reputations $x_0 > x^*$ than for $x_0 < x^*$. Hence for $x > x^*$, $\Delta_{x^*}(x)$ is an average of low dividends while $x_t > x^*$, and a continuation value $\Delta_{x^*}(x^*)$ when x_t hits x^* , which comes to less than $\Delta_{x^*}(x^*)$, as required.

6.2 Shirk-Work-Shirk Equilibria

Given that reputational updating is slow at $x \approx 0$ and $x \approx 1$, it seems natural to expect there to be shirk-work-shirk equilibria, where a firm works when its reputation is between two cutoffs, $x \in [\underline{x}, \bar{x}]$ and shirks elsewhere. In this type of equilibrium, a firm with a low reputation is trapped in a lower shirk-hole in which market learning is too slow to incentivise effort, while a firm above \underline{x} experiences convergent dynamics around \bar{x} . Simulations show that such equilibria may exist for certain parameter ranges. One such equilibrium is shown in figure 5.

Based on our simulations and understanding of the problem we make the following conjecture.

Conjecture 5 *There exists c^* such that for all $c \in (0, c^*)$ the work-shirk equilibrium is unique.*

The intuition why a work-shirk-work equilibrium does not exist for small c is as follows. The argument for more complicated equilibria is similar. When the cost is low, the dividend $D_H(x)$ for

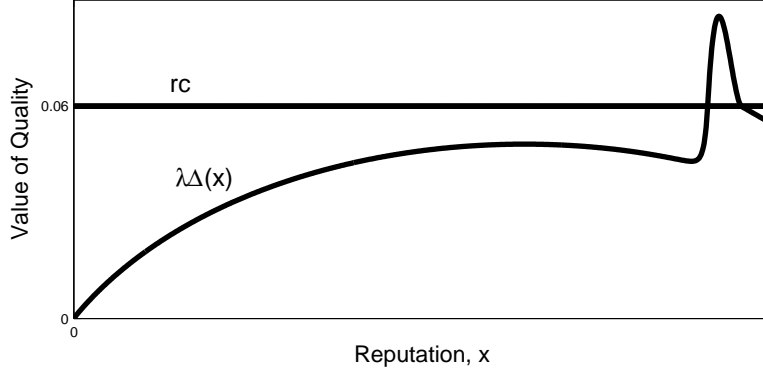


Figure 5: **Shirk-Work-Shirk Equilibrium.** This figure illustrates the asset value of quality in a work-shirk equilibrium, $\Delta_{x^*}(x)$. The straight line equals rc/λ . This figure assumes that $\mu = 1$, $\lambda = 1$, $r = 1$ and $c = 0.06$. The resulting cutoffs are $\underline{x} = 0.910$ and $\bar{x} = 0.958$.

intermediate reputations is high enough to incentivize work for $x \in [\varepsilon, 1 - \varepsilon]$. But a shirk-work-shirk profile with cutoffs $\underline{x} \in (0, \varepsilon)$ and $\bar{x} \in (1 - \varepsilon, 1)$ is unlikely to make both cutoff reputations \underline{x} and \bar{x} indifferent: While the work-shirk cutoff \bar{x} is an attractor of reputational dynamics, i.e. $\mathbb{E}[d|x - x^*] < 0$, the shirk-work cutoff \underline{x} is a deflector: $\mathbb{E}[d|x - x^*] > 0$. The diverging dynamics around \underline{x} create a spike in the marginal value of reputation $V'_\theta(x)$ and the dividend $D(x)$ at \underline{x} . Simulations show that this spike in the dividend creates a peak in the value of quality $\Delta(x)$ around \underline{x} . This can be consistent with $\lambda\Delta(\underline{x}) = \lambda\Delta(\bar{x}) = rc$ only if $[\underline{x}, \bar{x}]$ is small and the peak is to the right of \underline{x} . Yet this conflicts with the fact that the work region $[\underline{x}, \bar{x}]$ must be large, and with simulation results that for small values of \underline{x} , Δ peaks below \underline{x} .

6.3 Frequent Actions

We are also interested in analysing our model as the obsolescence rate grows, $\lambda \rightarrow \infty$.

Conjecture 6 *There exists λ^* such that $\lambda > \lambda^*$ there is no equilibrium with positive effort.*

To understand the intuition behind this conjecture, consider a work-shirk equilibrium. As λ grows, the reputation quickly converges to x^* , and the value functions become very flat. This does not immediately prove the result since $\Delta(x)$ may converge to 0, while $\lambda\Delta(x)$ remains greater than c . However equation (6.2) implies that

$$\lambda\Delta(x) \approx \frac{\lambda}{\lambda + r} \mu^2 x^* (1 - x^*) V'_H(x^*),$$

which converges to zero as $\lim_{\lambda \rightarrow \infty} V'_H(x^*) = 0$.

7 Conclusion

This paper develops a new model of reputation, where the firm invests in its quality, and the quality is imperfectly observed by the market. As customers experience the product, the firm's reputation evolves. This evolution, in turn, affects the incentives of the firm to invest in quality. The model forms a bridge between repeated games and classical models of reputation. In contrast to repeated games, different firms may have different capabilities. In contrast to classical models of reputation, firm's capabilities are a function of past decisions and are therefore endogenous. This model seems realistic: The current state of General Motors is not a result of some exogenous type, but because of its past hiring policies, investment decisions and organisational reorganisations, all of which are endogenous.

The results of the model depend on how the market learns about the firm's quality. When the market learns through good news, there is a unique work-shirk equilibrium and convergent dynamics. When the market learns through bad news, there is a continuum of shirk-work equilibria and divergent dynamics. The results for Brownian news look like those for good news: for low costs there is a work-shirk equilibrium, but no shirk-work equilibrium.

The model can be extended in many ways. Within the current framework, one would like to study more general Poisson processes, and allow investment costs to differ for high and low quality firms. More generally, one could consider multiple quality levels or allow for quality ladders. Finally, it would be interesting to analyse a model with multiple firms, or where firms could enter and exit.

A Good News

A.1 Derivation of Equation (4.3) in Proof of Proposition 2

For a low-quality firm, the value at reputation x^* is given by

$$V_L(x^*) = x^* \quad (\text{A.1})$$

For a high-quality firm the value at reputation $x^* \in [x^*, 1]$ is given by the differential equation

$$rV_H(x) = r\mu x + \frac{d}{dt}V_H(x) - \lambda[V_H(x) - V_L(x)] + \mu[V_H(1) - V_H(x)]$$

The firm's value at reputation x^* is therefore

$$V_H(x^*) = \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} [r\mu x^* + \lambda V_L(x^*) + \mu V_H(1)] dt \quad (\text{A.2})$$

Subtracting (A.1) from (A.2), we obtain

$$\Delta_{x^*}(x^*) = \frac{\mu}{r + \lambda + \mu} [V_H(1) - \mu x^*] \quad (\text{A.3})$$

We now wish to evaluate $V_H(1) - \mu x^*$. As in equation (A.2),

$$V_H(1) - \mu x^* = \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} [r\mu(x_t - x^*) + \lambda(V_L(x_t) - \mu x^*) + \mu(V_H(1) - \mu x^*)] dt \quad (\text{A.4})$$

Evaluating the second term on the right hand side,

$$\begin{aligned} \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} \lambda(V_L(x_t) - \mu x^*) dt &= \lambda \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} \left[\int_{s=t}^{\infty} e^{-r(s-t)} r\mu(x_t - x^*) dt \right] \\ &= \frac{\lambda}{\mu + \lambda} \int_{s=0}^{\infty} e^{-rs} r\mu(x_s - x^*) [1 - e^{-(\mu+\lambda)s}] dt \end{aligned} \quad (\text{A.5})$$

Plugging into (A.4),

$$\frac{r + \lambda}{r + \mu + \lambda} (V_H(1) - \mu x^*) = \int_{t=0}^{\infty} e^{-rt} r\mu(x_t - x^*) \left[\frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\mu+\lambda)t} \right] dt$$

Observe that $x_t = x^*$ after time $\tau(x^*)$. Using (A.3), we then obtain

$$\Delta_{x^*}(x^*) = \frac{\mu}{r + \lambda} \int_{t=0}^{\tau(x^*)} e^{-rt} r(x_t - x^*) \left[\frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\mu+\lambda)t} \right] dt$$

as required.

B Bad News

B.1 Derivation of Equations (5.5) and (5.6) in Proof of Proposition 3

Since $\lambda \geq \mu$, $V_L(0) = 0$. Equation (5.2) implies that

$$\Delta_{x^*}(x) = \int_{t=0}^{\infty} e^{-(r+\lambda)t} \mu V_L(x_t) \quad (\text{B.1})$$

Suppose $x > x^*$. Then the value of a low-quality firm is given by

$$V_L(x) = \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} [r(\mu x_t - c) + \lambda V_H(x_t) + \mu \cdot 0] dt \quad (\text{B.2})$$

The second term here is

$$\begin{aligned} \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} \lambda V_H(x_t) dt &= \lambda \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} \left[\int_{s=t}^{\infty} e^{-r(s-t)} r(\mu x_t - c) dt \right] \\ &= \frac{\lambda}{\mu + \lambda} \int_{s=0}^{\infty} e^{-rs} r(\mu x_s - c) [1 - e^{-(\lambda+\mu)s}] ds \end{aligned}$$

Plugging into (B.2),

$$V_L(x) = \int_{t=0}^{\infty} e^{-rt} r(\mu x_t - c) \left[\frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu)t} \right] dt$$

Using equation (B.1)

$$\begin{aligned} \Delta_{x^*}(x) &= \int_{t=0}^{\infty} e^{-(r+\lambda)t} \mu \left[\int_{s=t}^{\infty} e^{-r(s-t)} r(\mu x_s - c) \left[\frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu)(s-t)} \right] ds \right] dt \\ &= \frac{\mu}{\lambda + \mu} \int_{s=0}^{\infty} e^{-rs} r(\mu x_s - c) (1 - e^{-(\lambda+\mu)s}) ds \end{aligned}$$

which gives us equation (5.5).

Next, suppose $x < x^*$. A low-quality firm's value function is given by $V_L(x) = \int_{t=0}^{\infty} e^{-(r+\mu)t} r \mu x_t dt$.

Using equation (B.1)

$$\begin{aligned} \Delta_{x^*}(x) &= \int_{t=0}^{\infty} e^{-(r+\lambda)t} \mu \left[\int_{s=t}^{\infty} e^{-(r+\mu)(s-t)} r \mu x_s ds \right] dt \\ &= \frac{\mu}{\lambda - \mu} \int_{s=0}^{\infty} e^{-rs} r \mu x_s (e^{-\mu s} - e^{-\lambda s}) ds \end{aligned}$$

which gives us equation (5.6).

C Proof of Proposition 4

We will now show that for sufficiently small c there exists a cutoff x^* such that:

- (a) Cutoff is indifferent: $\lambda\Delta_{x^*}(x^*) = rc$
- (b) High reputations shirk: $\lambda\Delta_{x^*}(x) < rc$ for $x > x^*$.
- (c) Low reputations work: $\lambda\Delta_{x^*}(x) > rc$ for $x < x^*$

The proof is structured as follows. In Section C.1 we derive a useful formula for $V'_\theta(x)$. In Section C.2 we perform a change of variables, replacing reputation x , with the log-likelihood ratio ℓ .

Lemma 7 proves part (a): For every small c there exists a cutoff x^* close to 1 such that the cutoff is indifferent: $\lambda\Delta_{x^*}(x^*) = rc$. Lemma 8 proves part (b): Suppose x^* is high and $\lambda\Delta_{x^*,c}(x^*) = rc$, as guaranteed by Lemma 7. Then $\lambda\widehat{\Delta}_{x^*,c}(x) < rc$ for $x > x^*$. Similarly, Lemma 9 proves part (c): Suppose x^* is high and $\lambda\widehat{\Delta}_{x^*,c}(x^*) = rc$, as guaranteed by Lemma 7. Then $\lambda\widehat{\Delta}_{x^*,c}(x) > rc$ for $x < x^*$. Finally, Lemma 10 proves uniqueness of the work-shirk equilibrium: If c is sufficiently small then there can be at most one cutoff x^* satisfying $\lambda\Delta_{x^*}(x^*) = rc$.

C.1 Marginal Value of Reputation

This subsection generalizes some of the results from section 3.

Denote by h^t a history of the realizations of the stochastic processes up to time t . We can then write a general non-Markovian effort function $\eta = \eta(h^t, x_0, \theta_0)$ as a function of history and initial conditions. Where the initial conditions are unambiguous we write more concisely $\eta = \eta(h^t)$.

Lemma 5 *For any effort and belief function $\langle \eta, \tilde{\eta} \rangle$, and any (non-Markovian) alternative effort function $\eta'(h^t)$ the firm's value equals:*

$$V_\theta(x) = \mathbb{E}_{x, \theta, \eta', \tilde{\eta}} \left[\int_0^\infty e^{-rt} (r(\mu x_t - c\eta'(h^t)) - (\eta'(h^t) - \eta(x_t))(\lambda\Delta(x_t) - rc)) dt \right]$$

Proof. Consider η' that differs from η only on “one history h^t at one time $[t, t+dt]$ ”, say $\eta'(h^t) = 1$ while $\eta(x(h^t)) = 0$. When history h^t realizes, a firm that exerts effort according to η but whose quality process θ_{t+dt} is governed by η' gains $\lambda\Delta(x(h^t))$. Thus, we can write the firm's actual value as its value under the more favorable process η' minus the fair value $\lambda\Delta(x(h^t))$

$$\begin{aligned} V_\theta(x) &= \mathbb{E}_{x, \theta, \eta', \tilde{\eta}} \left[\int_0^\infty e^{-rt} r(\mu x_t - c\eta(x_t)) dt \right] \\ &\quad - \mathbb{E}_{x, \theta, \eta', \tilde{\eta}} \left[\int_0^\infty e^{-rt} (\eta'(h^t) - \eta(x_t)) \lambda\Delta(x_t) dt \right] \end{aligned}$$

For a “one-shot deviation”, it makes no difference whether we take the expectation in the last, corrective term conditional on η or η' . For “multi-period” deviations, we need to condition on η' because the firm “pays as it goes” for the favorable evolution θ_t (η'). Thus, in general we get

$$\begin{aligned} V_\theta(x) &= \mathbb{E}_{x,\theta,\eta',\tilde{\eta}} \left[\int_0^\infty e^{-rt} (r(\mu x_t - c\eta(x_t)) - (\eta'(h^t) - \eta(x_t)) \lambda \Delta(x_t)) dt \right] \\ &= \mathbb{E}_{x,\theta,\eta',\tilde{\eta}} \left[\int_0^\infty e^{-rt} (r(\mu x_t - c\eta'(h^t)) - (\eta'(h^t) - \eta(x_t)) (\lambda \Delta(x_t) - rc)) dt \right] \end{aligned}$$

as required. \square

This lemma has a certain intrinsic interest in formalizing one more way to tweak value functions by trading off changes in current payoffs $\lambda \Delta(x_t) - rc$ against the evolution of future state variables.

The real reason for deriving it, however, is to apply it to work-shirk effort profiles in situations where $(\eta'(h^t) - \eta(x_t)) (\lambda \Delta(x_t) - rc)$ is either small or we know its sign.

Lemma 6 (a) *In a work-shirk profile with cutoff x^* in which x^* weakly prefers to shirk, i.e. $\lambda \Delta_{x^*}(x^*) \leq rc$, the marginal value of reputation is strictly positive:*

$$V'_\theta(x) > 0 \text{ for all } x \in \mathbb{R}.$$

(b) *In a work-shirk profile with cutoff x^* in which x^* is indifferent, i.e. $\lambda \Delta_{x^*}(x^*) = rc$, the marginal value is given by:*

$$V'_\theta(x) = r\mu \int e^{-rt} \mathbb{E}_{x_0=x, \theta_0=\theta} \left[\frac{dx_t}{dx_0}(x) \right] dt > 0 \text{ for all } x \in \mathbb{R}.$$

(c) *In a work-shirk profile with cutoff x^* , the value of quality of x^* is strictly positive:*

$$\Delta_{x^*}(x^*) > 0.$$

Proof. For (a) and (b) fix a realization of the stochastic processes h^t and consider the trajectories $x_t(x)$ and $x_t(x+dx)$ starting dx apart. Let $\eta'(h^t, x)$ be the non-Markovian strategy of a firm starting at x mimicking the effort $\eta(x_t(x+dx))$ of a firm starting at $x+dx$, and $x'_t(x)$ the induced trajectory. By construction, we get that $x'_t(x) \leq x_t(x+dx)$ for all t and $\eta(x_t(x)) = 1 > 0 = \eta'(h^t, x)$ iff $x_t(x) < x^* < x_t(x+dx)$ (otherwise $\eta(x_t(x)) = \eta'(h^t, x)$)

We can then decompose the incremental value of reputation as follows:

$$\begin{aligned}
V_\theta(x+dx) - V_\theta(x) &= [V_{\theta,\eta}(x+dx) - V_{\theta,\eta'}(x)] + [V_{\theta,\eta'}(x) - V_{\theta,\eta}(x)] \\
&= r\mu \int e^{-rt} \mathbb{E} [x_t(x+dx) - x'_t(x)] dt \\
&\quad - \mathbb{E}_{x,\theta,\eta',\tilde{\eta}} \left[\int_{t:x_t(x) < x^* < x_t(x+dx)} e^{-rt} (\lambda\Delta_{x^*}(x_t) - rc) dt \right]
\end{aligned}$$

The first term in this expression is positive and of order dx . The second term is of order dx^2 if $\lambda\Delta_{x^*}(x^*) = rc$, proving (b), and positive of order dx if $\lambda\Delta_{x^*}(x^*) < rc$, proving part (a).

(c) If $\Delta_{x^*}(x^*) \leq 0$, the premise in part (a) would be satisfied, yielding $V'_H(x) > 0$ for all x . Thus (6.2) implies that $\Delta_{x^*}(x^*) > 0$, yielding a contradiction. \square

Part (b) has the flavor of the envelope theorem: when the firm's first-order condition holds at the cutoff, then a change in the initial reputation only affects his payoff through the reputational evolution. Intuitively, a firm with a lower initial reputation work more, leading to a gain of $\Delta(x)$ when a technology shock hits; however, this is exactly offset by the extra cost born by the firm. The marginal value of reputation $V'_\theta(x)$ is thus determined by the "durability" of the reputational increment $\frac{dx_t}{dx_0}$. Once, x_t reaches the reflecting cutoff x^* this reputational increment disappears.

C.2 Updating Log-likelihood Ratios

Define $\ell(x) = \log(x/(1-x)) \in \mathbb{R} \cup \{-\infty, \infty\}$ and note that $x(\ell) = \frac{e^\ell}{1+e^\ell}$ and $\frac{d\ell}{dx} = \frac{(1+e^\ell)^2}{e^\ell} = \frac{1}{x(1-x)}$. According to Bayes rule, market beliefs η affect ℓ via

$$\frac{d\ell}{dt} = \frac{d\ell}{dx} \lambda(\eta - x) = \lambda \frac{(1+e^\ell)^2}{e^\ell} \left(\eta - \frac{e^\ell}{1+e^\ell} \right) = \begin{cases} \lambda(1+e^{-\ell}) & \text{for } \eta = 1 \\ -\lambda(1+e^\ell) & \text{for } \eta = 0 \end{cases}.$$

Realized quality $dZ = \theta dt + dW$ affects ℓ via ¹¹

$$d_\theta \ell = \frac{d\ell}{dx} d_\theta x + \frac{1}{2} \frac{d^2\ell}{dx^2} \text{Var}(d_\theta x) dt = \begin{cases} \frac{\mu^2}{2} + \mu dW & \text{for } \theta = H \\ -\frac{\mu^2}{2} + \mu dW & \text{for } \theta = L \end{cases}.$$

Thus, in a work-shirk profile with cutoff $\ell^* \gg 0$, high reputations $\ell_t \gg 0$ approximately follow

¹¹Remember $\frac{d\ell}{dx} = \frac{1}{x(1-x)}$ and $\frac{d^2\ell}{dx^2} = \frac{2x-1}{x^2(1-x)^2}$. For $\theta = H$ we have $d_H x = \mu^2 x(1-x)^2 dt + \mu x(1-x) dW$ and $\text{Var}(d_H x) = \mu^2 x^2(1-x)^2$ so that

$$d_\theta \ell = \frac{d\ell}{dx} d_\theta x + \frac{1}{2} \frac{d^2\ell}{dx^2} \text{Var}(d_\theta x) dt = \mu^2(1-x) dt + \mu dW + \frac{1}{2} \mu^2(2x-1) = \frac{\mu^2}{2} + \mu dW$$

a Brownian motion with drift $\lambda \pm \frac{\mu^2}{2}$ reflected at ℓ^* :

$$d\ell^{H,L} \approx \begin{cases} \left(\lambda \pm \frac{\mu^2}{2}\right) dt + \mu dW & \text{for } \ell < \ell^* \\ -\infty & \text{for } \ell > \ell^* \end{cases}.$$

Finally, we can write NPVs $\widehat{V}_\theta(\ell) := V_\theta\left(\frac{e^\ell}{1+e^\ell}\right)$, value of quality $\widehat{\Delta}(\ell) := \Delta\left(\frac{e^\ell}{1+e^\ell}\right)$, and effort $\widehat{\eta}(\ell) := \eta\left(\frac{e^\ell}{1+e^\ell}\right)$ as functions of ℓ and obtain

$$\widehat{\Delta}(\ell) = \int_0^\infty e^{-(r+\lambda)t} \mathbb{E}_{\ell_0=\ell, \theta_s \leq t=L} [\widehat{D}(\ell_t)] dt. \quad (\text{C.1})$$

where the dividend is

$$\widehat{D}(\ell) = \mu^2 \widehat{V}'_H(\ell_t).$$

C.3 Indifference of Cutoff

We now show that for small costs there exists a high cutoff who satisfies the indifference condition. Since $\widehat{\Delta}$ and \widehat{V} depend on c , we subscript them with c where useful.

Lemma 7 *For every $\ell \in \mathbb{R}$ there exists $c > 0$ such that for all $c^* < c$ there exists $\ell^* > \ell$ such that $rc^* = \lambda \widehat{\Delta}_{\ell^*, c^*}(\ell^*)$*

Proof. Fix a cutoff $\ell \in \mathbb{R}$ and consider $\widehat{\Delta}_{\ell, c}(\ell)$ as a function of $c \in [0, \mu\lambda/(r+\lambda)]$. By Lemma 6(b) we have $\widehat{\Delta}_{\ell, c}(\ell) > 0$ for all c . Since $\widehat{\Delta}_{\ell, c}(\ell)$ is continuous in c , it takes on its minimum $\widehat{\Delta}_{\ell, c'}(\ell) > 0$ at some c' .

Define c by $rc = \lambda \widehat{\Delta}_{\ell, c'}(\ell)$ and fix $c^* \in (0, c)$. Using the definitions of c' and c^* ,

$$\lambda \widehat{\Delta}_{\ell, c^*}(\ell) \geq \lambda \widehat{\Delta}_{\ell, c'}(\ell) > rc^*$$

so the firm prefers to work. On the other hand, at $\ell = \infty$,

$$rc^* > r \widehat{\Delta}_{\infty, c^*}(\infty) = 0$$

so the firm prefers to shirk. By continuity of $\widehat{\Delta}_{\ell, c^*}(\ell)$ as a function of $\ell \in \mathbb{R} \cup \{-\infty, \infty\}$, there exists $\ell^* \in (\ell, \infty)$ with $rc^* = \lambda \widehat{\Delta}_{\ell^*, c^*}(\ell^*)$. \square

The daunting array of quantifiers in the statement of this lemma, guarantees that we can assume ℓ^* with $rc^* = \lambda \widehat{\Delta}_{\ell^*, c^*}(\ell^*)$ as large as necessary in the coming arguments.

C.4 High Reputations Shirk

Lemma 8 shows that firms with high reputations shirk. In proving this result we show that the dividend of quality $\widehat{D}(\ell)$ is much greater below ℓ^* than above ℓ^* . Intuitively incremental reputation above ℓ^* is less “durable” because it disappears when ℓ_t hits ℓ^* and reputational updating $\frac{d\ell}{dt}$ decelerates from $-\lambda(1+e^\ell) \approx -\infty$ to $\lambda(1+e^{-\ell}) \approx \lambda$.

Lemma 8 *Suppose ℓ^* is large and $\lambda\widehat{\Delta}_{\ell^*}(\ell^*) = rc$. Then $\lambda\widehat{\Delta}_{\ell^*}(\ell) < rc$ for all $\ell > \ell^*$.*

Proof. Fix $\ell > \ell^*$. Suppose ℓ_t hits the cutoff ℓ^* for the first time at $t = T$. We can then write,

$$\begin{aligned}\widehat{\Delta}_{\ell^*}(\ell) - \widehat{\Delta}_{\ell^*}(\ell^*) &= \mathbb{E} \left[\int_0^T e^{-(\lambda+r)t} \mathbb{E}_{\ell_0=\ell, \theta_s \leq t=L} \left[\mu^2 \widehat{V}'_H(\ell_t) \right] dt + e^{-(\lambda+r)T} \widehat{\Delta}_{\ell^*}(\ell^*) \right] - \widehat{\Delta}_{\ell^*}(\ell^*) \\ &= \int_0^T e^{-(\lambda+r)t} \left(\mathbb{E}_{\ell_0=\ell, \theta_s \leq t=L} \left[\mu^2 \widehat{V}'_H(\ell_t) \right] - (\lambda+r) \widehat{\Delta}_{\ell^*}(\ell^*) \right) dt\end{aligned}\quad (\text{C.2})$$

We wish to show that (C.2) is negative. We show this using two claims.

Claim 1. Fix $\alpha > \beta > 0$ and sufficiently high ℓ^* . Then there exists a $\gamma > 0$ such that the discounted probability that the future reputation ℓ_t is between $\ell^* - \alpha$ and $\ell^* - \beta$,

$$(r+\lambda) \int_0^\infty e^{-(r+\lambda)t} \Pr_{\ell_t=\ell^*, \theta_s \leq t=L} [\ell_t \in [\ell^* - \alpha; \ell^* - \beta]] dt \quad (\text{C.3})$$

is bounded below by γ independently of ℓ^* .

Proof. Remember that the reputational dynamics ℓ_t for high values of ℓ^* are approximated by a reflected Brownian motion with finite drift:

$$d\ell^L \approx \begin{cases} \left(\lambda - \frac{\mu^2}{2} \right) dt + \mu dW & \text{for } \ell < \ell^* \\ -\infty & \text{for } \ell > \ell^* \end{cases} \quad (\text{C.4})$$

The process ℓ_t therefore has a positive probability of lying in $[\ell^* - \alpha; \ell^* - \beta]$ at any time t , so (C.3) can be bounded below by $\varepsilon > 0$, for all sufficiently high ℓ^* .

Claim 2. Fix $\alpha > \beta > 0$ and $M > 0$. Suppose ℓ^* is sufficiently high and $rc = \lambda\widehat{\Delta}_{\ell^*,c}(\ell^*)$. Then

$$\widehat{V}'_{H,\ell^*}(\ell^* - \gamma) > M\widehat{V}'_{H,\ell^*}(\ell^* + \delta) \text{ for all } \gamma \in [\beta; \alpha] \text{ and all } \delta > 0.$$

Proof. Let $\gamma \in [\beta; \alpha]$. When ℓ^* is sufficiently high, the reputational dynamics are given by (C.4). The expected drift is bounded above, and the expected time T before the reputational dynamic starting at $\ell_0 = \ell^* - \gamma$ reaches the cutoff $\ell_T|_{\ell_0=\ell^*-\gamma} = \ell^*$, is bounded below independently of ℓ^* . Thus, $\mathbb{E} \left[\frac{d\ell_t}{d\ell_0}(\ell') \right]$ for $\ell' \in [\ell^* - \alpha; \ell^* - \beta]$ is bounded away from 0 as $\ell^* \rightarrow \infty$.

Next, consider $\ell^* + \delta$. The expected time T before $\ell_T|_{\ell_0=\ell^*+\delta} = \ell^*$ uniformly converges to 0. This is easier to see for the posterior x_t than for the log-likelihood-ratio ℓ , as $\mathbb{E}\left[\frac{d_H x}{dt}\right] = -\lambda x$ is bounded away from 0 while $1 - x^*$ converges to 0. Thus, $\mathbb{E}\left[\frac{d\ell_t}{d\ell_0}(\ell^* + \delta)\right]$ for $\delta > 0$ converges to 0 as $\ell^* \rightarrow \infty$.

For large values of ℓ^* we can ignore in $\widehat{V}'_{H,\ell^*}(\ell) \geq r\mu \int e^{-rt} \mathbb{E}\left[\frac{e^{\ell_t}}{(1+e^{\ell_t})^2} \frac{d\ell_t}{d\ell_0}(\ell)\right] dt$ all terms where ℓ_t and ℓ are on different sides of ℓ^* . Since $e^\ell/(1+e^\ell)^2$ is decreasing in $\ell > 0$ we get bounds $e^{\ell_t(\ell^*-\gamma)}/(1+e^{\ell_t(\ell^*-\gamma)})^2 \geq e^{\ell^*}/(1+e^{\ell^*})^2 \geq e^{\ell_t(\ell^*+\delta)}/(1+e^{\ell_t(\ell^*+\delta)})^2$ and equation (??) implies

$$\frac{\widehat{V}'_{H,\ell^*}(\ell^* - \gamma)}{\widehat{V}'_{H,\ell^*}(\ell^* + \delta)} \geq \frac{r\mu \left[\frac{e^{\ell^*}}{(1+e^{\ell^*})^2}\right] \int e^{-rt} \mathbb{E}\left[\frac{d\ell_t}{d\ell_0}(\ell^* - \gamma)\right] dt}{r\mu \left[\frac{e^{\ell^*}}{(1+e^{\ell^*})^2}\right] \int e^{-rt} \mathbb{E}\left[\frac{d\ell_t}{d\ell_0}(\ell^* + \delta)\right] dt} \geq \frac{\int e^{-rt} \mathbb{E}\left[\frac{d\ell_t}{d\ell_0}(\ell^* - \gamma)\right] dt}{\int e^{-rt} \mathbb{E}\left[\frac{d\ell_t}{d\ell_0}(\ell^* + \delta)\right] dt}$$

Therefore, $\widehat{V}'_{H,\ell^*}(\ell^* - \gamma)/\widehat{V}'_{H,\ell^*}(\ell^* + \delta)$ diverges to ∞ as $\ell^* \rightarrow \infty$, uniformly over all $\gamma \in [\beta; \alpha]$ and $\delta > 0$.

Proof of Lemma. We now show that the right hand side of equation (C.2) is negative. Let $\ell > \ell^* \gg 0$. Fix $\alpha > \beta > 0$ and $\varepsilon > 0$, and choose $M = \frac{1}{\varepsilon}$. Equation (6.2) implies that

$$\begin{aligned} (r + \lambda) \widehat{\Delta}_{\ell^*}(\ell^*) &= (r + \lambda) \int_0^\infty e^{-(r+\lambda)t} \mathbb{E}_{\ell_0=\ell^*, \theta_s \leq t=L} \left[\mu^2 \widehat{V}'_H(\ell_t) \right] dt \\ &\geq (r + \lambda) \int_0^\infty e^{-(r+\lambda)t} \left(\Pr_{\ell_0=\ell^*, \theta_s \leq t=L} (\ell_t \in [\ell^* - \beta; \ell^* - \alpha]) \right. \\ &\quad \left. * \min_{\ell' \in [\ell^* - \beta; \ell^* - \alpha]} \left\{ \mu^2 \widehat{V}'_H(\ell') \right\} \right) dt \\ &\geq \varepsilon M \max_{\ell \in [\ell^*, \infty)} \left\{ \mu^2 \widehat{V}'_H(\ell) \right\} \\ &= \max_{\ell \in [\ell^*, \infty)} \left\{ \mu^2 \widehat{V}'_H(\ell) \right\} \end{aligned}$$

where the second inequality uses Claims 1 and 2. Hence (C.2) is negative, as required. \square

C.5 Low Reputations Work

Lemma 9 shows that firms with low reputations work. For reputations $\ell \in (\ell_2, \ell^*)$ for some ℓ_2 defined below, the optimality of working follows directly by showing that $\widehat{V}'_\theta(\ell)$ and $\widehat{\Delta}'(\ell)$ are decreasing on (ℓ_2, ℓ^*) . For reputations $\ell < \ell_2$ the result follows from the closeness of $\widehat{\Delta}_{\ell^*}(\cdot)$ and $\widehat{\Delta}_\infty(\cdot)$.

Lemma 9 *Assume ℓ^* is large, costs c are small and $\lambda \widehat{\Delta}_{\ell^*}(\ell^*) = rc$. Then $\lambda \widehat{\Delta}_{\ell^*}(\ell) > rc$ for all $\ell < \ell^*$.*

Proof.

Claim 1. There exists ℓ_1 sufficiently large such that $\widehat{V}'_{H,\ell^*}(\cdot)$ is strictly decreasing on $[\ell_1, \ell^*]$ for any $\ell^* > \ell_1$.

Proof. For $0 \ll \ell_t < \ell^*$, the reputational dynamics ℓ_t are approximately a Brownian motion (C.4). Let ℓ_1 be sufficiently high and consider $\ell \in [\ell_1, \ell^*]$. When $\ell_t < \ell^*$ then $\frac{d\ell_t}{d\ell_0} \approx 1$; and when the trajectory hits ℓ^* then $\frac{d\ell_t}{d\ell_0} = 0$ since the boundary is reflecting. Using equation (C.1)

$$\widehat{V}'_H(\ell) = r\mu \int_0^\infty e^{-rt} \mathbb{E}_{\ell_0=\ell} \left[\frac{e^{\ell_t}}{(1+e^{\ell_t})^2} \frac{d\ell_t}{d\ell_0} \right] dt \approx r\mu \int_0^\infty e^{-rt} \mathbb{E}_{\ell_0=\ell} \left[\frac{e^{\ell_t}}{(1+e^{\ell_t})^2} \mathbf{1}_{t < T(\ell_0)} \right] dt \quad (\text{C.5})$$

where $T(\ell_0)$ is the time ℓ_t first hits ℓ^* . Since $e^{\ell_t}/(1+e^{\ell_t})^2$ is strictly decreasing for $\ell_t > 0$, and $T(\ell_0)$ is decreasing in ℓ_0 , equation (C.5) is strictly decreasing in ℓ on $[\ell_1, \ell^*]$.

Claim 2. There exists ℓ_2 sufficiently large such that $\widehat{\Delta}_{\ell^*}(\cdot)$ is strictly decreasing on $[\ell_2, \ell^*]$ for any $\ell^* > \ell_2$.

Proof. Pick ℓ_1 as in Claim 1. Since the reputational dynamics are determined by reflected Brownian motion with finite drift (C.4), we can pick $\ell_2 \gg \ell_1$ such that for any $\ell^* > \ell_2$, $\Pr_{\ell_0 \in [\ell_2, \ell^*]}(\ell_t \in [\ell_1, \ell^*]) \approx 1$. Claim 1 says that $\widehat{V}'_{H,\ell^*}(\cdot)$ is strictly decreasing on $[\ell_1, \ell^*]$. Equation (C.1) says that $\widehat{\Delta}_{\ell^*}(\ell)$ is the integral over $\widehat{V}'_{H,\ell^*}(\ell_t)$, yielding the result.

Claim 3. Assume that $\lambda \widehat{\Delta}_{\ell^*}(\ell^*) = rc$ and fix any ℓ_2 . Then $\widehat{\Delta}_{\ell^*}(\cdot)$ converges to $\widehat{\Delta}_\infty(\cdot)$ uniformly on $[-\infty, \ell_2]$ as $\ell^* \rightarrow \infty$.

Proof. As $\ell^* \rightarrow \infty$, $\widehat{\Delta}_{\ell^*}(\ell)$ converges pointwise to $\widehat{\Delta}_\infty(\ell)$ for all ℓ . Let $\ell^* \gg \ell_2$. For any $\ell < \ell_2$, equations (??) and (C.1) imply that, $\widehat{V}'_{\theta,\ell^*}(\ell)$, and thus $\widehat{\Delta}_{\ell^*}(\ell)$, depend on ℓ^* only on trajectories ℓ_t that reach ℓ^* . The future discounted probability of these trajectories converges to 0 as $\ell^* \rightarrow \infty$, so the convergence is uniform for $\ell < \ell_2$.

Proof of Lemma. Choose $0 \ll \ell_2 \ll \ell^*$. Claim 2 implies that $\widehat{\Delta}_{\ell^*}(\ell)$ is strictly decreasing in ℓ for $\ell \in [\ell_2, \ell^*]$. Since $\lambda \widehat{\Delta}_{\ell^*}(\ell^*) = rc$, we have

$$\lambda \widehat{\Delta}_{\ell^*}(\ell^*) > rc \quad \text{for } \ell \in [\ell_2, \ell^*].$$

The function $\widehat{\Delta}_1(\cdot)$ is bounded away from 0 on $[-\infty, \ell_2]$. Hence Claim 3 implies that $\widehat{\Delta}_{\ell^*}(\ell)$ is bounded away from zero. For small costs,

$$\lambda \widehat{\Delta}_{\ell^*}(\ell^*) > rc \quad \text{for } \ell \in (-\infty, \ell_2),$$

as required. \square

C.6 Uniqueness

Lemma 10 shows that, when costs are low, there is at most one work-shirk equilibrium. Intuitively, the cutoff cannot be too low since firms with reputations $\ell^* \approx 0$ would have too much incentive to work. When ℓ^* is larger, the dynamics of ℓ_t with a cutoff ℓ^* look exactly like the dynamics of $\ell_t + \epsilon$ with a cutoff $\ell^* + \epsilon$. The function $e^{\ell_t}/(1 + e^{\ell_t})^2 = x(1 - x)$ is decreasing in the firm's reputation, so $\widehat{\Delta}_{\ell^*}(\ell^*)$ is also decreasing and there can be at most one indifference point.

Lemma 10 *Suppose c is small. Then there is at most one cutoff ℓ^* which satisfies $\lambda \widehat{\Delta}_{\ell^*}(\ell^*) = rc$.*

Proof.

Claim 1. There exists an ℓ_1 such that $\widehat{\Delta}_{\ell^*}(\ell^*)$ is strictly decreasing in ℓ^* for $\ell^* \in [\ell_1, \infty]$.

Proof. For $0 \ll \ell_t < \ell^*$, the reputational dynamics ℓ_t are approximately a Brownian motion. Moreover, the dynamics are identical when we firm has reputation $\ell_t + \epsilon$ and the cutoff is $\ell^* + \epsilon$, where $\epsilon > 0$. Therefore when ℓ_1 is sufficiently high and $\ell^* > \ell_1$

$$\begin{aligned} \widehat{V}'_{H, \ell^*} \mathbb{E}_{\ell_0 = \ell} \left[\frac{e^{\ell_t}}{(1 + e^{\ell_t})^2} \frac{d\ell_t}{d\ell_0}(\ell; \ell^*) \right] dt &\geq r\mu \int e^{-rt} \mathbb{E}_{\ell_0 = \ell} \left[\frac{e^{\ell_t}}{(1 + e^{\ell_t})^2} \frac{d\ell_t}{d\ell_0}(\ell + \epsilon; \ell^* + \epsilon) \right] dt \\ &= \widehat{V}'_{H, \ell^* + \epsilon}(\ell + \epsilon) \end{aligned}$$

where the inequality follows from the fact that $e^{\ell_t}/(1 + e^{\ell_t})^2$ is decreasing in $\ell_t > 0$. Using equation (C.1), we conclude that

$$\widehat{\Delta}_{\ell^*}(\ell^*) \geq \widehat{\Delta}_{\ell^* + \epsilon}(\ell^* + \epsilon),$$

as required.

Claim 2. For low c , there is no work-shirk equilibrium with $\ell^* < \ell_1$

Proof. Let $c = 0$ and define k such that $2k = \lambda \widehat{\Delta}_{-\infty}(0)$. By continuity there exists $\ell' < 0$ such that $\lambda \widehat{\Delta}_{-\ell^*}(0) > k'$ for $\ell^* \in [-\infty, \ell']$. By continuity, there exists $c' < 0$ such that $\lambda \widehat{\Delta}_{\ell^*}(0) - rc > 0$ for $\ell^* \in [-\infty, \ell']$ when $c < c'$. If $\ell^* < \ell'$ is the cutoff from a work-shirk equilibrium then $\widehat{\Delta}_{\ell^*}(\ell^*) \geq \widehat{\Delta}_{\ell^*}(0)$, so that

$$0 = \lambda \widehat{\Delta}_{\ell^*}(\ell^*) - rc \geq \lambda \widehat{\Delta}_{\ell^*}(0) - rc.$$

We thus have a contradiction and $\ell^* \in [-\infty, \ell']$ cannot be an equilibrium when $c < c'$.

When $c = 0$, $\widehat{\Delta}_{-\infty}(0)$ is bounded below by k'' on $[\ell', \ell_1]$. This follows from Lemma 6 and continuity of $V'_H(x)$, which imply that $D(x)$ is bounded below on $[\ell', \ell_1]$. By continuity, $\lambda \widehat{\Delta}_{\ell}(\ell) - rc > 0$ on $[\ell', \ell_1]$ for all $c < c''$.

Proof of Lemma. When costs are small there is no equilibrium with $\ell < \ell_1$. Suppose there are two equilibria with cutoffs $\ell_1^*, \ell_2^* \geq \ell_1$. Then $\lambda \widehat{\Delta}_{\ell_1^*}(\ell_1^*) = \lambda \widehat{\Delta}_{\ell_2^*}(\ell_2^*)$, which contradicts Claim 1. \square

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