

Dynamic Managerial Compensation: a Mechanism Design Approach

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Abstract

We characterize the optimal incentive scheme for a manager who faces costly effort decisions and whose ability to generate profits for the firm varies stochastically over time. The optimal contract is obtained as the solution to a dynamic mechanism design problem with hidden actions and persistent shocks to the agent's private information. When the agent is risk-neutral, the optimal contract can often be implemented with a simple pay package that is linear in the firm's profits. Furthermore, the power of the incentive scheme typically increases over time, thus providing a possible justification for the frequent practice of putting more stocks and options in the package of managers with a longer tenure in the firm. Contrary to other explanations proposed in the literature (e.g. declining disutility of effort, career concerns), the optimality of seniority-based reward schemes is not driven by any particular assumption on the agent's preferences/technology. It results from an optimal allocation of the manager's informational rents over time. Building on the insights from the risk-neutral case, we then explore the properties of optimal incentive schemes for risk-averse managers. Contrary to the risk-neutral case, the optimal pay package is typically non-linear in the firm's profits (although, there are instances where it is a convex function of a linear aggregator). Furthermore, we find that risk-aversion may contribute to reducing (but not necessarily eliminate) the benefit of offering incentives whose power increase, on average, over time.

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1 Introduction

This paper contributes to the literature on managerial compensation by adopting a mechanism design approach to characterize the dynamics of the optimal incentives contract.

We consider an environment in which the firm's shareholders (the principal) hire a manager (the agent) whose ability to generate profits for the firm varies stochastically over time. This could reflect, for example, the possibility that the value of the manager's expertise/competence changes in response to variations in the business environment. It could also be the result of learning by doing. We assume that both the manager's ability to generate profits (his type) as well as his effort choices are the manager's private information. The firm's shareholders simply observe the dynamics of profits (equivalently, the value of their shares), which we assume to be verifiable, and pay the manager on the basis of this information.

Contrary to the literature on renegotiation (e.g. Laffont and Tirole, 1988, 1990), we assume that the firm's shareholders perfectly understand the value of commitment and hence adhere to the incentive scheme they offered when they hired the manager, even if, after certain contingencies, such a scheme need not be optimal anymore. However, contrary to this literature, we do not impose restrictions on the process governing the evolution of the agent's private information. In particular, we do not restrict the agent's type to be constant over time, nor do we restrict the shocks to the agent's type to be independent. Allowing for general processes is important for it permits us to shed light on certain interesting properties of the optimal scheme that are obscured, if not completely eliminated, by assuming perfectly correlated, or independent types (more below).

Our baseline model features an environment where both the firm's shareholders and the manager are risk-neutral. Because the firm contracts with the manager at the time the latter is already privately informed about his type, interesting dynamics emerge even without introducing risk aversion. In particular, we show that the power of incentives typically increases over time, which can explain the frequent practice of putting more stocks and options in the package of managers with a longer tenure in the firm. Contrary to other explanations proposed in the literature (e.g. declining

disutility of effort, career concerns), in our model, the optimality of seniority-based reward schemes is not driven by variations in the agent’s preferences, nor by variations in his outside option. It results from an optimal allocation of the manager’s informational rents over time. In other words, it originates from the firm’s desire to minimize the manager’s compensation while preserving his incentives for both effort and information revelation.

The driving assumption behind this result is the assumption that the effect of the manager’s initial type on the distribution of his future types declines over time. This assumption seems quite reasonable and is satisfied, for instance, when the agent’s private information evolves according to an ARIMA process with coefficients of linear dependence smaller than one in absolute value. As documented in other recent works on dynamic mechanism design (e.g. Battaglini, 2005, Pavan, Segal, and Toikka (2008)) this assumption implies that, to minimize the agent’s rents, it is more efficient to distort decisions downwards in the early stages of the relationship than in later ones. The reason is that an agent’s ability to guarantee himself a rent by mimicking another type depends on the different expectations the two types have about their future types. When this difference declines with the time horizon, distorting decisions in the distant future becomes very ineffective at reducing informational rents. When applied to the situation in this paper, this principle of “vanishing distortions” leads to an effort policy that is closer to the first-best in the long run than in the short run. This follows from the fact that a type’s rent increases in the effort of lower types, as shown by Laffont and Tirole (1986) in a static setting.

A second prediction of the model is that the optimal contract under risk neutrality often takes the form of a simple (state-contingent) linear contract. In other words, in each period, the firm pays the manager a fixed salary plus a bonus that is linear in the firm’s profits (or, equivalently, in the firm’s stock price, provided the latter also depends on the manager’s effort). When the manager’s type follows an ARIMA process (more generally, any process whose future distributions exhibit a certain separability with respect to the initial type), then the slope of the linear scheme changes deterministically over time, i.e. it depends on the manager’s initial type, but does not need to respond to future shocks.

More generally, the optimal contract requires that the manager be given the possibility of proposing changes to his pay package over time, in response to the shocks to his type (equivalently, to any privately observed shock to the environment that affects his ability to generate profits for the firm). The idea that a manager may be given the possibility to propose changes to his reward package seems appealing in light of the recent empirical literature on managerial compensation where it is found that this practice has become more frequent in the last decade (see, among others, Kuhnen and Zwiebel (2008), and Bebchuck and Fried (2004)).

While, under risk neutrality, the optimality of linear schemes holds across a variety of specifications of the process governing the evolution of the manager's productivity, there are instances where the optimal effort policy requires the use of stronger incentive schemes according to which the manager is paid a bonus only when the firm's profits exceed a certain target, where this target may depend on the (history of the) manager's reports about his type. While the power of these schemes is stronger, contrary to linear schemes, these "bonus" schemes would not be appropriate when profits are the result not only of the manager's type and effort, but also of unobservable shocks (noise) whose distribution is unaffected by the manager's effort.

Building on the insights from the risk-neutral case, in the second part of the paper we explore the properties of optimal incentive schemes for risk-averse managers. Contrary to the risk-neutral case, we find that the optimal pay package is typically non-linear in the firm's profits. To eliminate (or at least reduce) the effects of the risk associated with the manager's compensation, the principal needs to use a pay scheme that is convex in (a linear aggregator of) the firm's profits.

We also find that risk-aversion tends to reduce (but not necessarily eliminate) the benefits of seniority-based incentive schemes whose power increase, on average, over time. The reason is that the volatility of the agent's future types given his current type increases with the time horizon. In other words, while the agent's current type is a fairly good predictor of his type in the next period, it is a fairly poor predictor of his type, say, five periods into the future. Since the agent's exposure to risk is increasing in effort (because of incentive compatibility), other things equal, risk aversion makes it more attractive for the principal to induce higher effort in the early stages of

the relationship, when the agent is subject to little risk about his ability to generate profits, than in later periods, where the uncertainty about the agent’s type (as perceived from the moment the contract is signed) is higher.

Thus, quite interestingly, the same property of “declining correlation” which makes seniority-based schemes attractive under risk neutrality, now contributes to making such schemes costly under risk aversion. Whether risk-averse managers with a longer tenure receive more or less high-powered incentives then depends on which of the two effects prevails.

Related literature.¹ The literature on managerial compensation is way too large to be successfully summarized within the context of this paper. We refer to Prendergast (1999) for an excellent review and to Edmans and Gabaix (2009) for a survey of some more recent developments.

In terms of model, the work which is closest to this paper is Laffont and Tirole (1986) and Edmans and Gabaix (2008). In our model, like in these two papers, the firm’s profits are a deterministic function of the manager’s effort and of the manager’s type, with the manager observing his type prior to choosing his effort. Laffont and Tirole (1986) were the first to show how, in a static setting, this modelling choice permits one to use techniques from the mechanism design literature to solve for the optimal contract and to show how the latter can often be implemented with linear schemes. In independent work, Edmans and Gabaix (2008) show how this approach can be applied to a dynamic setting, allowing for risk aversion. Contrary to Laffont and Tirole and the current paper, Edmans and Gabaix (2008), however, do not characterize the optimal effort policy. Another difference is that Edmans and Gabaix assume contracting occurs at a time at which the agent does not possess any private information. This assumption has strong implications. For example, under risk neutrality, it implies that the principal always finds it optimal to induce the agent to choose the first-best level of effort in each period, thus precluding interesting dynamics. Lastly, while our schemes are often similar (modulo certain important details) to the ones proposed in Edmans and Gabaix, the stochastic processes and the effort policies considered here are significantly more

¹This part is even more preliminary than the rest. We apologize to those who believe their work should have been cited here and that we omitted to discuss.

general (for example, we allow effort to depend on the entire history of shocks experienced by the manager while Edmans and Gabaix restrict attention to policies that either depend only on time or at most on the current shocks).

The paper is also related to the empirical literature on the use of seniority-based incentives. This literature finds mixed evidence as to the effect of tenure on performance-related pay. While the early literature has suggested that managers with a longer tenure tend to have weaker incentives and explained this by the fact that the board of directors may tend to be captured by CEOs over time (e.g. Hill and Phan, 1991), more recent evidence points to the contrary (see, e.g. Lippert and Porter, 1997, but also Gibbons and Murphy, 1991). As one would expect, often these different findings originate in different choices about the relevant incentives measurements. At the theoretical level, our paper contributes to this literature by offering a new trade-off for the optimality of seniority-based incentives that, to the best of our knowledge, was not noticed before.

Lastly, from a methodological standpoint, the paper uses recent results in the dynamic mechanism design literature to arrive to a characterization of the necessary and sufficient conditions for incentive-compatibility. In particular, the approach here builds on the techniques developed in Pavan, Segal, and Toikka (2008)—hereafter referred to as PST.

The rest of the paper is organized as follows. Section 2 presents the baseline model. Section 2.2 characterizes the optimal mechanism. Section 3 extends the analysis to settings where the optimal effort policy is contingent on the entire history of shocks. Section 4 examines optimal schemes for risk-averse agents. All proofs omitted in the text are in the Appendix.

2 The Baseline Model

2.1 The environment

The firm’s shareholders (hereafter referred to as the principal) hire a manager (the agent) to work on a project over T periods, where T may be either finite or infinite. In each period t , the agent receives some private information $\theta_t \in \Theta_t$ about the environment or, equivalently, about his ability

to generate profits for the firm, and then chooses effort level $e_t \in E \subset \mathbb{R}$. We will assume that $\Theta_t \subset \mathbb{R}$ is either equal to $[\underline{\theta}_t, \bar{\theta}_t]$ or, in case $\bar{\theta}_t = +\infty$ to $[\underline{\theta}_t, \bar{\theta}_t) \subset \mathbb{R}$ for some $-\infty < \underline{\theta}_t \leq \bar{\theta}_t \leq +\infty$.²

To simplify the exposition (and facilitate the characterization of the optimal effort policy) we will assume that $E = \mathbb{R}$.³

The principal's profits π_t in period t , gross of any agent compensation, depend on the the sequence of effort decisions $e^t \equiv (e_s)_{s=1}^t$ exerted by the agent in previous periods and on the agent's current "type" θ_t .⁴ In particular, we assume that⁵

$$\pi_t = \theta_t + e_t + \sum_{\tau=1}^{t-1} \rho^\tau e_{t-\tau}$$

for some constant $\rho \geq 0$ that captures the persistence of the effect of the manager's effort on the firm's profits. The set of possible period- t profits will be denoted by

$$\Pi_t \equiv \left\{ \pi_t \in \mathbb{R} : \pi_t = \theta_t + e_t + \sum_{\tau=1}^{t-1} \rho^\tau e_{t-\tau}, \theta_t \in \Theta_t, e_s \in E, \forall s \leq t \right\}$$

Both θ^t and e^t are the agent's private information. On the contrary, the stream of profits π^t are assumed to be verifiable, which implies that the agent can be rewarded as a function of the firm's profits.

As is common in the literature, we equate the agent's period- t consumption c_t with the payment from the principal (in other words, we assume away the possibility of hidden savings). Such a restriction is however without loss of generality under the assumption of risk-neutrality considered in this section.

In each period, the principal may condition the agent's payment on the entire history of profits π^t . By choosing effort e_t in period t , the agent suffers a disutility $\psi(e_t)$. To ensure interior solutions

²As it will become clear from the analysis in the subsequent sections, that Θ_t is bounded from below is to guarantee that expected payoffs, when expressed taking incentives into account, are well defined.

³That effort can take negative values should not raise concerns: because e here simply stands for the effect of the agent's activity on the firm's performance, there is no reason to restrict e to be positive.

⁴From now on, we adopt the convention of denoting sequences of variables by their superscripts.

⁵Note that because θ_t is *not* restricted to be independent of the past shocks $\theta^{t-1} \equiv (\theta_1, \dots, \theta_{t-1})$, there is no loss of generality in assuming that π_t depends only on θ_t , as opposed to the entire history $\theta^t = (\theta_1, \dots, \theta_t)$. To see this, suppose that $\pi_t = f_t(\theta^t) + h_t(e^t)$ for some functions $f_t : \mathbb{R}^t \rightarrow \mathbb{R}$ and $h_t : \mathbb{R}^t \rightarrow \mathbb{R}$. It then suffices to change variables and simply let $\theta_t^{new} = f_t(\theta^t)$.

and to validate a certain dynamic envelope theorem (more below), we will assume that ψ is a continuously differentiable function and that there exist scalars $\bar{e} \in \mathbb{R}_{++}$ and $K > 1 + \sum_{s=1}^{T-1} (\delta\rho)^s$ such that $\psi(e) = 0$ for all $e < 0$, ψ is thrice continuously differentiable over $(0, \bar{e})$ with $\psi''(e) > 0$ and $\psi'''(e) \geq 0$ for all $e \in (0, \bar{e})$ and $\psi(e) = Ke$ for all $e > \bar{e}$.⁶

The agent's preferences over (lotteries over) streams of consumption levels c^T and streams of effort choices e^T are described by an expected utility function with (Bernoulli) utility given by

$$U^A(c^T, e^T) = \sum_{t=1}^T \delta^{t-1} [c_t - \psi(e_t)] \quad (1)$$

where $\delta < 1$ is a discount factor. As standard, the aforementioned specification presumes time-consistency. In what follows, we will thus assume that, after each history h_t , the agent maximizes the expectation of $U^A(c^T, e^T)$, where the expectation is taken with respect to whatever information is available to the agent after history h_t .

Throughout, we will also assume that $\delta\rho < 1$. The principal's payoff is given by the discounted sum of the firm's profits, net of the agent's compensation:

$$U^P(\pi^T, c^T) = \sum_{t=1}^T \delta^{t-1} [\pi_t - c_t].$$

The function U^P also corresponds to the principal's Bernoulli function used to evaluate lotteries over (π^T, c^T) .

In each period t , θ_t is drawn from a cumulative distribution function $F_t(\cdot|\theta^{t-1})$ with support Θ_t . Below, we will often find it convenient to describe the evolution of the agent's type through a collection of functions of independent shocks. More precisely, let $(\tilde{\varepsilon}_t)_{t=2}^T$ denote a collection of random variables, jointly independent and independent of θ_1 , each with support $\mathcal{E}_t \subset \mathbb{R}$ where $\mathcal{E}_t = [\underline{\varepsilon}_t \bar{\varepsilon}_t]$ if $\bar{\varepsilon}_t < +\infty$ and $[\underline{\varepsilon}_t, \bar{\varepsilon}_t)$ if $\bar{\varepsilon}_t = +\infty$, for some $-\infty < \underline{\varepsilon}_t \leq \bar{\varepsilon}_t \leq +\infty$.⁷ Then let $(z_t(\cdot))_{t=2}^T$ denote a collection of real-valued functions such that, for any $t \geq 2$, any θ_1 and any $\varepsilon^{t-1} \in \mathcal{E}^{t-1} \equiv \times_{s=2}^{t-1} \mathcal{E}_s$, the distribution of $z_t(\theta_1, (\varepsilon^{t-1}, \tilde{\varepsilon}_t))$ given $(\theta_1, \varepsilon^{t-1})$ is the same as that of θ_t

⁶These conditions are satisfied e.g. when $\bar{e} = K$ and $\psi(e) = (1/2)e^2$ for all $e \in [0, \bar{e}]$.

⁷The reason for restricting $\underline{\varepsilon}_t > -\infty$ is the same as for restricting Θ_t to be bounded from below; it guarantees that the agent's payoff in any incentive compatible mechanism can be conveniently expressed in integral form.

given $\theta^{t-1} = z^{t-1}(\theta_1, \varepsilon^{t-1})$, where $z^{t-1}(\theta_1, \varepsilon^{t-1}) \equiv (\theta_1, z_2(\theta_1, \varepsilon_2), \dots, z_{t-1}(\theta_1, \varepsilon^{t-1}))$. As indicated in PST (2008), any stochastic process (i.e. any collection of kernels $F = \langle F_t(\cdot|\cdot) \rangle_{t=1}^T$) admits at least one such representation.

We initially restrict attention to processes for which each z_t is separable in its first component.

Definition 1 Let $(\tilde{\varepsilon}_t)_{t=2}^T$ be a collection of random variables, jointly independent, and each independent of θ_1 , such that, for each $t \geq 2$, $\tilde{\varepsilon}_t$ is distributed according to the c.d.f. G_t strictly increasing on the interval $\mathcal{E}_t \subset \mathbb{R}$. The process for $(\tilde{\theta}_t)_{t=1}^T$ given by the kernels

$$F \equiv \langle F_t : \Theta^{t-1} \rightarrow \Delta(\Theta_t) \rangle_{t=1}^T$$

is separable in the first component (SFC) if it admits an independent-shock representation such that for each $t \geq 2$, the function $z_t : \Theta_1 \times \mathcal{E}^t \rightarrow \Theta_t$ takes the form

$$z_t(\theta_1, \varepsilon^t) = \gamma_t(\theta_1) + \phi_t(\varepsilon_2, \dots, \varepsilon_t)$$

for some functions $\gamma_t : \Theta_1 \rightarrow \mathbb{R}$ and $\phi_t : \mathcal{E}^t \rightarrow \mathbb{R}$.

The set of SFC processes is quite large and it includes for example all moving average processes, and more generally any ARIMA process with arbitrary parameters. We also assume the following.

Condition 1 (finite expectations) The distribution F_1 satisfies $\int |\tilde{\theta}_1| dF_1(\tilde{\theta}_1) < +\infty$ and the collection of distributions $(G_t)_{t=2}^T$ satisfies $\int |\phi_t(\tilde{\varepsilon}_2, \dots, \tilde{\varepsilon}_t)| dG_2(\tilde{\varepsilon}_2) \times \dots \times dG_t(\tilde{\varepsilon}_t) < +\infty$ for each t .⁸

Condition 2 (monotone hazard rate) The inverse hazard rate $\eta(\theta_1) \equiv \frac{1-F_1(\theta_1)}{f_1(\theta_1)}$ is non-increasing.

Condition 3 (smoothness+fosd) For each t , $\gamma_t(\cdot)$ is nondecreasing and differentiable. Furthermore, there exists a $M \in \mathbb{R}_+$ such that $\sup_t \{|\gamma_t'(\theta_1)|\} \leq M$ for all θ_1 .

⁸This condition guarantees that for any $1 \leq s < t \leq T$, any history $(\theta_1, \varepsilon^s)$, $\mathbb{E}[|z_t(\theta_1, \varepsilon^s, \tilde{\varepsilon}_{s+1}, \dots, \tilde{\varepsilon}_t)|] < +\infty$.

2.2 The mechanism design problem

The principal's problem consists of choosing a mechanism detailing for each period t a recommendation for the agent's effort e_t and a level of consumption c_t that depend on the sequence of realized profits π^t and (possibly) on a sequence of messages about the environment sent by the agent over time.

By the revelation principle, we restrict attention to direct mechanisms for which a truthful and obedient strategy is optimal for the agent. Let $\Theta^t \equiv \times_{\tau=1}^t \Theta_\tau$ and $\Pi^t = \times_{\tau=1}^t \Pi_\tau$. A (deterministic) direct mechanism $\Omega = \langle \xi_t, s_t \rangle_{t=1}^T$ consists of a collection of functions $\xi_t : \Theta^t \times \Pi^{t-1} \rightarrow E$ and $s_t : \Theta^t \times \Pi^t \rightarrow \mathbb{R}$ such that $\xi_t(\theta^t, \pi^{t-1})$ is the recommended level of effort for period t given the agent's reports θ^t and the observed past profits π^{t-1} , while $s_t(\theta^t, \pi^{t-1}, \pi_t)$ is the principal's payment (i.e. the agent's consumption) at the end of period t given the reports θ^t and the observed profits $\pi^t = (\pi^{t-1}, \pi_t)$. Note that $s_t(\theta^t, \pi^{t-1}, \pi_t)$ depends also on the current performance π_t . Equivalently, the mechanism Ω specifies for each period t and each history (θ^t, π^{t-1}) a recommended effort level $\xi_t(\theta^t, \pi^{t-1})$ along with a contingent payment scheme $s_t(\theta^t, \pi^{t-1}, \cdot) : \Pi_t \rightarrow \mathbb{R}$. With a slight abuse of notation, henceforth we will denote by $e_t(\theta^t) \equiv \xi_t(\theta^t, \pi^{t-1}(\theta^{t-1}))$ and by $c_t(\theta^t) = s_t(\theta^t, \pi^t(\theta^t))$ respectively the equilibrium effort and the equilibrium consumption level for period t given θ^t , where $\pi^t(\theta^t) = (\pi_s(\theta^s))_{s=1}^t$ with $\pi_s(\theta^s)$ defined recursively by $\pi_s(\theta^s) = \theta_s + \sum_{\tau=0}^{s-1} \rho^\tau \xi_{s-\tau}(\theta^{s-\tau}, \pi^{s-\tau-1}(\theta^{s-\tau-1}))$.

The timing of play in each period t is the following:

- At the beginning of period t , the agent learns $\theta_t \in \Theta_t$;
- The agent then sends a report $\hat{\theta}_t \in \Theta_t$;
- Finally, the mechanism reacts by prescribing an effort choice $e_t = \xi_t(\theta^t, \pi^{t-1})$ and a reward scheme $s_t(\theta^t, \pi^{t-1}, \cdot) : \Pi_t \rightarrow \mathbb{R}$.

The mechanism Ω is offered to the agent at date 1, after the agent has observed the first realization θ_1 of the process governing the evolution of θ_t . If the agent refuses to participate in the mechanism Ω , then both the agent and the principal receive their outside options, which we assume

to be equal to zero. If, instead, the agent accepts Ω , then he is obliged to stay in the relationship in all subsequent periods.⁹

Because we will often find it convenient to describe the evolution of the agent's type through an independent-shock representation (described above), hereafter, we will also consider direct mechanisms in which the agent reports the shocks ε_t in each period $t \geq 2$ instead of his period- t type θ_t . We will then denote such mechanisms by $\hat{\Omega} = \langle \hat{\xi}_t, \hat{s}_t \rangle_{t=1}^T$ where $\hat{\xi}_t : \Theta_1 \times \mathcal{E}^t \times \Pi^{t-1} \rightarrow E$ and $\hat{s}_t : \Theta_1 \times \mathcal{E}^t \times \Pi^t \rightarrow \mathbb{R}$ have the same interpretation as the mappings ξ_t and s_t in the primitive representation (the one in terms of the θ^t). Likewise, we will denote by $\hat{c}_t(\theta_1, \varepsilon^t)$ and by $\hat{e}_t(\theta_1, \varepsilon^t)$ the consumption and effort choices that are implemented in equilibrium given $(\theta_1, \varepsilon^t)$.

The optimal mechanism

To ease the understanding of the properties of the optimal mechanism, we start by considering the optimal effort policy in the absence of any private information.¹⁰

Proposition 1 *Assume the agent does not possess any private information, i.e. both the evolution of the environment (as captured by the process for θ_t) and the agent's effort choices e^T are publicly observable and verifiable. The optimal contract for the principal then implements the following effort policy:*

$$\psi'(e_t^{FB}) = 1 + \sum_{s=1}^{T-t} (\delta\rho)^s \quad \forall t, \forall (\theta_1, \varepsilon^t)$$

In particular, when $T = +\infty$, the optimal effort is stationary over time and is implicitly given by $\psi'(e^{FB}) = 1/[1 - \delta\rho]$. Likewise, the optimal effort is constant and given by $\psi'(e^{FB}) = 1$ when the manager's effort has only a transitory effect on the firm's performance, i.e. when $\rho = 0$. That the first-best effort policy is independent of any variation in the underlying environment is a consequence of the assumption of separability of the agent's disutility of effort from the underlying state θ_t .

⁹That participation must be guaranteed only in period one is clearly not restrictive when the principal can ask the agent to post bonds. Below, we will discuss also situations/implementations where, even in the absence of bonding, participation can be guaranteed after any history.

¹⁰Given the assumptions on ψ , $e_t^{FB} \in (0, \bar{e})$ for all t .

Clearly, the same first-best effort policy is implemented in any environment in which the agent's initial type θ_1 is publicly observed and verifiable (equivalently, in any environment in which the agent contracts with the principal before learning θ_1), irrespective of the observability of effort choices and future shocks θ_t .¹¹

Next, consider the case where the agent possesses relevant private information. Thus assume that both the evolution of the environment (as captured by the process for θ_t) and the agent's effort choices are the agent's private information. In addition, suppose that contracting between the agent and the principal occurs at a time at which the agent is already informed about his period-1 type θ_1 . The following proposition presents the main characterization result for this environment.

Proposition 2 *Assume conditions 1-3 hold. For any θ_1 , let $D_{1,1}(\theta_1) \equiv 1$ and for any $t \geq 2$, let $D_{1,t}(\theta_1) \equiv \gamma'_t(\theta_1) - \rho\gamma'_{t-1}(\theta_1)$, with, $\gamma'_1(\theta_1) \equiv 1$. Assume that $D_{1,t}(\theta_1) \geq 0$ for any θ_1 . Then consider the effort policy \hat{e}^* implicitly defined, for all t all θ_1 , by¹²*

$$\psi'(\hat{e}_t^*(\theta_1)) = 1 + \sum_{s=1}^{T-t} (\delta\rho)^s - \eta(\theta_1)D_{1,t}(\theta_1)\psi''(\hat{e}_t^*(\theta_1)) \quad \forall \theta_1, \forall t \geq 1, \quad (2)$$

unless $\psi''_+(0) \geq \left[1 + \sum_{s=1}^{T-t} (\delta\rho)^s\right] / [\eta(\theta_1)D_{1,t}(\theta_1)]$ in which case $\hat{e}_t^*(\theta_1) = 0$.

1. For any t and any θ_1 let

$$\alpha_t(\theta_1) \equiv \psi'(\hat{e}_t^*(\theta_1)) - \delta\rho\psi'(\hat{e}_{t+1}^*(\theta_1))$$

[if T is finite, then $\alpha_T(\theta_1) \equiv \psi'(\hat{e}_T^*(\theta_1))$]. Suppose the policy \hat{e}^* satisfies the following single-crossing condition

$$\left[\sum_{t=1}^T \delta^{t-1} \gamma'_t(\theta_1) [\alpha_t(\theta_1) - \alpha_t(\hat{\theta}_1)] \right] [\theta_1 - \hat{\theta}_1] \geq 0 \quad \forall \theta_1, \hat{\theta}_1 \in \Theta_1. \quad (3)$$

Then the recommendation policy

$$\hat{\xi}_t^*(\theta_1, \varepsilon^t, \pi^{t-1}) = \hat{e}_t^*(\theta_1) \quad \forall (\theta_1, \varepsilon^t, \pi^{t-1}) \in \Theta_1 \times \mathcal{E}^t \times \Pi^{t-1}$$

¹¹As we will show below, this property is however a consequence of the assumption of transferable utility, i.e. of the fact that both the agent's and the principal's preferences are linear in the transfers c_t .

¹²Throughout, ψ''_+ will denote the second right derivative of ψ .

together with the output-contingent reward scheme \hat{s} defined below are part of an optimal mechanism. The reward scheme \hat{s} is such that

$$\hat{s}_1^*(\theta_1, \pi_1) = S_1(\theta_1) + \alpha_1(\theta_1)\pi_1$$

while for any $t \geq 2$,

$$\hat{s}_t^*(\theta_1, \varepsilon^t, \pi^t) = \alpha_t(\theta_1)\pi_t$$

where

$$S_1(\theta_1) \equiv \sum_{t=1}^T \delta^{t-1} \left[\psi(\hat{e}_t^*(\theta_1)) + \int_{\underline{\theta}_1}^{\theta_1} D_{1,t}(s) \psi'(\hat{e}_t^*(s)) ds - \mathbb{E} [\alpha_t(\theta_1) \hat{\pi}_t^*(\theta_1, \varepsilon^t)] \right]$$

with $\hat{\pi}_t^*(\theta_1, \varepsilon^t) \equiv z_t(\theta_1, \varepsilon^t) + \hat{e}_t^*(\theta_1) + \sum_{\tau=1}^{t-1} \rho^\tau \hat{e}_{t-\tau}^*(\theta_1)$.

2. Suppose that for any t , either (a) $\rho = 0$ and the function $\eta(\cdot)D_{1,t}(\cdot)$ is non-increasing, or (b) $\psi(e) = ke^2/2$ for all $e \in [0, \bar{e}]$ and $\eta(\cdot)[D_{1,t}(\cdot) - \delta\rho D_{1,t+1}(\cdot)]$ is non-increasing [if T is finite, then for $t = T$, $\eta(\cdot)D_{1,T}(\cdot)$ is non-increasing]. Then the effort policy \hat{e}^* implicitly given by (2) satisfies the single-crossing condition (3).

Because this is one of the main results in the paper and because many of the subsequent results follow from arguments/techniques similar to those used to establish Proposition 2, the proof for this result is given below instead of being relegated to the Appendix. The reader interested only in the predictions of the model can however skip this proof and continue with the reading at page 20.

Proof. The structure of the proof is the following. Lemma 1 provides a necessary condition for incentive compatibility based on the application of a dynamic envelope theorem (as in Proposition 3 in PST) to the agent's optimization problem. Lemma 2 characterizes the effort policy \hat{e}^* that solves the principal's *relaxed* problem, where the latter considers only the necessary condition established in Lemma 1 (along with a certain participation constraint) and ignores all remaining constraints. Lemma 3 shows that, when the solution to the relaxed program satisfies the single-crossing condition of (3), then (i) it can be implemented by the linear scheme described in the proposition, (ii) under this scheme all types find it optimal to participate, and (iii) the lowest type

θ_1 receives a zero expected payoff in equilibrium. As discussed more in detail below, together these properties guarantee that the effort policy \hat{e}^* (equivalently, the recommendation policy $\hat{\xi}^*$) along with the linear reward scheme \hat{s}^* are part of an optimal mechanism. Finally, Lemma 4 completes the proof by establishing the result in Part 2.

Given the mechanism $\hat{\Omega} = \langle \hat{\xi}, \hat{s} \rangle$, let $V^{\hat{\Omega}}(\theta_1)$ denote the value function when the agent's period one type is θ_1 . This is simply the supremum of the agent's expected payoff over all possible reporting and effort strategies. The mechanism $\hat{\Omega}$ is incentive compatible if $V^{\hat{\Omega}}(\theta_1)$ coincides with the agent's expected payoff under a truthful and obedient strategy for every $\theta_1 \in \Theta_1$. We then have the following result.

Lemma 1 *The mechanism $\hat{\Omega}$ is incentive compatible only if $V^{\hat{\Omega}}(\theta_1)$ is Lipschitz continuous and for almost every $\theta_1 \in \Theta_1$*

$$\frac{dV^{\hat{\Omega}}(\theta_1)}{d\theta_1} = \mathbb{E} \left[\sum_{t=1}^T \delta^{t-1} D_{1,t}(\theta_1) \psi'(\hat{e}_t(\theta_1, \hat{\varepsilon}^t)) \right].$$

Proof of the lemma. Consider the following fictitious environment. At any point in time, the agent can misreport his private information but is then “forced” to choose effort so as to perfectly “hide” his lies. That is, at any period t , and for any given sequence of reports $(\hat{\theta}_1, \hat{\varepsilon}^t)$, the agent must exert effort e_t so that $\pi_t = \hat{\pi}_t(\hat{\theta}_1, \hat{\varepsilon}^t)$, where $\hat{\pi}_t(\hat{\theta}_1, \hat{\varepsilon}^t)$ is the equilibrium profit for period t given $(\hat{\theta}_1, \hat{\varepsilon}^t)$, as defined in the Proposition. Now let

$$\begin{aligned} \hat{e}_t(\theta_1, \varepsilon^t; \hat{\theta}_1, \hat{\varepsilon}^t) &= \hat{\pi}_t(\hat{\theta}_1, \hat{\varepsilon}^t) - z_t(\theta_1, \varepsilon^t) - \rho \sum_{\tau=1}^{t-1} \rho^{\tau-1} \hat{e}_{t-\tau}(\theta_1, \varepsilon^{t-\tau}; \hat{\theta}_1, \hat{\varepsilon}^{t-\tau}) \\ &= \hat{\pi}_t(\hat{\theta}_1, \hat{\varepsilon}^t) - z_t(\theta_1, \varepsilon^t) \\ &\quad - \rho \left(\hat{e}_{t-1}(\theta_1, \varepsilon^{t-1}; \hat{\theta}_1, \hat{\varepsilon}^{t-1}) + \sum_{\tau=1}^{t-2} \rho^\tau \hat{e}_{t-1-\tau}(\theta_1, \varepsilon^{t-1-\tau}; \hat{\theta}_1, \hat{\varepsilon}^{t-1-\tau}) \right) \\ &= \hat{\pi}_t(\hat{\theta}_1, \hat{\varepsilon}^t) - z_t(\theta_1, \varepsilon^t) - \rho \left(\hat{\pi}_{t-1}(\hat{\theta}_1, \hat{\varepsilon}^{t-1}) - z_{t-1}(\theta_1, \varepsilon^{t-1}) \right) \end{aligned} \quad (4)$$

denote the effort the agent must exert in period t to meet the target $\hat{\pi}_t(\hat{\theta}_1, \hat{\varepsilon}^t)$ when his true type is $(\theta_1, \varepsilon^t)$ given that he met the targets $(\hat{\pi}_s(\hat{\theta}_1, \hat{\varepsilon}^s))_{s=1}^{t-1}$ in all preceding periods, with $e_1(\theta_1; \hat{\theta}_1) = \hat{\pi}_1(\hat{\theta}_1) - \theta_1$.

Now fix $(\hat{\theta}_1, \hat{\varepsilon}^T)$ and let $(\hat{c}^T, \hat{\pi}^T)$ be the stream of equilibrium payments and profits that, given the mechanism $\hat{\Omega}$, correspond to the sequence of reports $(\hat{\theta}_1, \hat{\varepsilon}^T)$. For any $(\hat{\theta}_1, \hat{\varepsilon}^T)$ and given any sequence of true shocks $(\theta_1, \varepsilon^T)$, the agent's payoff in this fictitious environment is given by

$$\begin{aligned}
\hat{U}^A(\theta_1, \varepsilon^T; \hat{\theta}_1, \hat{\varepsilon}^T) &= \sum_{t=1}^T \delta^{t-1} [\hat{c}_t - \psi(\hat{e}_t(\theta_1, \varepsilon^t; \hat{\theta}_1, \hat{\varepsilon}^t))] \\
&= \hat{c}_1 - \psi(\hat{\pi}_1 - \theta_1) \\
&\quad + \sum_{t=2}^T \delta^{t-1} [\hat{c}_t - \psi(\hat{\pi}_t - z_t(\theta_1, \varepsilon^t) - \rho(\hat{\pi}_{t-1} - z_{t-1}(\theta_1, \varepsilon^{t-1})))] \\
&= \hat{c}_1 - \psi(\hat{\pi}_1 - \theta_1) \\
&\quad + \sum_{t=2}^T \delta^{t-1} [\hat{c}_t - \psi(\hat{\pi}_t - \gamma_t(\theta_1) - \phi_t(\varepsilon^t) - \rho(\hat{\pi}_{t-1} - \gamma_{t-1}(\theta_1) - \phi_{t-1}(\varepsilon^{t-1})))]
\end{aligned}$$

Condition 3 implies that \hat{U}^A is equi-Lipschitz continuous and differentiable in θ_1 . Now suppose the mechanism $\hat{\Omega}$ is incentive compatible in the unrestricted world where the agent is free to choose any effort he wants at any point in time. It is then necessarily incentive compatible also in this fictitious world where effort is pinned down by $(\theta_1, \varepsilon^T; \hat{\theta}_1, \hat{\varepsilon}^T)$ according to (4). The result in the Lemma then follows directly from Proposition 3 in PST: Letting $\hat{U}^A(\theta_1, \varepsilon^T)$ denote the agent's payoff when he follows a truthtelling and obedient strategy, we have that $\hat{\Omega}$ is incentive compatible only if $V^{\hat{\Omega}}$ is Lipschitz continuous and, for almost every $\theta_1 \in \Theta_1$,

$$\begin{aligned}
\frac{dV^{\hat{\Omega}}(\theta_1)}{d\theta_1} &= \mathbb{E} \left[\frac{\partial \hat{U}^A(\theta_1, \tilde{\varepsilon}^T)}{\partial \theta_1} \right] \\
&= \mathbb{E} \left[\psi'(\hat{e}_1(\theta_1)) + \sum_{t=2}^T \delta^{t-1} [\gamma'_t(\theta_1) - \rho \gamma'_{t-1}(\theta_1)] \psi'(\hat{e}_t(\theta_1, \tilde{\varepsilon}^t)) \right] \\
&= \mathbb{E} \left[\sum_{t=1}^T \delta^{t-1} D_{1,t}(\theta_1) \psi'(\hat{e}_t(\theta_1, \tilde{\varepsilon}_t)) \right],
\end{aligned}$$

which establishes the result. ■

Now, one can think of the principal's problem as consisting in choosing a pair of contingent policies $\langle \hat{\pi}, \hat{c} \rangle$ so as to maximize her expected payoff

$$\mathbb{E}[\hat{U}^P] = \mathbb{E} \left[\sum_{t=1}^T \delta^{t-1} \left[\hat{\pi}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t) - \hat{c}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t) \right] \right]$$

subject to all IC and IR constraints. Because both the principal's and the agent's preferences are quasilinear, $\mathbb{E}[\hat{U}^P]$ can be rewritten as expected total surplus, net of the agent's expected (intertemporal) rent:

$$\mathbb{E}[\hat{U}^P] = \mathbb{E} \left[\sum_{t=1}^T \delta^{t-1} \hat{\pi}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t) - \psi(\hat{e}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t)) \right] - \mathbb{E}[V^{\hat{\Omega}}(\tilde{\theta}_1)] \quad (5)$$

Using the result in the previous Lemma and integrating by parts, the agent's expected (intertemporal) rent can in turn be written as

$$\begin{aligned} \mathbb{E}[V^{\hat{\Omega}}(\tilde{\theta}_1)] &= V^{\hat{\Omega}}(\theta_1) + \mathbb{E} \left[\frac{1 - F(\tilde{\theta}_1)}{f(\tilde{\theta}_1)} \frac{dV^{\hat{\Omega}}(\tilde{\theta}_1)}{d\theta_1} \right] \\ &= V^{\hat{\Omega}}(\theta_1) + \mathbb{E} \left[\eta(\tilde{\theta}_1) \sum_{t=1}^T \delta^{t-1} D_{1,t}(\tilde{\theta}_1) \psi'(\hat{e}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t)) \right] \end{aligned} \quad (6)$$

Finally, substituting (6) into (5), we have that

$$\begin{aligned} \mathbb{E}[\hat{U}^P] &= \mathbb{E} \left[\sum_{t=1}^T \delta^{t-1} \left(\hat{\pi}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t) - \psi(\hat{e}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t)) - \eta(\tilde{\theta}_1) D_{1,t}(\tilde{\theta}_1) \psi'(\hat{e}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t)) \right) \right] - V^{\hat{\Omega}}(\theta) \quad (7) \\ &= \mathbb{E} \left[\sum_{t=1}^T \delta^{t-1} \left(z_t(\tilde{\theta}_1, \tilde{\varepsilon}^t) + \hat{e}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t) + \sum_{\tau=1}^{t-1} \rho^\tau \hat{e}_{t-\tau}(\tilde{\theta}_1, \tilde{\varepsilon}^{t-\tau}) - \psi(\hat{e}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t)) \right. \right. \\ &\quad \left. \left. - \eta(\tilde{\theta}_1) D_{1,t}(\tilde{\theta}_1) \psi'(\hat{e}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t)) \right) \right] \\ &\quad - V^{\hat{\Omega}}(\theta) \end{aligned}$$

Next, consider a relaxed program for the principal that consists of choosing an effort policy \hat{e} and a constant $V^{\hat{\Omega}}(\theta) \geq 0$ so as to maximize $\mathbb{E}[\hat{U}^P]$. The solution to this relaxed program is given in the following lemma.

Lemma 2 *Suppose that, for any $\theta_1 \in \Theta_1$ any t , $D_{1,t}(\theta_1) \geq 0$. The (almost-unique) solution to the principal's relaxed program is then given by $V^{\hat{\Omega}}(\theta) = 0$ along with the effort policy \hat{e}^* as in the Proposition.*

Proof of the Lemma. The result follows directly from pointwise maximization of (7). The assumptions that ψ is a continuously differentiable function with $\psi(e) = 0$ for all $e < 0$, $\psi''(e) > 0$ and $\psi'''(e) \geq 0$ for all $e \in [0, \bar{e}]$, $\psi'(e) = K$ for all $e > \bar{e}$, together with $D_{1,t}(\theta_1) \geq 0$ for all θ_1 , imply that, for all t all $(\theta_1, \varepsilon^t)$, the principal's payoff \hat{U}^P is strictly increasing in e_t for all

$e_t < \hat{e}_t^*(\theta_1)$, and strictly decreasing in e_t for all $e_t > \hat{e}_t^*(\theta_1)$, where $\hat{e}_t^*(\theta_1)$ is implicitly given by (2) when $\psi_+''(0) < \left[1 + \sum_{s=1}^{T-t} (\delta\rho)^s\right] / [\eta(\theta_1)D_{1,t}(\theta_1)]$ and $\hat{e}_t^*(\theta_1) = 0$ otherwise. ■

To prove the result in part 1, it then suffices to show that, when the effort policy in (2) satisfies the single-crossing condition (3), it can be implemented by the linear scheme proposed in the Proposition. That is, it suffices to show that, under this scheme, (i) the agent finds it optimal to participate in period one, (ii) the agent finds it optimal to report all his private information truthfully and obey to the principal's recommendations, and (iii) the lowest period-1 type's expected payoff is equal to his outside option, i.e. $V^{\hat{\Omega}}(\underline{\theta}) = 0$. This is shown in the following lemma.

Lemma 3 *Assume the effort policy \hat{e}^* that solves the relaxed program (as implicitly given by (2)) satisfies the single-crossing condition (3). Then the mechanism $\hat{\Omega} = \langle \hat{\xi}_t, \hat{s}_t \rangle_{t=1}^T$ where $\hat{\xi}_t$ and \hat{s}_t are, respectively, the recommendation policy and the (output) contingent reward scheme described in the Proposition, implements the effort policy \hat{e}^* , it induces any type θ_1 to participate and gives the lowest period-1 type $\underline{\theta}_1$ a zero expected payoff.*

Proof of the Lemma. Because neither $\hat{\xi}_t$ nor \hat{s}_t depend on ε^t , it is immediate that the agent finds it optimal to report the shocks truthfully. Furthermore, conditional upon reporting $\hat{\theta}_1$ in period 1, it is also immediate that, at any period $t \geq 1$ the agent finds it optimal to follow the principal's recommendation and choose effort $\hat{e}_t^*(\hat{\theta}_1)$, irrespective of his true period-1 type θ_1 , the true shocks ε^t and the history of past performances π^{t-1} . To see this, note that at any period $t \geq 1$, and for any history $(\theta_1, \varepsilon^t, \hat{\theta}_1, \hat{\varepsilon}^t, \pi^{t-1}, e^{t-1})$, the problem that the agent faces in period t is to choose a (possibly contingent) plan $(e_t, e_{t+1}(\cdot) \dots, e_T(\cdot))$ to maximize

$$\mathbb{E} \left[\sum_{\tau=t}^T \delta^{\tau-t} \left(\alpha_{\tau}(\hat{\theta}_1) \left(\tilde{e}_{\tau} + \sum_{\kappa=1}^{\tau-1} \rho^{\kappa} \tilde{e}_{\tau-\kappa} + z_{\tau}(\theta_1, \tilde{\varepsilon}^{\tau}) \right) - \psi(\tilde{e}_{\tau}) \right) \mid \theta_1, \varepsilon^t \right]$$

The solution to this problem is given by the (non-contingent) effort policy implicitly defined by

$$\psi'(e_t) = \alpha_t(\hat{\theta}_1) + \sum_{\tau=1}^T (\delta\rho)^{\tau} \alpha_{t+\tau}(\hat{\theta}_1)$$

When the sequence $\left(\alpha_t(\hat{\theta}_1)\right)_{t=1}^T$ is the one specified in the Proposition, the effort policy that solves these conditions is the policy \hat{e}^* that solves the relaxed program.

It remains to show that each type θ_1 finds it optimal to report truthfully and to participate, and that type $\hat{\theta}_1$ expects a zero payoff from the relationship. That each type θ_1 finds it optimal to participate is guaranteed by the fact that his expected payoff (under a truthful and obedient strategy) is given by

$$\sum_{t=1}^T \delta^{t-1} \int_{\theta_1}^{\theta_1} D_{1,t}(s) \psi'(e_t^*(s)) ds$$

which is non-negative because $D_{1,t}(\theta_1) \geq 0$ and $\psi'(e) \geq 0$. To see that each type θ_1 finds it optimal to report truthfully let

$$\begin{aligned} U(\theta_1; \hat{\theta}_1) &\equiv \sum_{t=1}^T \delta^{t-1} \int_{\theta_1}^{\hat{\theta}_1} D_{1,t}(s) \psi'(e_t^*(s)) ds \\ &\quad + \mathbb{E} \left[\sum_{t=1}^T \delta^{t-1} \alpha_t(\hat{\theta}_1) [z_t(\theta_1, \tilde{\varepsilon}^t) - z_t(\hat{\theta}_1, \tilde{\varepsilon}^t)] \right] \end{aligned}$$

The function $U(\theta_1; \hat{\theta}_1)$ simply represents the payoff that type θ_1 obtains by mimicking type $\hat{\theta}_1$.

Next note that

$$\frac{\partial U(\theta_1; \hat{\theta}_1)}{\partial \theta_1} = \sum_{t=1}^T \delta^{t-1} \alpha_t(\hat{\theta}_1) \gamma'_t(\theta_1).$$

The single-crossing condition in the Proposition guarantees that

$$\left[\frac{dU(\theta_1; \theta_1)}{d\theta_1} - \frac{\partial U(\theta_1; \hat{\theta}_1)}{\partial \theta_1} \right] [\theta_1 - \hat{\theta}_1] \geq 0$$

To see this note that

$$\begin{aligned}
\frac{dU(\theta_1; \theta_1)}{d\theta_1} - \frac{\partial U(\theta_1; \hat{\theta}_1)}{\partial \theta_1} &= \sum_{t=1}^T \delta^{t-1} D_{1,t}(\theta_1) \psi'(\hat{e}_t^*(\theta_1)) - \sum_{t=1}^T \delta^{t-1} \alpha_t(\hat{\theta}_1) \gamma'_t(\theta_1) \\
&= \sum_{t=1}^T \delta^{t-1} \left\{ \begin{array}{l} [\gamma'_t(\theta_1) - \rho \gamma'_{t-1}(\theta_1)] \psi'(\hat{e}_t^*(\theta_1)) \\ -\gamma'_t(\theta_1) [\psi'(\hat{e}_t^*(\hat{\theta}_1)) - \delta \rho \psi'(\hat{e}_{t+1}^*(\hat{\theta}_1))] \end{array} \right\} \\
&= \psi'(\hat{e}_1^*(\theta_1)) - \psi'(\hat{e}_1^*(\hat{\theta}_1)) + \delta \rho \psi'(\hat{e}_2^*(\theta_1)) \\
&\quad + \delta \gamma'_2(\theta_1) \psi'(\hat{e}_2^*(\theta_1)) - \delta \rho \psi'(\hat{e}_2^*(\theta_1)) \\
&\quad + \dots \\
&= [\psi'(\hat{e}_1^*(\theta_1)) - \delta \rho \psi'(\hat{e}_2^*(\theta_1))] - [\psi'(\hat{e}_1^*(\hat{\theta}_1)) - \delta \rho \psi'(\hat{e}_1^*(\hat{\theta}_1))] \\
&\quad + \delta \gamma'_2(\theta_1) [\psi'(\hat{e}_2^*(\theta_1)) - \delta \rho \psi'(\hat{e}_3^*(\theta_1))] \\
&\quad - \delta \gamma'_2(\theta_1) [\psi'(\hat{e}_2^*(\hat{\theta}_1)) - \delta \rho \psi'(\hat{e}_3^*(\hat{\theta}_1))] \\
&\quad + \dots \\
&= \sum_{t=1}^T \delta^{t-1} \gamma'_t(\theta_1) [\alpha_t(\theta_1) - \alpha_t(\hat{\theta}_1)].
\end{aligned}$$

The result then follows from Lemma 12 in PST.

Lemma 4 *Suppose that, for any t , either (a) $\rho = 0$ and the function $\eta(\cdot)D_{1,t}(\cdot)$ is non-increasing, or (b) $\psi(e) = ke^2/2$ for all $e \in [0, \bar{e}]$ and $\eta(\cdot)[D_{1,t}(\cdot) - \delta \rho D_{1,t+1}(\cdot)]$ is non-increasing [if T is finite, then $\eta(\cdot)D_{1,T}(\cdot)$ is non-increasing]. Then the effort policy \hat{e}^* implicitly given by (2) satisfies condition (3), i.e., for any $\theta_1, \hat{\theta}_1 \in \Theta_1$:*

$$\left[\sum_{t=1}^T \delta^{t-1} \gamma'_t(\theta_1) [\alpha_t(\theta_1) - \alpha_t(\hat{\theta}_1)] \right] [\theta_1 - \hat{\theta}_1] \geq 0$$

Proof of the lemma. We establish the result by showing that, under the assumptions in the lemma, $\alpha_t(\theta_1)$ is non-decreasing in θ_1 , for each $t \geq 1$. Consider first case (a). When $\rho = 0$, $\alpha_t(\theta_1) = \psi'(\hat{e}_t^*(\theta_1))$. It then suffices to show that the effort policy $\hat{e}_t^*(\theta_1)$ implicitly given by (2) is non-decreasing. To see that this is indeed the case, it is enough to recognize that the dynamic virtual surplus (as defined in 7) has increasing differences in e_t and $-\eta(\theta_1)D_{1,t}(\theta_1)$ and, by assumption, $\eta(\theta_1)D_{1,t}(\theta_1)$ is non-increasing.¹³

¹³The relevant terms of the dynamic virtual surplus are $e_t + \sum_{s=1}^{T-t} (\delta \rho)^s e_t - \psi(e_t) - \eta(\theta_1)D_t(\theta_1)\psi'(e_t)$.

Next, consider case (b). For any $t < T$ and any $\theta'_1 > \theta''_1$,

$$\begin{aligned}
\alpha_t(\theta'_1) - \alpha_t(\theta''_1) &= [\psi'(\hat{e}_t^*(\theta'_1)) - \delta\rho\psi'(\hat{e}_{t+1}^*(\theta'_1))] - [\psi'(\hat{e}_t^*(\theta''_1)) - \delta\rho\psi'(\hat{e}_{t+1}^*(\theta''_1))] \\
&= \left[1 + \sum_{s=1}^{T-t} (\delta\rho)^s - \eta(\theta'_1)D_{1,t}(\theta'_1)k - \delta\rho \left(1 + \sum_{s=1}^{T-t-1} (\delta\rho)^s - \eta(\theta'_1)D_{1,t+1}(\theta'_1)k \right) \right] \\
&\quad - \left[1 + \sum_{s=1}^{T-t} (\delta\rho)^s - \eta(\theta''_1)D_{1,t}(\theta''_1)k - \delta\rho \left(1 + \sum_{s=1}^{T-t-1} (\delta\rho)^s - \eta(\theta''_1)D_{1,t+1}(\theta''_1)k \right) \right] \\
&= k [\eta(\theta''_1) (D_{1,t}(\theta''_1) - \delta\rho D_{1,t+1}(\theta''_1)) - \eta(\theta'_1) (D_{1,t}(\theta'_1) - \delta\rho D_{1,t+1}(\theta'_1))] \\
&\geq 0
\end{aligned}$$

where the inequality follows from the assumption that $\eta(\cdot)[D_{1,t}(\cdot) - \delta\rho D_{1,t+1}(\cdot)]$ is non-increasing.

Likewise, when T is finite, then

$$\alpha_T(\theta'_1) - \alpha_T(\theta''_1) = \psi'(\hat{e}_T^*(\theta'_1)) - \psi'(\hat{e}_T^*(\theta''_1)) = k [\eta(\theta''_1)D_{1,T}(\theta''_1) - \eta(\theta'_1)D_{1,T}(\theta'_1)] \geq 0$$

where the inequality follows from the assumption that $\eta(\cdot)D_{1,T}(\cdot)$ is non-increasing. ■

This completes the proof. ■

Note that, because the agent is indifferent over the way the constant term $S(\theta_1)$ is distributed over time, an equivalent (linear) implementation consists in paying the agent in each period t a fixed wage

$$\psi(\hat{e}_t^*(\theta_1)) + \int_{\underline{\theta}_1}^{\theta_1} D_{1,t}(s)\psi'(\hat{e}_t^*(s))ds - \mathbb{E} [\alpha_t(\theta_1)\hat{\pi}_t^*(\theta_1, \tilde{\varepsilon}^t)]$$

plus a fraction $\alpha_t(\theta_1)$ of the current profits π_t , with $S_1(\theta_1)$ now defined by

$$S_1(\theta_1) = \psi(\hat{e}_1^*(\theta_1)) + \int_{\underline{\theta}_1}^{\theta_1} \psi'(\hat{e}_1^*(s))ds - \alpha_1(\theta_1)\hat{\pi}_1^*(\theta_1).$$

While the particular way the constant term $S_1(\theta_1)$ is distributed over time is clearly inconsequential for incentives, certain choices may have the advantage of guaranteeing that, if the agent has the option to leave the relationship at any point in time, he does not find it optimal to do so. To see this, suppose that $T = +\infty$ and that all shocks are strictly positive, i.e. $\theta_1, \varepsilon_s > 0$ all s . Then front-loading the payment

$$- \sum_{t=1}^{\infty} \delta^{t-1} \mathbb{E} [\alpha_t(\theta_1)\hat{\pi}_t^*(\theta_1, \tilde{\varepsilon}^t)]$$

and then paying in each period

$$\psi(\hat{e}_t^*(\theta_1)) + \int_{\underline{\theta}_1}^{\theta_1} D_{1,t}(s)\psi'(\hat{e}_t^*(s))ds + \alpha_t(\theta_1)\pi_t$$

guarantees participation in each period, at any truthful history.

We now turn to the properties of the optimal effort policy. Because $D_{1,t} \geq 0$ and ψ' is convex, the optimal effort policy involves downward distortions. These distortions in turn depend on the first-period inverse hazard rate $\eta(\theta_1)$ and on the function $D_{1,t}$, which captures the effect of θ_1 on both θ_t and θ_{t-1} , taking into account the persistent effect of effort. When the process for θ_t satisfies condition SFC, these distortions are independent of the realizations of the shocks ε_t and of their distributions G_t . Whether $\hat{e}_t^*(\theta_1)$ increases or decreases with t then depends entirely on the dynamics of $D_{1,t}(\theta_1)$ as illustrated in the following examples, where the conditions of Part 2 of Proposition 2 are clearly satisfied.

Example 1 Suppose that $T = \infty$ and that θ_t evolves according to an AR(1) process

$$\theta_t = \beta\theta_{t-1} + \varepsilon_t$$

for some $\beta \in (0, 1)$ with $\beta > \rho \geq 0$. Then $D_{1,t}(\theta_1) = \beta^{t-2}(\beta - \rho)$ for all $\theta_1 \in \Theta_1$. It follows that $\hat{e}_t^*(\theta_1)$ increases over time and

$$\lim_{t \rightarrow \infty} \hat{e}_t^*(\theta_1) = 1/[1 - \delta\rho] = e^{FB} \quad \forall \theta_1.$$

Example 2 Assume that each θ_t is i.i.d., so that $D_{1,t}(\theta_1) = 0$ for all $t \geq 2$ and all θ_1 . Then effort is distorted only in the first period, i.e. $\hat{e}_1^*(\theta_1) < e_1^{FB}$ and $\hat{e}_t^* = e_t^{FB}$ for all $t \geq 2$.

Example 3 Suppose θ_t follows a random walk, i.e.

$$\theta_t = \theta_{t-1} + \varepsilon_t$$

and that effort has only a contemporaneous effect on the firm's profits (i.e. $\rho = 0$). Then $\hat{e}_t^*(\theta_1)$ is constant over time and coincides with the static optimal effort.

The result in Example 1 is actually quite general; most ARIMA(k,q,m) processes have the property that $\lim_{t \rightarrow \infty} D_{1,t} = 0$, where $D_{1,t}$ are nonnegative scalars decreasing in t that depend on the parameters (k,q,m) of the ARIMA process.

Example 2 is the case considered by Edmans and Gabaix (2008) where it is also assumed that $\rho = 0$. However, contrary to the case considered here, they assume that contracting occurs before the agent learns his first-period type. As discussed above, together with risk neutrality this implies that the sequence of effort decisions is always efficient.

Finally, the random walk case of Example 3 is also a process that is sometimes considered in the literature (especially in continuous-time Brownian models). In this case, because effort is constant over time, the optimal mechanism can be implemented by offering in period one the same menu of linear contracts that the principal would offer in a static relationship, and then committing to using the contract selected in period one in each subsequent period. Each linear contract (indexed by θ_1) has a fixed payment of

$$S(\theta_1) \equiv \psi(\hat{e}^*(\theta_1)) + \int_{\theta_1}^{\theta_1} \psi'(\hat{e}^*(s)) ds - \alpha(\theta_1)[\theta_1 + \hat{e}^*(\theta_1)]$$

together with a piece-rate $\alpha(\theta_1)$. These contracts are reminiscent of those derived in Laffont and Tirole (1986) in a static regulatory setting. Contrary to the static case, the entire linear scheme $S(\theta_1) + \alpha(\theta_1)\tilde{\pi}_t$ — as opposed to the point $S(\theta_1) + \alpha(\theta_1)[\theta_1 + \hat{e}^*(\theta_1)]$ — is now used over time. This is a direct consequence of the fact that the firm's performance $\tilde{\pi}_t$ now changes stochastically over time in response to the shocks $\tilde{\varepsilon}^t$. Also note that while the optimal mechanism can be implemented by using in each period the static optimal contract for period one, this does not mean that the dynamic optimal mechanism coincides with a sequence of static optimal contracts, as in Baron and Besanko (1984). Rather the opposite. In fact, because the agent's type θ_t (and its distribution) changes over time, the sequence of static optimal contracts entails a different choice of effort for each period. What the result then implies is that, despite the lack of stationarity, it is optimal for the principal to commit to the same reward scheme (and to induce the same effort) as if the agent's type were constant over time.

Out of curiosity, also note that the optimal reward scheme (and the corresponding effort dynamics) when θ_t follows a random walk coincide with the one that the principal would offer in an environment in which the shocks have only a transitory (as opposed to permanent) effect on the firm’s performance. More generally, assuming $\mathbb{E}[\tilde{\varepsilon}_t] = 0$ for all $t > 1$ and letting $(a_s^t)_{s,t}$ denote arbitrary scalars, the optimal contract is the same when $\theta_t = \gamma_t' \theta_1 + \sum_{s=2}^t a_s^t \varepsilon_s$ as when $\theta_t = \gamma_t' \theta_1 + \varepsilon_t$.

Seniority. While the examples above highlight interesting properties for the dynamics of effort, they also have important implications for the dynamics of the optimal reward scheme. What these examples have in common is the fact that the effect of the agent’s first-period type on his future types declines over time (strictly in the first example). We find this property of “declining correlation” to be reasonable for many stochastic processes describing the evolution of the agent’s productivity. As anticipated in the introduction, this property has implications for the dynamics of the optimal reward scheme. In particular, it helps understand why it may be optimal to reward managers with a longer tenure with a more high-powered incentive scheme, e.g. by giving them more equity in the firm. To illustrate, consider the case presented in Example 1 above, and note that in this case

$$\alpha_t(\theta_1) = 1 - \eta(\theta_1) (\beta - \rho) \beta^{t-2} [\psi''(\hat{e}_t^*(\theta_1)) - \delta \rho \beta \psi''(\hat{e}_{t+1}^*(\theta_1))]. \quad (8)$$

This term, which captures the power of the incentive scheme, is typically increasing in t (it is easy to see that this is the case, for example, when $\rho = 0$ —in which case $\alpha_t(\theta_1)$ reduces to $\psi'(\hat{e}_t^*(\theta_1))$ —or when ψ is quadratic).

Note that the reason why the power of the incentive scheme here increases over time is not driven by variations of the manager’s preferences. It is merely a consequence of the fact that, when he was hired, the manager possessed relevant private information about his ability to generate profits for the firm. In the case of an AR(1) process, the correlation between the manager’s initial type and his future types declines over time. This implies that, to minimize the informational rents that the firms’ shareholders must leave to the manager, it is optimal to (downward) distort the agent’s effort more when he is “young” than when he is “old”. Because the manager’s effort

is increasing in the sensitivity α_t of his reward scheme to the firm's performance π_t , this in turn implies that it is optimal to give the manager a more "high powered" incentive scheme when he is "senior" than when he is a "young".

Clearly, as mentioned in the introduction, other explanations for seniority have been suggested in the literature. Gibbons and Murphy (1991), for example, argue that career-concern incentives decline over time and, by implication, managers with a higher tenure must be provided with stronger "explicit contracts", i.e. with more high-powered incentive schemes. In their model, explicit incentives are a substitute for career-concern incentives.¹⁴

Another explanation for the correlation between seniority and the power of the incentive scheme may come from the fact that the disutility of effort may decline over time, most notably as the result of learning by doing. While we find such explanations plausible in certain environments, what our results indicate is that, even in the absence of any assumption of time-variant preferences/technologies/career concerns, seniority may arise quite naturally as the result of an optimal intertemporal screening problem in settings in which the correlation between the manager initial type/talent and his future ones declines over time. We believe this is a plausible assumption for most environments of interest.

3 Fully-contingent effort policies

Consider now an environment in which the process for θ_t does not satisfy the SFC condition. When this is the case, the optimal effort policy typically depends not only on θ_1 but also on the realization of the shocks ε^t . In many cases of interest, the optimal mechanism can still be implemented by a menu of linear contracts, but the agent must now be allowed to change the slope of these contracts over time in response to the shocks. To illustrate, assume that $\rho = 0$, so that effort has only a transitory effect on the firm's performance, that $T < +\infty$,¹⁵ that the stochastic process governing the evolution of θ_t is *Markov* so that each kernel $F_t(\cdot|\theta^{t-1})$ depends on θ^{t-1} only through θ_{t-1} .

¹⁴For a detailed analysis of career concerns incentives, see Dewatripont, Jewitt and Tirole (1999).

¹⁵The results in this section actually extend to $T = +\infty$ under mild additional conditions.

Finally, assume that, for any t any θ_{t-1} , $F_t(\cdot|\theta_{t-1})$ is absolutely continuous and strictly increasing over Θ_t with density $f_t(\theta_t|\theta_{t-1}) > 0$ for all $\theta_t \in (\underline{\theta}_t, \bar{\theta}_t)$, and that, for each t , there exists an integrable function $B_t : \Theta_t \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ such that, for any $\theta^t \in \Theta^t$, $\partial F_t(\theta_t|\theta_{t-1})/\partial\theta_{t-1}$ exists and $|\partial F_t(\theta_t|\theta_{t-1})/\partial\theta_{t-1}| \leq B_t(\theta_t)$.¹⁶

Following steps similar to those used in the proof of Proposition 2, it is easy to see that the solution to the principal's relaxed program is an effort policy \hat{e}^* that is implicitly defined by the following conditions¹⁷

$$\psi'(\hat{e}_t^*(\theta_1, \varepsilon^t)) = 1 - \eta(\theta_1) \frac{\partial z_t(\theta_1, \varepsilon^t)}{\partial \theta_1} \psi''(\hat{e}_t^*(\theta_1, \varepsilon^t)) \quad (9)$$

where (z, G) is *any* independent shock representation for the process that corresponds to the kernels $F = \langle F_t(\cdot|\cdot) \rangle_{t=1}^T$.

Equivalently, this condition can be expressed in terms of the primitive representation F as follows. Consider the mechanism Ω where in each period the agent reports θ_t (as opposed to ε_t). Following steps similar to those in the proof of Proposition 2 (see also Proposition 2 in PST), one can show that, in any IC mechanism, after almost every truthful history¹⁸ h_{t-1} , the value function $V^\Omega(\theta^{t-1}, \theta_t)$ is Lipschitz continuous in θ_t and, for almost every θ_t ,

$$\frac{\partial V^\Omega(\theta^t)}{\partial \theta_t} = \mathbb{E}_{\tilde{\theta}^T|\theta^t} \left[\sum_{\tau=t}^T \delta^{\tau-t} J_\tau^T(\tilde{\theta}^\tau) \psi'(e_\tau(\tilde{\theta}^\tau)) \right] \quad (10)$$

where for all t , $J_t^t(\theta^t) \equiv 1$, and for any $\tau > t$,

$$J_t^\tau(\theta^\tau) \equiv \sum_{\substack{K \in \mathbb{N}, l \in \mathbb{N}^{K+1}; \\ t=l_0 < \dots < l_K = \tau}} \prod_{k=1}^K I_{l_{k-1}}^{l_k}(\theta^{l_k}),$$

with

$$I_l^m(\theta^m) \equiv - \frac{\partial F_m(\theta_m|\theta^{m-1})/\partial \theta_l}{f_m(\theta_m|\theta^{m-1})}.$$

¹⁶Throughout, if $\theta_{t-1} = \underline{\theta}_{t-1}$, then $\partial F_t(\theta_t|\underline{\theta}_{t-1})/\partial\theta_{t-1}$ denotes the right derivative of F_t with respect to θ_{t-1} .

¹⁷Again, this presumes that the RHS of (9) evaluated at $e = 0$ is positive, which is the case when $\psi''_+(0)$ is not too high. When this is not the case, then $\hat{e}_t^*(\theta_1, \varepsilon^t) = 0$.

¹⁸A truthful history h_{t-1} is one that is reached by reporting θ^{t-1} truthfully and following the principal's effort recommendations in each period $s = 1, \dots, t-1$. For simplicity, whenever there is no risk of confusion, we will denote a truthful history simply by θ^{t-1} .

The function $J_t^T(\theta^\tau)$ is an impulse-response function that captures the total effect of a variation of θ_t on the distribution of θ_τ taking into account all effects on intermediate types $(\theta_{t+1}, \dots, \theta_{\tau-1})$.

While condition (10) applies to *any* (differentiable) process, in the case of a Markov process, because each $I_l^m(\theta^m)$ is equal to zero for all $l < m-1$ and depends on θ^m only through (θ_m, θ_{m-1}) , the impulse response $J_t^T(\theta^\tau)$ reduces to a function of $(\theta_t, \dots, \theta_\tau)$ only and can be written as $J_t^T(\theta_t, \theta_{t+1}, \dots, \theta_\tau) = \prod_{k=t+1}^\tau I_{k-1}^k(\theta_k, \theta_{k-1})$, with each I_{k-1}^k given by

$$I_{k-1}^k(\theta_k, \theta_{k-1}) = \frac{-\partial F_k(\theta_k|\theta_{k-1})/\partial \theta_{k-1}}{f_k(\theta_k|\theta_{k-1})}.$$

Applying condition (10) to $t = 1$, we then have that

$$V^\Omega(\theta_1) = \mathbb{E}_{(\tilde{\theta}_2, \dots, \tilde{\theta}_T)|\theta_1} \left[\sum_{t=1}^T \delta^{t-1} \int_{\underline{\theta}_1}^{\theta_1} J_1^t(s, \tilde{\theta}_2, \dots, \tilde{\theta}_t) \psi'(e_t(s, \tilde{\theta}_2, \dots, \tilde{\theta}_t)) ds \right].$$

Integrating by parts, this implies that the expected ex-ante surplus for the agent is given by

$$\mathbb{E} \left[V^\Omega(\tilde{\theta}_1) \right] = \mathbb{E}_{\tilde{\theta}^T} \left[\eta(\tilde{\theta}_1) \sum_{t=1}^T \delta^{t-1} J_1^t(\tilde{\theta}^t) \psi'(e_t(\tilde{\theta}^t)) \right].$$

The principal's expected payoff is thus given by

$$\mathbb{E}[U^P] = \mathbb{E} \left[\sum_{t=1}^T \delta^{t-1} \left\{ \tilde{\theta}_t + \hat{e}_t(\tilde{\theta}^t) - \psi(\hat{e}_t(\tilde{\theta}^t)) - \eta(\tilde{\theta}_1) J_1^t(\tilde{\theta}^t) \psi'(e_t(\tilde{\theta}^t)) \right\} \right] - V^\Omega(\underline{\theta}_1).$$

Provided that $J_1^t(\theta^t) \geq 0$ for each t all θ^t , which is the case under FOSD, the optimal effort policy can then be obtained by pointwise maximization of $\mathbb{E}[U^P]$ and is given by

$$\psi'(e_t^*(\theta^t)) = 1 - \eta(\theta_1) J_1^t(\theta^t) \psi''(e_t^*(\theta^t))$$

if $1 - \eta(\theta_1) J_1^t(\theta^t) \psi''(0) > 0$ and by $e_t^*(\theta^t) = 0$ otherwise.

This condition is the analogue of (9) expressed in terms of the primitive representation (the one where the agent reports θ_t as opposed to ε^t). From the same arguments as in the previous section, it then follows that, if there exists a payment scheme s that implements the effort policy e^* and gives zero expected surplus to the lowest period-one type (i.e. such that $V^\Omega(\underline{\theta}_1) = 0$) then, together with the effort policy e^* , such a payment scheme is part of an optimal mechanism.

Now consider the following class of payment schemes. In each period t the principal pays the agent a fixed amount $S_t(\theta^t)$ and a linear bonus $\alpha_t(\theta^t)\pi_t$, where both S_t and α_t are now allowed to depend on the entire history of reports θ^t (equivalently, S_t and α_t are chosen by the agent out of a menu, as a function of the observed shocks θ^t). In what follows we show that when the desired effort policy e^* satisfies a certain single-crossing condition, which is the analogue of condition (3) in the previous section, then the policy e^* can be implemented by a reward scheme in this class. To see this, for any t , let

$$\alpha_t(\theta^t) = \psi'(e_t^*(\theta^t)).$$

The sequence of fixed payments $S_t(\theta^t)$ is then defined recursively as follows. For $t = T$, let

$$S_T(\theta^T) \equiv \psi(e_T^*(\theta^T)) + \int_{\underline{\theta}_T}^{\theta_T} \psi'(e_T^*(\theta^{T-1}, s)) ds - \alpha_T(\theta^T)\pi_T^*(\theta^T)$$

while for any $t < T$,

$$\begin{aligned} S_t(\theta^t) &\equiv \psi(e_t^*(\theta^t)) - \alpha_t(\theta^t)\pi_t^*(\theta^t) \\ &+ \int_{\underline{\theta}_t}^{\theta_t} \mathbb{E}_{(\tilde{\theta}_{t+1}, \dots, \tilde{\theta}_T) | s} \left[\sum_{\tau=t}^T \delta^{\tau-t} J_t^\tau \left(s, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau \right) \psi'(e_t^*(\theta^{t-1}, s, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau)) \right] ds \\ &- \mathbb{E}_{\tilde{\theta}^T | \theta^t} \left[\sum_{\tau=t+1}^T \delta^{\tau-t} \left(S_\tau(\tilde{\theta}^\tau) + \alpha_\tau(\tilde{\theta}^\tau)\pi_\tau^*(\tilde{\theta}^\tau) - \psi(e_\tau^*(\tilde{\theta}^\tau)) \right) \right] \end{aligned} \quad (11)$$

where, for any $j = 1, \dots, T$, any $\theta^j \in \Theta^j$, $\pi_j^*(\theta^j) \equiv \theta_j + e_j^*(\theta^j)$.

Now suppose $t = T$ and that the history of past reports is $\hat{\theta}^{T-1}$. It is then immediate that, irrespective of the true shocks θ^T , if the agent reports $\hat{\theta}_T$ in period T he then finds it optimal to choose effort $e_T^*(\hat{\theta}^{T-1}, \hat{\theta}_T)$. Because the environment is Markov, it is also immediate that, irrespective of whether the history of past reports $\hat{\theta}^{T-1}$ was truthful, an agent whose period- T type is θ_T always finds it optimal to report truthfully in period T . This follows from arguments similar to those used to establish Proposition 2. To see this, note that the *continuation* payoff that type θ_T obtains by reporting $\hat{\theta}_T$ is simply¹⁹

$$u_T(\theta_T, \hat{\theta}_T; \hat{\theta}^{T-1}) \equiv \int_{\underline{\theta}_T}^{\hat{\theta}_T} \psi'(e_T^*(\hat{\theta}^{T-1}, s)) ds + \alpha_T(\hat{\theta}^{T-1}, \hat{\theta}_T)[\theta_T - \hat{\theta}_T]. \quad (12)$$

¹⁹In what follows, by continuation payoff, we mean the discounted sum of the future flow payoffs.

Now, let

$$u_T(\theta_T; \hat{\theta}^{T-1}) \equiv u_T(\theta_T, \theta_T; \hat{\theta}^{T-1}) = \int_{\underline{\theta}_T}^{\theta_T} \psi'(e_T^*(\hat{\theta}^{T-1}, s)) ds \quad (13)$$

denote the continuation payoff that type θ_T obtains by reporting truthfully. It is then immediate that

$$\begin{aligned} \left[\frac{du_T(\theta_T; \hat{\theta}^{T-1})}{d\theta_T} - \frac{\partial u_T(\theta_T, \hat{\theta}_T; \hat{\theta}^{T-1})}{\partial \theta_T} \right] &= \psi'(e_T^*(\hat{\theta}^{T-1}, \theta_T)) - \alpha_T(\hat{\theta}^{T-1}, \hat{\theta}_T) \\ &= \psi'(e_T^*(\hat{\theta}^{T-1}, \theta_T)) - \psi'(e_T^*(\hat{\theta}^{T-1}, \hat{\theta}_T)) \end{aligned}$$

and hence

$$\left[\frac{du_T(\theta_T; \hat{\theta}^{T-1})}{d\theta_T} - \frac{\partial u_T(\theta_T, \hat{\theta}_T; \hat{\theta}^{T-1})}{\partial \theta_T} \right] [\theta_T - \hat{\theta}_T] \geq 0 \quad (14)$$

if and only if $e_T^*(\hat{\theta}^{T-1}, \cdot)$ is increasing. As it is well known, condition (14) guarantees that truthtelling is optimal (see e.g. Garcia, 2005).

Now, by induction, suppose that, irrespective of whether he has reported truthfully in the past, at any period $\tau > t$, the agent finds it optimal to report θ_τ truthfully. Then, consider the agent's incentives in period t . Take any history of reports $\hat{\theta}^{t-1}$. Again, because the environment is Markov, it is irrelevant whether this history corresponds to the truth or not. Then suppose the agent's true type in period t is θ_t and he announces $\hat{\theta}_t$. His continuation payoff is then given by

$$\begin{aligned} u_t(\theta_t, \hat{\theta}_t; \hat{\theta}^{t-1}) &= u_t(\hat{\theta}_t; \hat{\theta}^{t-1}) + \alpha_t(\hat{\theta}^{t-1}, \hat{\theta}_t)[\theta_t - \hat{\theta}_t] \\ &\quad + \mathbb{E}_{\tilde{\theta}_{t+1}|\theta_t} [u_{t+1}(\tilde{\theta}_{t+1}; \hat{\theta}^{t-1}, \hat{\theta}_t)] - \mathbb{E}_{\tilde{\theta}_{t+1}|\hat{\theta}_t} [u_{t+1}(\tilde{\theta}_{t+1}; \hat{\theta}^{t-1}, \hat{\theta}_t)] \end{aligned} \quad (15)$$

where, for any period $l \geq 1$ and any $(\theta_l, \hat{\theta}^{l-1})$,

$$u_l(\theta_l; \hat{\theta}^{l-1}) = \int_{\underline{\theta}_l}^{\theta_l} \mathbb{E}_{(\tilde{\theta}_{l+1}, \dots, \tilde{\theta}_T)|s} \left[\sum_{\tau=l}^T \delta^{\tau-l} J_\tau^\tau(s, \tilde{\theta}_{l+1}, \dots, \tilde{\theta}_\tau) \psi'(e_l^*(\hat{\theta}^{l-1}, s, \tilde{\theta}_{l+1}, \dots, \tilde{\theta}_\tau)) \right] ds \quad (16)$$

is the equilibrium continuation payoff under a truthful and obedient strategy starting from period l onwards, given the current type θ_l and the history of past reports $\hat{\theta}^{l-1}$. It follows that

$$\frac{du_t(\theta_t; \hat{\theta}^{t-1})}{d\theta_t} = \mathbb{E}_{(\tilde{\theta}_{t+1}, \dots, \tilde{\theta}_T)|\theta_t} \left[\sum_{\tau=t}^T \delta^{\tau-t} J_\tau^\tau(\theta_t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau) \psi'(e_t^*(\hat{\theta}^{t-1}, \theta_t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau)) \right]$$

and that²⁰

$$\begin{aligned} \frac{\partial u_t(\theta_t, \hat{\theta}_t; \theta^{t-1})}{\partial \theta_t} &= \alpha_t(\hat{\theta}^{t-1}, \hat{\theta}_t) + \frac{\partial \mathbb{E}_{\tilde{\theta}_{t+1}|\theta_t} [u_{t+1}(\tilde{\theta}_{t+1}; \hat{\theta}^{t-1}, \hat{\theta}_t)]}{\partial \theta_t} \\ &= \mathbb{E}_{(\tilde{\theta}_{t+1}, \dots, \tilde{\theta}_T)|\theta_t} \left[\sum_{\tau=t}^T \delta^{\tau-t} J_t^\tau \left(\theta_t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau \right) \psi'(e_t^*(\hat{\theta}^{t-1}, \hat{\theta}_t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau)) \right]. \end{aligned} \quad (17)$$

Once again, a sufficient condition for $u_t(\theta_t; \hat{\theta}^{t-1}) \geq u_t(\theta_t, \hat{\theta}_t; \hat{\theta}^{t-1})$ for any $\hat{\theta}_t$ is that

$$\left[\frac{du_t(\theta_t; \hat{\theta}^{t-1})}{d\theta_t} - \frac{\partial u_t(\theta_t, \hat{\theta}_t; \hat{\theta}^{t-1})}{\partial \theta_t} \right] [\theta_t - \hat{\theta}_t] \geq 0, \quad (18)$$

or equivalently that

$$\mathbb{E}_{\tilde{\theta}_{t+1}, \dots, \tilde{\theta}_T|\theta_t} \left[\sum_{\tau=t}^T \delta^{\tau-t} J_t^\tau \left(\theta_t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau \right) [\alpha_\tau(\hat{\theta}^{t-1}, \theta_t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau) - \alpha_\tau(\hat{\theta}^{t-1}, \hat{\theta}_t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau)] \right] [\theta_t - \hat{\theta}_t] \geq 0. \quad (19)$$

This condition is the equivalent of condition (3) in the previous section. Note that, this condition is satisfied, for example, when the effort policy is *strongly monotone*, i.e. when at any period t , $e_t^*(\theta^t)$ is nondecreasing in θ^t . We then have the following result.

Proposition 3 *Assume the evolution of θ_t is governed by a Markov process satisfying the assumptions described above and that, for each period t , $\pi_t = \theta_t + e_t$.*

1. *Any policy e satisfying the single-crossing condition (19) for any t , any $(\hat{\theta}^{t-1}, \hat{\theta}_t, \theta_t)$, can be implemented by the following linear pay package s : In every period t , given any history or reports θ^t and any history of observed performances π^t , the principal pays the agent*

$$s_t(\theta^t, \pi^t) = S_t(\theta^t) + \alpha_t(\theta^t)\pi_t$$

where $\alpha_t(\theta^t) \equiv \psi'(e_t(\theta^t))$ and where the fixed payment $S_t(\cdot)$ is as in (11).

2. *Let e^* be the effort policy implicitly defined, for all t and all $\theta^t \in \Theta^t$, by*

$$\psi'(e_t^*(\theta^t)) = 1 - \eta(\theta_1) J_1^t(\theta^t) \psi''(e_t^*(\theta^t)) \quad (20)$$

²⁰The expression in (17) is obtained by integration by parts, using (16).

unless $\psi''(0) \geq 1/[\eta(\theta_1)J_1^t(\theta^t)]$ in which case $e_t^*(\theta^t) = 0$. Assume e^* satisfies the single-crossing condition of (19). Then e^* , together with the linear pay package s^* described above are part of an optimal mechanism.

A few remarks are in order. First note that the result in Proposition 3 complements that in Proposition 2: while Proposition 3 does not restrict the process for θ_t to satisfy the SFC condition, it restricts θ_t to follow a Markov process, a property that is not required by Proposition 2.

Second, note that the linear scheme in Proposition 3 has the appealing property of guaranteeing that, even if the agent has the option of leaving the relationship at any point in time, he never finds it optimal to do so, i.e. it guarantees participation at any period, after any history.

Third note that a key distinction between the linear scheme of Proposition 3 and that of Proposition 2 is that the agent must now be allowed to propose changes to his pay package over time. These changes are in response to the shocks θ^t . This finding is consistent with some of the recent literature on managerial compensation which documents that CEO compensation is often proposed by CEOs themselves (see e.g. Bebchuck and Fried, 2004). In our setting, the firm's shareholders (the principal) set in advance broad restrictions on the CEO's pay package but then delegate to the latter the choice of the specific terms of the reward scheme so as to permit him to respond to (unverifiable) variations in the environment. In particular, the optimal mechanism involves offering the CEO a menu of linear contracts with *memory*, in the sense that the set of possible packages available for period t depend on the reward packages selected in past periods (as indexed by θ^{t-1}).

Fourth, note that a form of seniority is likely to hold also in this environment, albeit only in expectation: by inspecting (20) one can see that the power of the incentive scheme, as captured by e_t increases, on average, with the manager's tenure, provided that $\mathbb{E}_{\tilde{\theta}_t|\theta^{t-1}} \left[J_1^t(\theta^{t-1}, \tilde{\theta}_t) \right] \leq J_1^{t-1}(\theta^{t-1})$. As discussed in the introduction, this property is satisfied by many stochastic processes that feature a correlation between θ_1 and θ_t declining in t .

Lastly note that, while the possibility of implementing the policy e^* that solves the relaxed

program (as given by (20)) with a menu of linear schemes is certainly appealing, such a possibility cannot be taken for granted. In fact, in many cases of interest e^* does not satisfy the single-crossing condition of (19). To see this, assume that, for any $t > 1$ and any θ_{t-1} , $I_{t-1}^t(\cdot, \theta_{t-1})$ is continuous and $\lim_{\theta_t \rightarrow \underline{\theta}_t} I_{t-1}^t(\theta_t, \theta_{t-1}) = \lim_{\theta_t \rightarrow \bar{\theta}_t} I_{t-1}^t(\theta_t, \theta_{t-1}) = 0$.²¹ Then for any $1 < s \leq \tau$, any θ_{-s}^τ , $\lim_{\theta_s \rightarrow \underline{\theta}_s} J_1^\tau(\theta_1, \dots, \theta_s, \dots, \theta_\tau) = \lim_{\theta_s \rightarrow \bar{\theta}_s} J_1^\tau(\theta_1, \dots, \theta_s, \dots, \theta_\tau) = 0$. This in turn implies that $\lim_{\theta_s \rightarrow \bar{\theta}_s} e_\tau^*(\theta_{-s}^\tau, \theta_s) = \lim_{\theta_s \rightarrow \underline{\theta}_s} e_\tau^*(\theta_{-s}^\tau, \theta_s) = e_s^{FB}$. The policy $e_\tau^*(\theta_{-s}^\tau, \theta_s)$ is then typically non-monotone in θ_s , for any $\tau \geq s$ any θ_{-s}^τ , which makes it difficult (if not impossible) to satisfy (19).

Motivated by the aforementioned considerations about the possible difficulties of implementing the optimal effort policy with linear schemes, we now consider an alternative implementation based on the “trick” used to establish Lemma 1 in the proof of Proposition 2. The idea is to charge the agent a sufficiently large penalty L whenever, given the announcements θ^t , the observed profits are different from the equilibrium ones $\pi_t^*(\theta^t)$. To see how this permits one to relax condition (19), suppose that in all periods $t < T$ the principal uses the same reward scheme as in Part 1 in Proposition 3, whereas at $t = T$, she uses the following scheme

$$s_T(\theta^T, \pi^T) = \begin{cases} \psi(e_T^*(\theta^T)) + \int_{\underline{\theta}_T}^{\hat{\theta}_T} \psi'(e_T^*(\theta^{T-1}, s)) ds & \text{if } \pi_T = \pi_T^*(\theta^T) \\ -L & \text{otherwise} \end{cases} \quad (21)$$

Note that, conditional on “meeting the target”, under the new scheme, for any sequence of reports θ^T , the agent receives exactly the same compensation he would have obtained under the original linear scheme by choosing effort in period t so as to attain profits $\pi^T(\theta^T)$. Provided that L is large enough, it is then immediate that deviations from the equilibrium strategy are less profitable under the new scheme than under the original linear one. In particular, the agent’s continuation payoff in period T , after he has reported $(\hat{\theta}^T)$ and experienced a shock θ_T in period t , is now given by

$$\begin{aligned} \hat{u}_T(\theta_T, \hat{\theta}_T; \hat{\theta}^{T-1}) &\equiv \int_{\underline{\theta}_T}^{\hat{\theta}_T} \psi'(e_T^*(\hat{\theta}^{T-1}, s)) ds + \psi\left(\pi_T(\hat{\theta}^{T-1}, \hat{\theta}_T) - \hat{\theta}_T\right) - \psi\left(\pi_T(\hat{\theta}^{T-1}, \hat{\theta}_T) - \theta_T\right) \\ &= \int_{\underline{\theta}_T}^{\hat{\theta}_T} \psi'(e_T^*(\hat{\theta}^{T-1}, s)) ds + \psi(e_T^*(\hat{\theta}^{T-1}, \hat{\theta}_T)) - \psi(e_T^*(\hat{\theta}^{T-1}, \hat{\theta}_T) + \hat{\theta}_T - \theta_T) \end{aligned}$$

²¹Note that, under our assumption of full support (i.e. F_t strictly increasing) over Θ_t , these conditions hold e.g. when $\bar{\theta}_t < +\infty$ and when F_t is an atomless distribution with density strictly positive over $[\underline{\theta}_t, \bar{\theta}_t]$.

rather than $u_T(\theta_T, \hat{\theta}_T; \hat{\theta}^{T-1})$ as in Equation (12). Irrespective of whether $\hat{\theta}^{T-1}$ was truthful or not, incentive compatibility is then ensured in period T (i.e. the agent finds it optimal to report θ_T truthfully and then choose the equilibrium level of effort $e_T^*(\hat{\theta}^{T-1}, \theta_T)$) if the effort policy e^* satisfies the analogue of condition (14) with $u_T(\theta_T, \hat{\theta}_T; \theta^{T-1})$ now replaced by the function $\hat{u}_T(\theta_T, \hat{\theta}_T; \theta^{T-1})$, that is, if²²

$$\left[\psi'(e_T^*(\hat{\theta}^{T-1}, \theta_T)) - \psi'(e_T^*(\hat{\theta}^{T-1}, \hat{\theta}_T) + \hat{\theta}_T - \theta_T) \right] \left[\theta_T - \hat{\theta}_T \right] \geq 0. \quad (22)$$

Note that condition (22) is clearly weaker than condition (19) which requires $[\psi'(e_T^*(\hat{\theta}^{T-1}, \theta_T)) - \psi'(e_T^*(\hat{\theta}^{T-1}, \hat{\theta}_T))][\theta_T - \hat{\theta}_T] \geq 0$. Moving from the linear scheme to this alternative scheme thus permits one to implement effort policies that are not necessarily monotone in the shock θ_T . It is easy to see that condition (22) is equivalent to requiring that the profit function $\pi_T(\hat{\theta}^{T-1}, \cdot)$ (as opposed to the effort policy $e_T^*(\hat{\theta}^{T-1}, \cdot)$) being non-decreasing. Absent the dependence on history, this is the same result found by Laffont and Tirole (1993, A1.4) for the static case.

Now suppose the principal replaces the entire linear scheme s^* with the incentive scheme s recursively defined, for each t , as follows

$$s_t(\theta^t, \pi^t) = \begin{cases} \psi(e_t^*(\theta^t)) + \int_{\hat{\theta}_t}^{\theta_t} \mathbb{E}_{(\tilde{\theta}_{t+1}, \dots, \tilde{\theta}_T) | s} \left[\sum_{\tau=t}^T \delta^{\tau-t} J_t^\tau(s, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau) \psi'(e_t^*(\theta^{t-1}, s, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau)) \right] ds \\ - \mathbb{E}_{\hat{\theta}^T | \theta^t} \left[\sum_{\tau=t+1}^T \delta^{\tau-t} \left(s_\tau(\tilde{\theta}^\tau, \pi^{\tau*}(\tilde{\theta}^\tau)) - \psi(e_\tau^*(\tilde{\theta}^\tau)) \right) \right] & \text{if } \pi_t = \pi_t^*(\theta^t) \\ -L \text{ otherwise.} \end{cases} \quad (23)$$

where $\pi^{\tau*}(\tilde{\theta}^\tau) \equiv (\pi_s^*(\theta^s))_{s=1}^\tau$. Note that, for $t = T$, this scheme is the same as the one in (21). Now suppose, by induction, that under the scheme s defined above, truthful reporting is optimal for the agent in each period $\tau > t$, irrespective of the period- τ history (recall that, because the environment is Markov, if truthful reporting is optimal on the equilibrium path, i.e. at a truthful period- τ history, then it is optimal at *all* period- τ histories). Provided L is large enough, the agent's period- t continuation payoff under this scheme when his period- t type is θ_t , he reports $\hat{\theta}_t$,

²²As mentioned above, note that the payoff under truth-telling under the new scheme is exactly the same as under the original scheme. That is $u_T(\theta_T; \hat{\theta}^{T-1})$ continues to be as in (13).

and the sequence of past reports is $\hat{\theta}^{t-1}$, is then given by

$$\begin{aligned} \hat{u}_t(\theta_t, \hat{\theta}_t; \hat{\theta}^{t-1}) &= u_t(\hat{\theta}_t; \hat{\theta}^{t-1}) + \psi(e_t^*(\hat{\theta}^{t-1}, \hat{\theta}_t)) - \psi(e_t^*(\hat{\theta}^{t-1}, \hat{\theta}_t) + \hat{\theta}_t - \theta_t) \\ &\quad + \mathbb{E}_{\tilde{\theta}_{t+1}|\theta_t} \left[u_{t+1}(\tilde{\theta}_{t+1}; \hat{\theta}^{t-1}, \hat{\theta}_t) \right] - \mathbb{E}_{\tilde{\theta}_{t+1}|\hat{\theta}_t} \left[u_{t+1}(\tilde{\theta}_{t+1}; \hat{\theta}^{t-1}, \hat{\theta}_t) \right], \end{aligned}$$

where, for any period $l \geq 1$ and any $(\theta_l, \hat{\theta}^{l-1})$, $u_l(\theta_l; \hat{\theta}^{l-1})$ continues to denote the equilibrium continuation payoff, as defined in (16). Incentive compatibility is then guaranteed in period t if condition (18) holds, that is, if

$$\mathbb{E}_{\tilde{\theta}_{t+1}, \dots, \tilde{\theta}_T | \theta_t} \left[\begin{aligned} &\psi'(e_t^*(\hat{\theta}^{t-1}, \theta_t)) - \psi'(e_t^*(\hat{\theta}^{t-1}, \hat{\theta}_t) + \hat{\theta}_t - \theta_t) \\ &+ \sum_{\tau=t+1}^T \delta^{\tau-t} J_\tau^T \left(\theta_t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau \right) \\ &\times [\psi'(e_\tau^*(\hat{\theta}^{t-1}, \theta_t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau)) - \psi'(e_\tau^*(\hat{\theta}^{t-1}, \hat{\theta}_t, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau))] \end{aligned} \right] [\theta_t - \hat{\theta}_t] \geq 0. \quad (24)$$

Note that this condition is the same as that in (19) with the initial term $\psi'(e_t^*(\hat{\theta}^{t-1}, \theta_t)) - \psi'(e_t^*(\hat{\theta}^{t-1}, \hat{\theta}_t))$ replaced by $\psi'(e_t^*(\hat{\theta}^{t-1}, \theta_t)) - \psi'(e_t^*(\hat{\theta}^{t-1}, \hat{\theta}_t) + \hat{\theta}_t - \theta_t)$. We then have the following result.

Proposition 4 *Any effort policy e satisfying the single-crossing condition (24) for any t , any $(\hat{\theta}^{t-1}, \hat{\theta}_t, \theta_t)$, can be implemented by the non-linear pay scheme s given in (23).*

As an illustration of how the scheme s given in (23) may help implementing effort policies e^* that solve the principal's relaxed program but that cannot be implemented with the linear scheme s^* of Proposition 3, consider the following example.

Example 4 *Suppose that, for any $e \in [0, \bar{e}]$, $\psi(e) = e^2/2$. Let θ_1 be a non-negative random variable with distribution F strictly increasing and absolutely continuous on the interval $[\underline{\theta}_1, \bar{\theta}_1] \subset \mathbb{R}_{++}$ whose hazard rate satisfies $\eta(\theta_1) \leq \theta_1$ for each θ_1 .²³ Now suppose that, for any $t \geq 2$, $\theta_t = \theta_1 \times \prod_{\tau=2}^t \varepsilon_\tau$, where $\varepsilon^T \equiv (\varepsilon_\tau)_{\tau=2}^T$ is a collection of jointly independent random variables, each independent of θ_1 , each distributed according to the function G strictly increasing and absolutely continuous with density g strictly positive over \mathbb{R}_+ . Let e^* be the effort policy that solves the relaxed program as given in (20). Then the policy e^* cannot be implemented by the linear scheme s^* of Proposition 3 but it can be implemented by the non-linear scheme s of Proposition 4.*

²³This condition is satisfied, for instance, when θ_1 is distributed uniformly over the interval $[1, 3/2]$.

4 Risk aversion

We now show how the optimal mechanism must be adjusted to accommodate the possibility that the agent is risk averse. We restrict attention here to the case where T is finite. To simplify the notation, we omit discounting, i.e. set $\delta = 1$. We start by assuming that the agent's preferences are represented by a Bernoulli function

$$U^A(c^T, e^T) = \mathcal{V} \left(\sum_{t=1}^T c_t \right) - \sum_{t=1}^T \psi(e_t)$$

where \mathcal{V} is a strictly increasing and (weakly) concave function. This representation is quite common in the literature (e.g. Holmstrom and Milgrom's (1987) seminal paper on linearity and aggregation in dynamic contracting). As well known, this representation permits one to introduce risk aversion while at the same time avoiding any complication stemming from the desire of consumption smoothing: it is thus appropriate for a setting where the agent cares only about his total compensation and not the way this is distributed over time. We will come back to an alternative representation that accommodates preferences for consumption smoothing at the end of the section.

For the stochastic process for θ_t , we adopt a general independent-shock representation and assume it satisfies the regularity conditions spelled out in PST, generalizing Conditions 1-3 to processes that need not satisfy the SFC condition.

Following steps similar to those used to establish Proposition 2, one can then show that the characterization of incentive compatibility is unaffected by the introduction of risk aversion and that the agent's value function in period one remains equal to

$$V^{\hat{\Omega}}(\theta_1) = V^{\hat{\Omega}}(\underline{\theta}_1) + \mathbb{E} \left[\sum_{t=1}^T \int_{\underline{\theta}_1}^{\theta_1} D_{1,t}(s, \tilde{\varepsilon}^t) \psi'(\hat{e}_t(s, \tilde{\varepsilon}^t)) ds \right],$$

where $D_{1,1}(\theta_1) \equiv 1$ and, for any $t > 1$,

$$D_{1,t}(\theta_1, \varepsilon^t) \equiv \frac{\partial z_t(\theta_1, \varepsilon^t)}{\partial \theta_1} - \rho \frac{\partial z_{t-1}(\theta_1, \varepsilon^{t-1})}{\partial \theta_1}$$

with $z_1(\theta_1) \equiv \theta_1$. Note that these $D_{1,t}(\theta_1, \varepsilon^t)$ functions reduce to the corresponding $D_{1,t}(\theta_1)$ functions of Section 2.2 when the stochastic process for θ_t satisfies the SFC condition.

A similar characterization applies to each period $t > 1$. For example, incentive compatibility at any *truthful* history²⁴ $h_{T-1} = (\theta_1, \varepsilon^{T-1})$ implies that $V^{\hat{\Omega}}(\theta_1, \varepsilon^{T-1}, \varepsilon_T)$ is Lipschitz continuous in ε_T and for a.e. ε_T ,

$$\frac{\partial V^{\hat{\Omega}}(\theta_1, \varepsilon^{T-1}, \varepsilon_T)}{\partial \varepsilon_T} = \frac{\partial z_T(\varepsilon^{T-1}, \varepsilon_T)}{\partial \varepsilon_T} \psi'(\hat{e}_T(\theta_1, \varepsilon^{T-1}, \varepsilon_T)),$$

which in turn implies that

$$V^{\hat{\Omega}}(\theta_1, \varepsilon^{T-1}, \varepsilon_T) = V^{\hat{\Omega}}(\theta_1, \varepsilon^{T-1}, \underline{\varepsilon}_T) + \int_{\underline{\varepsilon}_T}^{\varepsilon_T} \frac{\partial z_T(\varepsilon^{T-1}, s)}{\partial \varepsilon_T} \psi'(\hat{e}_T(\theta_1, \varepsilon^{T-1}, s)) ds.$$

Furthermore, using the fact that incentive compatibility implies that $V^{\hat{\Omega}}(\theta_1, \varepsilon^{T-1}, \varepsilon_T)$ must coincide with the equilibrium payoff with probability one, we have that, for almost every (h_{T-1}, ε_T) ,

$$\mathcal{V} \left(\sum_{t=1}^T \hat{c}_t(\theta_1, \varepsilon^t) \right) - \sum_{t=1}^T \psi(\hat{e}_t(\theta_1, \varepsilon^t)) = V^{\hat{\Omega}}(\theta_1, \varepsilon^{T-1}, \underline{\varepsilon}_T) + \int_{\underline{\varepsilon}_T}^{\varepsilon_T} \frac{\partial z_T(\varepsilon^{T-1}, s)}{\partial \varepsilon_T} \psi'(\hat{e}_T(\theta_1, \varepsilon^{T-1}, s)) ds.$$

This implies that in almost every state $(\theta_1, \varepsilon^T)$ the utility $\mathcal{V} \left(\sum_{t=1}^T \hat{c}_t(\theta_1, \varepsilon^t) \right)$ that the agent assigns to the total payment $\sum_{t=1}^T \hat{c}_t(\theta_1, \varepsilon^t)$ is uniquely determined by the effort policy \hat{e} up to a constant $V^{\hat{\Omega}}(\theta_1, \varepsilon^{T-1}, \underline{\varepsilon}_T)$ which may depend on $(\theta_1, \varepsilon^{T-1})$ but is independent of ε_T . Iterating backwards, and using the fact that in each period t and for any history h_t we have

$$V^{\hat{\Omega}}(h_t) = \mathbb{E}[V^{\hat{\Omega}}(h_t, \tilde{\varepsilon}_{t+1})],$$

one can then show that the dependence of the constant $V^{\hat{\Omega}}(\theta_1, \varepsilon^{T-1}, \underline{\varepsilon}_T)$ on the history $(\theta_1, \varepsilon^{T-1})$ is also uniquely determined by the effort policy \hat{e} up to a scalar K that does not depend on anything—the proof is in the Appendix; see also Proposition 9 in PST for a similar result for quasi-linear settings. Now, for any $1 < t \leq T$, let $D_{t,t}(\theta_1, \varepsilon^t) \equiv \partial z_t(\theta_1, \varepsilon^t) / \partial \varepsilon_t$ and, for any $s > t$,

$$D_{t,s}(\theta_1, \varepsilon^s) \equiv \frac{\partial z_s(\theta_1, \varepsilon^s)}{\partial \varepsilon_t} - \rho \frac{\partial z_{s-1}(\theta_1, \varepsilon^{s-1})}{\partial \varepsilon_t}.$$

We then have the following result.

²⁴Note that incentive compatibility at a truthful history h_t means that the agent's value function in the mechanism $\hat{\Omega}$ after reaching history h_t is equal to the agent's expected payoff when, starting from history h_t the agent follows a truthful and obedient strategy in each period $\tau \geq t$. Also recall that incentive-compatibility in period one, given θ_1 , implies incentive compatibility at almost all (i.e. with probability one) truthful period t -history, $t = 1, \dots, T$.

Proposition 5 *In any IC mechanism $\hat{\Omega}$, the total payment to the agent in each state $(\theta_1, \varepsilon^T)$ is given by:*

$$\sum_{t=1}^T \hat{c}_t(\theta_1, \varepsilon^t) = \mathcal{V}^{-1} \left(\sum_{t=1}^T \psi(\hat{e}_t(\theta_1, \varepsilon^t)) + V^{\hat{\Omega}}(\underline{\theta}_1) + \mathbb{E}_{\tilde{\varepsilon}^T} \left[\int_{\underline{\theta}_1}^{\theta_1} \sum_{t=1}^T D_{1,t}(s, \tilde{\varepsilon}^t) \psi'(\hat{e}_t(s, \tilde{\varepsilon}^t)) ds \right] + \sum_{t=2}^T \hat{H}_t(\theta_1, \varepsilon^t) \right)$$

with

$$\begin{aligned} \hat{H}_t(\theta_1, \varepsilon^t) &\equiv \mathbb{E}_{(\tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_T)} \left[\int_{\underline{\varepsilon}_t}^{\varepsilon_t} \sum_{\tau=t}^T D_{t,\tau}(\theta_1, \varepsilon^{t-1}, s, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_\tau) \psi'(\hat{e}_\tau(\theta_1, \varepsilon^{t-1}, s, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_\tau)) ds \right] \\ &\quad - \mathbb{E}_{(\tilde{\varepsilon}_t, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_T)} \left[\int_{\underline{\varepsilon}_t}^{\tilde{\varepsilon}_t} \sum_{\tau=t}^T D_{t,\tau}(\theta_1, \varepsilon^{t-1}, s, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_\tau) \psi'(\hat{e}_\tau(\theta_1, \varepsilon^{t-1}, s, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_\tau)) ds \right] \end{aligned}$$

Using the characterization in Proposition 5, we then have that, in any incentive-compatible mechanism, the principal's expected payoff can be expressed as:

$$\begin{aligned} \mathbb{E}[\hat{U}^P] &= \mathbb{E} \left[\sum_{t=1}^T \hat{\pi}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t) \right] - \mathbb{E} \left[\sum_{t=1}^T \hat{c}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t) \right] \\ &= \mathbb{E} \left[\sum_{t=1}^T \left(z_t(\tilde{\theta}_1, \tilde{\varepsilon}^t) + \hat{e}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t) \right) \right] \\ &\quad - \mathbb{E} \left[\mathcal{V}^{-1} \left(\sum_{t=1}^T \psi(\hat{e}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t)) + V^{\hat{\Omega}}(\underline{\theta}_1) + \mathbb{E}_{\tilde{\varepsilon}^T} \left[\int_{\underline{\theta}_1}^{\tilde{\theta}_1} \sum_{t=1}^T D_{1,t}(s, \tilde{\varepsilon}^t) \psi'(\hat{e}_t(s, \tilde{\varepsilon}^t)) ds \right] + \sum_{t=2}^T \hat{H}_t(\tilde{\theta}_1, \tilde{\varepsilon}^t) \right) \right] \end{aligned} \quad (25)$$

The expression in (25) is the analogue of *dynamic virtual surplus* for the case of a risk-averse agent (it is easy to see that, when \mathcal{V} is the identity function and the process for θ_t satisfies the SFC condition, (25) reduces to the same expression as in (7) by standard integration by parts).

We now turn to the possibility of using “quasi-linear” schemes (i.e. pay packages that are convex in a linear aggregator of the firm's profits) to implement a desired effort policy. We start with the following result.

Proposition 6 *Let \hat{e} be any policy that depends only on time t and on the agent's first period type θ_1 . Suppose that $\mathbb{E}D_{1,t}(\theta_1, \tilde{\varepsilon}^t) \geq 0$ for any t any θ_1 and that the policy \hat{e} satisfies the following single-crossing condition*

$$\left[\sum_{t=1}^T \mathbb{E}D_{1,t}(\theta_1, \tilde{\varepsilon}^t) [\alpha_t(\theta_1) - \alpha_t(\hat{\theta}_1)] \right] [\theta_1 - \hat{\theta}_1] \geq 0 \quad (26)$$

for any $\theta_1, \hat{\theta}_1 \in \Theta_1$, where for any $t < T$ and any θ_1

$$\alpha_t(\theta_1) \equiv \psi'(\hat{e}_t^*(\theta_1)) - \rho\psi'(\hat{e}_{t+1}^*(\theta_1))$$

while for $t = T$, $\alpha_T(\theta_1) \equiv \psi'(\hat{e}_T^*(\theta_1))$. Then the effort policy \hat{e} can be implemented by a “quasi-linear” pay scheme \hat{s}^* according to which the total payment the agent receives when he reports $(\theta_1, \varepsilon^t)$ and the sequence of observed profits is π^T is given by

$$\sum_{t=1}^T \hat{s}_t^*(\theta_1, \varepsilon^t, \pi^t) = \mathcal{V}^{-1} \left(S(\theta_1) + \sum_{t=1}^T \alpha_t(\theta_1) \pi_t \right) \quad (27)$$

where

$$S_1(\theta_1) \equiv \sum_{t=1}^T \left[\psi(\hat{e}_t^*(\theta_1)) + \int_{\underline{\theta}_1}^{\theta_1} \mathbb{E} D_{1,t}(s, \tilde{\varepsilon}^t) \psi'(\hat{e}_t^*(s)) ds - \alpha_t(\theta_1) \mathbb{E} [\hat{\pi}_t(\theta_1, \tilde{\varepsilon}^t)] \right]$$

with $\hat{\pi}_t(\theta_1, \varepsilon^t) \equiv z_t(\theta_1, \varepsilon^t) + \hat{e}_t(\theta_1) + \sum_{\tau=1}^{t-1} \rho^\tau \hat{e}_{t-\tau}(\theta_1)$.

The proof follows from steps similar to those that establish Proposition 2 adjusted for the fact that the stochastic process for θ_t is here not restricted to satisfy the SFC condition and for the fact that the agent’s payoff is now allowed to be concave in his total reward.

The value of the proposition is twofold. Firstly, it guarantees a form of continuity in the optimal mechanism and in the players’ payoff with respect to the agent’s preferences. In particular, it implies that when \mathcal{V} is sufficiently close to the identity function, the principal can guarantee herself a payoff arbitrarily close to the one she obtains under risk neutrality by choosing to implement the same effort policy as in Proposition 2 and by adjusting the reward scheme as indicated in (27). More generally, the proposition shows how one can adjust the linear reward scheme identified in the baseline model to implement any effort policy that depends only on time t and the agent’s first period report θ_1 , provided that such a policy satisfies the single-crossing condition of (26).

We now turn to the characterization of the optimal effort policy. It turns out that when the agent is *strictly* risk averse, then a policy that depends only on the agent’s initial type need not be fully optimal, even if the process for θ_t satisfies the SFC condition of Section 2. Due to the complexity of the optimization problem, we illustrate this in a two-period model where $z_t(\theta_1, \varepsilon_2) = \gamma\theta_1 + \varepsilon_2$ for $\gamma \in (0, 1]$. That is, the process is AR(1). We first characterize the effort policy that maximizes

the principal's expected payoff among those that depend only on θ_1 and then address the issue of the optimality of these policies.

Example 5 Suppose that $T = 2$ and that $\rho = 0$ so that $\pi_t = \theta_t + e_t$, $t = 1, 2$. In addition, suppose that the process for θ_t satisfies the SFC condition with $z_2(\theta_1, \varepsilon_2) = \gamma\theta_1 + \varepsilon_2$, $\gamma \in (0, 1]$, with θ_1 uniformly distributed over $[0, 1]$ and ε_2 distributed according to the c.d.f. G_2 strictly increasing on an arbitrary compact interval $\mathcal{E}_2 \subset \mathbb{R}$. Let the agent's utility function over consumption be given by $\mathcal{V}(c) = \frac{1}{\alpha}\sqrt{2\alpha c + \beta^2} - \frac{\beta}{\alpha}$ with $\alpha, \beta > 0$ and note that this function is chosen so that $\mathcal{V}^{-1}(u) = \frac{\alpha}{2}u^2 + \beta u$. Let $k < \frac{1}{\beta}$, and choose $\bar{e} > \frac{1}{k\beta}$. Put $K = k\bar{e}$ and suppose that, for any $e \in (0, K)$, $\psi(e) = \frac{ke^2}{2}$, while for any $e \geq K$, $\psi(e) = \frac{\bar{e}^2}{2} + (e - \bar{e})K$. Provided α is sufficiently small, the policy \hat{e}^* that maximizes the principal's expected payoff (as given by 25) among the policies that are contingent only on θ_1 is implicitly defined by

$$\begin{aligned} & \psi'(\hat{e}_1^*(\theta_1)) \left(\frac{\partial \mathcal{V}^{-1} \left(\psi(\hat{e}_1^*(\theta_1)) + \psi(\hat{e}_2^*(\theta_1)) + \int_0^{\theta_1} [\psi'(\hat{e}_1^*(s)) + \gamma\psi'(\hat{e}_2^*(s))] ds \right)}{\partial u} \right) \\ &= 1 - \psi''(\hat{e}_1^*(\theta_1)) \int_{\theta_1}^1 \left(\frac{\partial \mathcal{V}^{-1} \left(\psi(\hat{e}_1^*(q)) + \psi(\hat{e}_2^*(q)) + \int_0^q [\psi'(\hat{e}_1^*(s)) + \gamma\psi'(\hat{e}_2^*(s))] ds \right)}{\partial u} \right) dq \end{aligned} \quad (28)$$

and

$$\begin{aligned} & \psi'(\hat{e}_2^*(\theta_1)) \left(\frac{\partial \mathcal{V}^{-1} \left(\psi(\hat{e}_1^*(\theta_1)) + \psi(\hat{e}_2^*(\theta_1)) + \int_0^{\theta_1} [\psi'(\hat{e}_1^*(s)) + \gamma\psi'(\hat{e}_2^*(s))] ds \right)}{\partial u} \right) \\ &= 1 - \gamma\psi''(\hat{e}_2^*(\theta_1)) \int_{\theta_1}^1 \frac{\partial \mathcal{V}^{-1} \left(\psi(\hat{e}_1^*(q)) + \psi(\hat{e}_2^*(q)) + \int_0^q [\psi'(\hat{e}_1^*(s)) + \gamma\psi'(\hat{e}_2^*(s))] ds \right)}{\partial u} dq \\ & \quad - \alpha\psi'(\hat{e}_2^*(\theta_1))\psi''(\hat{e}_2^*(\theta_1))\text{Var}(\varepsilon_2). \end{aligned} \quad (30)$$

The policy \hat{e}^* satisfies the single-crossing condition (26) and hence can be implemented by the “quasi-linear” payment scheme of Proposition 6.

Equations (28) and (29) show that whether effort in period $t = 1, 2$ is optimally increasing or decreasing in θ_1 depends on a trade-off between two concerns: (1) insuring the agent against risk and (2) limiting the agent's intertemporal informational rent. Equation (28), for instance, shows that, at the optimum, the marginal disutility of an extra unit of period-1 effort for type θ_1 (the

left-hand side) must equal to the gain in profits (equal to 1) less the marginal cost of the additional rent that must be given to types $\hat{\theta}_1 > \theta_1$.

Noticing that $\eta(\theta_1) = (1 - \theta_1)$ in the case of a uniform distribution, it is easy to see that (28) reduces to condition (2) for the risk-neutral case when $\alpha = 0$ and $\beta = 1$. Also note that, because the set of types for which the agent must be given an additional rent is decreasing in θ_1 , as in the risk-neutral case, the policy \hat{e}_1^* is increasing in θ_1 . However, contrary to the risk-neutral case, distortions do not vanish at the top. In fact, when applied to $\bar{\theta}_1$, condition (28) becomes

$$\psi'(\hat{e}_1^*(\bar{\theta}_1)) \left(\frac{\partial \mathcal{V}^{-1} \left(\psi(\hat{e}_1^*(\bar{\theta}_1)) + \psi(\hat{e}_2^*(\bar{\theta}_1)) + \int_{\theta_1}^{\bar{\theta}_1} [\psi'(\hat{e}_1^*(s)) + \gamma \psi'(\hat{e}_2^*(s))] ds \right)}{\partial u} \right) = 1 \quad (31)$$

while efficiency (under risk aversion) requires that

$$\psi'(\hat{e}_1^*(\bar{\theta}_1)) \left(\frac{\partial \mathcal{V}^{-1} \left(\psi(\hat{e}_1^*(\bar{\theta}_1)) + \psi(\hat{e}_2^*(\bar{\theta}_1)) \right)}{\partial u} \right) = 1$$

To further appreciate the distinctions/similarities between the risk-neutral and the risk-averse case, take the specification of the example, and let $k = \frac{1}{2}$. Then note that, under risk-neutrality (say $\alpha = 0$, $\beta = 1$) the (fully)²⁵ optimal policies are given by $\hat{e}_1^*(\theta_1) = 2 - (1 - \theta_1)$ and $\hat{e}_2^*(\theta_1) = 2 - (1 - \theta_1)\gamma$, with $e_1^{FB} = e_2^{FB} = 2$. As discussed in Section 2.2, to minimize the agent's informational rents, the principal finds it optimal to distort downward both e_1 and e_2 . Furthermore, because the correlation between the agent's initial type and his future ones is imperfect (and declining with time), it is optimal to distort more in the early stages of the relationship than in the later ones. This property leads to the seniority effect discussed in the previous sections. This effect can be seen easily within the context of this example: the smaller γ is (i.e. the lower the correlation between θ_1 and θ_2) the stronger the seniority effect, with $\hat{e}_2^*(\theta_1) = \hat{e}_1^*(\theta_1)$ [i.e. no seniority] when $\gamma = 1$ [random walk case] and $\hat{e}_2^*(\theta_1) = \hat{e}^{FB}$ when $\gamma = 0$ [θ_1 and θ_2 independent].

To see how risk aversion affects the choice of effort, Figure 4 below depicts the optimal (shock-independent) policies of Example 5 for $\alpha = \beta = 1$, $\gamma = k = \frac{1}{2}$, and ε_2 uniformly distributed over $[0, 1]$.²⁶

²⁵Recall, from Proposition 2, that with risk-neutrality restricting the policy \hat{e} to depend only on θ_1 is without loss

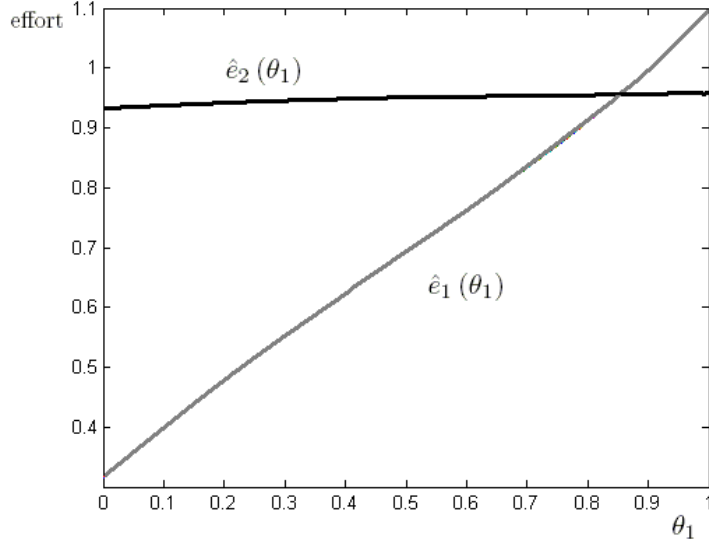


Figure 1: Effort policies (risk-averse case)

While a form of seniority continues to hold (\hat{e}_2 is on average higher than \hat{e}_1) risk aversion tends to depress \hat{e}_2 thus reducing the optimality of seniority-based reward schemes. Furthermore, now there exist values of θ_1 for which $\hat{e}_2(\theta_1) < \hat{e}_1(\theta_1)$. To see the reason for this, note that, evaluated at $\bar{\theta}_1$, equations (28) and (29) are symmetric except for the term $-\alpha\psi'(\hat{e}_2^*(\bar{\theta}_1))\psi''(\hat{e}_2^*(\bar{\theta}_1))Var(\varepsilon_2)$. This term captures an additional cost associated with a high second-period effort stemming from the volatility of the agent's payment generated by the shock ε_2 to his second-period productivity. To better appreciate where this term comes from, recall, from Proposition 5, that incentive-compatibility requires that the total payment the agent receives in each state $(\theta_1, \varepsilon_2)$ be given by

$$C(\theta_1, \varepsilon_2) = \mathcal{V}^{-1} \left(\psi(\hat{e}_1(\theta_1)) + \psi(\hat{e}_2(\theta_1)) + \psi'(\hat{e}_2(\theta_1))[\varepsilon_2 - \mathbb{E}[\tilde{\varepsilon}_2]] + \int_{\underline{\theta}_1}^{\theta_1} [\psi'(\hat{e}_1(s))ds + \gamma\psi'(\hat{e}_2(s))] ds \right).$$

It is then immediate that reducing $\hat{e}_2(\theta_1)$ permits the principal to reduce the agent's exposure to the risk generated by ε_2 . For high values of θ_1 , this new effect dominates the rent-extraction effect

of optimality.

²⁶We approximated the solution using a sixth-order polynomial.

documented in the previous section resulting in $\hat{e}_2(\theta_1) < \hat{e}_1(\theta_1)$.

When the correlation between θ_1 and θ_2 is low enough, however, this new effect mitigates but does not eliminate the optimality of seniority-based incentive schemes. When instead θ_1 and θ_2 are highly correlated, then this new effect can completely reverse the optimality of incentive schemes whose power increases with tenure. In the limit, when the shocks to the agent’s productivity become fully persistent (i.e. θ_t follows a random walk, with $\gamma = 1$), it is then immediate from (28) and (29) that $\hat{e}_2(\theta_1) < \hat{e}_1(\theta_1)$ for all θ_1 , in which case the power of the incentive scheme decreases with tenure.

Another way the principal could mitigate the effect of the volatility of ε_2 on the agent’s compensation is by conditioning the second-period effort on the realized value of ε_2 . To gauge the effect of this additional flexibility, consider again the situation examined in Example 5. While a complete analytical characterization of the fully optimal policy escapes us because of the complexity of the optimization problem, we could approximate the optimal policy with 6th degree polynomials. The result is depicted in Figure 2 below, where we considered the same parametrization as in Figure 1, but now allowed the second-period effort to depend on the shock ε_2 . Again, when the correlation between θ_1 and θ_2 is not too high (in the example, $\lambda = 1/2$) the optimality of seniority-based schemes is maintained: $\mathbb{E}\hat{e}_2(\theta_1, \tilde{\varepsilon}_2)$ is on average higher than $\hat{e}_1(\theta_1)$, but there are high values of θ_1 for which $\mathbb{E}\hat{e}_2(\theta_1, \tilde{\varepsilon}_2) < \hat{e}_1(\theta_1)$.

It is also interesting to note that, in this example, second-period effort, as illustrated in Figure 3 below, can be decreasing in the shock ε_2 . This permits the principal to mitigate some of the variability in the agent’s payment, as discussed above.²⁷

This suggests that, in certain environments such as the one considered in this example, it may be difficult to sustain the fully optimal policy with linear or even “quasi-linear” schemes along the lines of those of Proposition 6. When this is the case, one may need to resort to the type of schemes

²⁷Figure 3 is drawn to display two features of the policies: firstly, that the optimal second-period effort is decreasing in the shock ε_2 is clear; secondly, fixing ε_2 , it turns out that the increase in θ_1 is less steep for low values of ε_2 than for higher ones. For $\varepsilon_2 = 0$ it decreases slightly for high values of θ_1 .

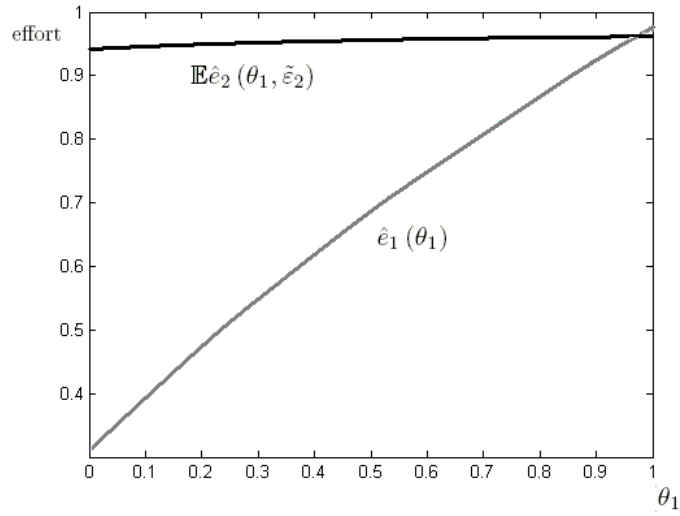


Figure 2: First-period and average second-period efforts

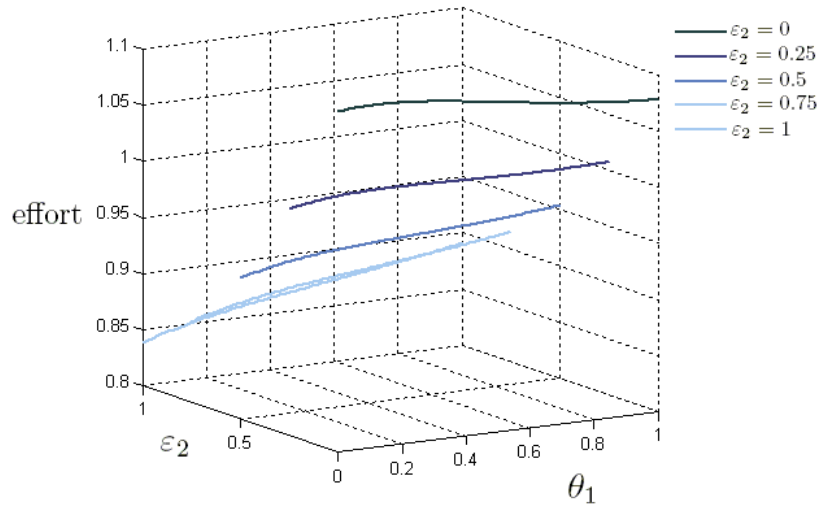


Figure 3: Second-period shock dependent effort

introduced in Proposition 4, adapted to the presence of risk aversion as indicated in Proposition 7 below. To facilitate the comparison with the results in the previous section, we revert here to the primitive representation where the agent reports θ_t , as opposed to the shocks ε_t . The following proposition then generalizes the results in Propositions 3 and 4 to the case of a (weakly) risk averse agent.

Proposition 7 *Suppose the agent's type θ_t evolves according to a Markov process and that effort has only a transitory effect on performance, so that $\pi_t = \theta_t + e_t$, all t .*

1. Any policy e satisfying the single-crossing condition of (19) for any t , any $(\hat{\theta}^{t-1}, \hat{\theta}_t, \theta_t)$, can be implemented by the following “quasi-linear” scheme s^* : given the reports θ^T and the observed performances π^T , the principal pays the agent a total reward

$$\sum_{t=1}^T s_t^*(\theta^t, \pi^t) = \mathcal{V}^{-1} \left(\sum_{t=1}^T [S_t(\theta^t) + \alpha_t(\theta^t)\pi_t] \right)$$

where the functions $S_t(\cdot)$ are as in (11) and where $\alpha_t(\theta^t) \equiv \psi'(e_t^*(\theta^t))$.

2. Any effort policy e satisfying the single-crossing condition (24) for any t , any $(\hat{\theta}^{t-1}, \hat{\theta}_t, \theta_t)$, can be implemented by the following “bonus” scheme s : given the reports θ^T and the observed performances π^T , the principal pays the agent a total reward

$$\sum_{t=1}^T s_t(\theta^t, \pi^t) = \mathcal{V}^{-1} \left(\begin{array}{l} \sum_{t=1}^T \psi(\hat{e}_t(\theta^t)) \\ + \mathbb{E}_{(\tilde{\theta}_2, \dots, \tilde{\theta}_T) | \theta_1} \left[\int_{\underline{\theta}_1}^{\theta_1} \sum_{t=1}^T J_1^t(s, \tilde{\theta}_2, \dots, \tilde{\theta}_t) \psi'(\hat{e}_t(s, \tilde{\theta}_2, \dots, \tilde{\theta}_t)) ds \right] \\ + \sum_{t=2}^T H_t(\theta^t) \end{array} \right)$$

if $\pi_t \geq \pi_t^*(\theta^t) \forall t$, and $\sum_{t=1}^T s_t(\theta^t, \pi^t) = -L$ otherwise, where, for any $t \geq 2$, any θ^t ,

$$\begin{aligned} H_t(\theta^t) &\equiv \mathbb{E}_{(\tilde{\theta}_{t+1}, \dots, \tilde{\theta}_T) | \theta_t} \left[\int_{\underline{\theta}_t}^{\theta_t} \sum_{\tau=t}^T J_t^\tau(\theta^{t-1}, s, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau) \psi'(\hat{e}_\tau(\theta^{t-1}, s, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau)) ds \right] \\ &\quad - \mathbb{E}_{(\tilde{\theta}_t, \dots, \tilde{\theta}_T)} \left[\int_{\underline{\theta}_t}^{\tilde{\theta}_t} \sum_{\tau=t}^T J_t^\tau(\theta^{t-1}, s, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau) \psi'(\hat{e}_\tau(\theta^{t-1}, s, \tilde{\theta}_{t+1}, \dots, \tilde{\theta}_\tau)) ds \right] \end{aligned}$$

Depending on whether the desired effort policy e satisfies the stronger single-crossing condition of (24) or the weaker single crossing condition of (19), it can be implemented either by the quasi-linear scheme s^* of part (1) where the agent's compensation is a convex function of the linear

aggregator $\sum_{t=1}^T [S_t(\theta^t) + \alpha_t(\theta^t)\pi_t]$, or by the bonus scheme s of part (2) according to which the agent receives a positive bonus only upon meeting the firm's targets in each period.

4.1 Consumption smoothing

Finally, to see how the results in the previous sections may be affected by the agent's preferences for consumption smoothing, consider an alternative setting where the agent's payoff is given by

$$U^A(c^T, e^T) = \sum_{t=1}^T \delta^{t-1} [v(c_t) - \psi(e_t)]$$

For simplicity, assume here that effort has only a transitory effect on performance, i.e. $\pi_t = \theta_t + e_t$ for all t , and that θ_t evolves according to a Markov process as in the previous section. Following the same steps used to establish Proposition 5, one can show that, in each state θ^T —equivalently, $(\theta_1, \varepsilon^T)$ —the utility of the total payment to the agent is uniquely pinned down by the policy e up to a constant $V^\Omega(\underline{\theta}_1)$. The characterization of the optimal reward scheme in this environment then proceeds as follows. Given any effort policy e , for each state θ^T , let

$$\begin{aligned} \sum_{t=1}^T \delta^{t-1} v(c_t(\theta^t)) &= \sum_{t=1}^T \delta^{t-1} \psi(e_t(\theta^t)) + V^{\hat{\Omega}}(\underline{\theta}_1) \\ &+ \mathbb{E}_{\tilde{\theta}^T | \theta_1} \left[\int_{\underline{\theta}_1}^{\theta_1} \sum_{t=1}^T \delta^{t-1} J_1^t(\tilde{\theta}^t) \psi'(e_t(\tilde{\theta}^t)) ds \right] \\ &+ \sum_{t=2}^T H_t(\theta^t) \end{aligned} \quad (32)$$

denote the total utility of money that is necessary to sustain the policy e , where for all $t \geq 2$, all θ^t

$$H_t(\theta^t) \equiv \mathbb{E}_{\tilde{\theta}^T | \theta^t} \left[\int_{\underline{\theta}_t}^{\theta_t} \sum_{\tau=t}^T \delta^{t-1} J_t^\tau(\tilde{\theta}^\tau) \psi'(e_\tau(\tilde{\theta}^\tau)) ds \right] - \mathbb{E}_{\tilde{\theta}^T | \theta^{t-1}} \left[\int_{\underline{\theta}_t}^{\tilde{\theta}_t} \sum_{\tau=t}^T \delta^{t-1} J_t^\tau(\tilde{\theta}^\tau) \psi'(e_\tau(\tilde{\theta}^\tau)) ds \right].$$

Then let $c^{opt}(\cdot; e)$ denote the reward scheme that minimizes the expected payment to the principal, among all schemes that satisfy conditions (32), naturally adapted to the filtration generated by the history of reports θ^t . We then have the following result.

Proposition 8 *Suppose the agent's type θ_t evolves according to a Markov process and that effort has only a transitory effect on performance so that $\pi_t = \theta_t + e_t$, for all t . Let e^* denote any effort*

policy that maximizes

$$\mathbb{E}_{\tilde{\theta}^T} \left[\sum_{t=1}^T \delta^{t-1} \left(\tilde{\theta}_t + e(\tilde{\theta}^t) - c_t^{opt}(\tilde{\theta}^t; e) \right) \right]$$

1. For any t and any θ^t , let $\alpha_t(\theta^t) \equiv \psi'(e_t^*(\theta^t))$. Suppose the policy e^* satisfies the single-crossing condition of (19) for any t , any $(\hat{\theta}^{t-1}, \hat{\theta}_t, \theta_t)$. Then e^* together with the “quasi-linear” reward scheme s^* defined below are part of an optimal mechanism. The scheme s^* is such that, in each period t , given the reports θ^t and the observed performances π^t , the principal pays the agent a reward $s_t^*(\theta^t, \pi^t) = v^{-1}(S_t^*(\theta^t) + \alpha_t(\theta^t)\pi_t)$, where the fixed payment $S_t^*(\theta^t)$ is now implicitly defined by

$$v^{-1}(S_t^*(\theta^t) + \alpha_t(\theta^t)\pi_t^*(\theta^t)) = c_t^{opt}(\theta^t; e^*)$$

with $\pi_t^*(\theta^t) \equiv \theta_t + e_t^*(\theta^t)$ for any t and any θ^t .

2. Suppose instead that the policy e^* does not satisfy the single-crossing condition of (19) but satisfies the single-crossing condition of (24) for any t , any $(\hat{\theta}^{t-1}, \hat{\theta}_t, \theta_t)$. Then e^* can be implemented by the following “bonus” scheme s : in each period t , given the reports θ^t and the observed performances π^t , the principal pays the agent a reward $s_t(\theta^t, \pi^t) = c_t^{opt}(\theta^t; e^*)$ if $\pi_t = \pi_t^*(\theta^t) \equiv \theta_t + e_t^*(\theta^t)$ and charges the agent a penalty $L > 0$ otherwise.

Both parts (1) and (2) follow directly from the preceding results along with the definition of $c^{opt}(\cdot; e)$. The only difference between this environment and the one examined in the rest of this section is that, while in that setting the way the total payment to the agent is distributed over time is irrelevant for the agent (and hence for the principal), in the environment considered here it is essential to distribute the payments optimally over the entire relationship. The payment schemes described in the Proposition guarantee that the agent has the right incentives to report his information truthfully and then exert the right level of effort, while at the same time inducing the level of intertemporal consumption smoothing that maximizes the agent’s utility and hence is of minimal cost for the principal.

To get a sense of how the principal allocates optimally the agent’s consumption over time (which is instrumental to the characterization of the optimal effort policy) one can use directly Rogerson’s

(1985) necessary conditions for optimality. Adapted to our environment, these conditions can be stated as follows.

Proposition 9 *Suppose that e can be implemented by the reward scheme s and let c be the corresponding consumption policy. If s implements e at minimum cost for the principal, then the following inverse Euler equation must hold for any two adjacent periods $1 \leq t, t+1 \leq T$, almost every θ^t ,*

$$\frac{1}{v'(c_t(\theta^t))} = \mathbb{E}_{\tilde{\theta}_{t+1}|\theta^t} \left[\frac{\delta}{v'(c_{t+1}(\theta^t, \tilde{\theta}_{t+1}))} \right]. \quad (33)$$

[optimal effort: to be completed]

5 Conclusion

[to be completed]

6 Appendix

Proof of Proposition 1. Because the agent's participation constraint clearly binds at the optimum, the principal's payoff coincides with the total surplus generated by the relationship, which is given by

$$W = \sum_{t=1}^T \delta^{t-1} [\pi_t(\theta_t, e^t) - \psi(e_t)] = \sum_{t=1}^T \delta^{t-1} \left[\theta_t + e_t + \sum_{\tau=1}^{t-1} \rho^\tau e_{t-\tau} - \psi(e_t) \right]$$

The result then follows from pointwise maximization of $\mathbb{E}[W]$ with respect to each $e_t(\theta_1, \varepsilon^t)$. ■

Proof of Example 4. First note that this environment satisfies all the conditions on the kernels

F assumed at the beginning of the section and that, for any $t \geq 2$, any $\theta_t \in \Theta_t = \mathbb{R}_+$, any θ_{t-1} ,

$$F_t(\theta_t|\theta_{t-1}) = G\left(\frac{\theta_t}{\theta_{t-1}}\right).$$

Therefore, for each $k \geq 2$, each (θ_k, θ_{k-1}) ,

$$I_{k-1}^k(\theta_k, \theta_{k-1}) = \frac{-\partial G\left(\frac{\theta_k}{\theta_{k-1}}\right) / \partial \theta_{k-1}}{\frac{1}{\theta_{k-1}} g\left(\frac{\theta_k}{\theta_{k-1}}\right)} = \frac{\theta_k}{\theta_{k-1}}.$$

It follows that for each $\tau > t$,

$$J_t^\tau(\theta_t, \dots, \theta_\tau) = \frac{\theta_\tau}{\theta_t}.$$

In each period t , the effort policy that solves the relaxed program is thus given by

$$e_t^*(\theta^t) = \max \left\{ 1 - \eta(\theta_1) \frac{\theta_t}{\theta_1}; 0 \right\}.$$

It is immediate to see that, because $e_t^*(\theta^t)$ is decreasing in θ_t , it violates condition (19). Furthermore, by taking e.g. $t = T$, one can easily see that for any $\hat{\theta}_1$ and any $\hat{\theta}_T < \theta_T < \hat{\theta}_1 / \eta(\hat{\theta}_1)$

$$u_T(\theta_T, \hat{\theta}_T; \hat{\theta}^{T-1}) > u_T(\theta_T; \hat{\theta}^{T-1}) \equiv u_T(\theta_T, \theta_T; \hat{\theta}^{T-1})$$

where

$$u_T(\theta_T; \hat{\theta}^{T-1}) = \int_0^{\theta_T} \psi'(e_T^*(\hat{\theta}^{T-1}, s)) ds = \int_0^{\theta_T} [1 - \eta(\hat{\theta}_1) \frac{s}{\hat{\theta}_1}] ds$$

and

$$\begin{aligned} u_T(\theta_T, \hat{\theta}_T; \hat{\theta}^{T-1}) &= \int_{\hat{\theta}_T}^{\hat{\theta}_T} \psi'(e_T^*(\hat{\theta}^{T-1}, s)) ds + \alpha_T(\hat{\theta}^{T-1}, \hat{\theta}_T)[\theta_T - \hat{\theta}_T] \\ &= \int_0^{\hat{\theta}_T} [1 - \eta(\hat{\theta}_1) \frac{s}{\hat{\theta}_1}] ds + \left[1 - \eta(\hat{\theta}_1) \frac{\hat{\theta}_T}{\hat{\theta}_1} \right] [\theta_T - \hat{\theta}_T]. \end{aligned}$$

are, respectively, the continuation payoff that type θ_T obtains by reporting θ_T truthfully and the continuation payoff he obtains by reporting $\hat{\theta}_T < \theta_T$, under the linear scheme s^* of Proposition 3. This proves that e^* cannot be implemented with the linear scheme.

Finally, to see that e^* can be implemented by the scheme s of Proposition 4, it suffices to show that it satisfies the single-crossing condition (24) holds for any t , any $(\hat{\theta}^{t-1}, \hat{\theta}_t, \theta_t)$. To see this, note that, when $t = 1$, (24) is equivalent to

$$\mathbb{E}_{\hat{\theta}_2, \dots, \hat{\theta}_T | \theta_1} \left[\max \{1 - \eta(\theta_1); 0\} - \max \left\{ \max \left\{ 1 - \eta(\hat{\theta}_1); 0 \right\} + \hat{\theta}_1 - \theta_1; 0 \right\} + \sum_{t=2}^T \delta^{t-1} \frac{\hat{\theta}_t}{\hat{\theta}_1} \left[\max \left\{ 1 - \eta(\theta_1) \frac{\hat{\theta}_t}{\hat{\theta}_1}; 0 \right\} - \max \left\{ 1 - \eta(\hat{\theta}_1) \frac{\hat{\theta}_t}{\hat{\theta}_1}; 0 \right\} \right] \right] [\theta_1 - \hat{\theta}_1] \geq 0.$$

This holds by the assumption that η is non-decreasing. Now take any period $t > 1$. Condition (24) then requires that

$$\left[\max \left\{ 1 - \eta \left(\hat{\theta}_1 \right) \frac{\theta_t}{\hat{\theta}_1}; 0 \right\} - \max \left\{ \max \left\{ 1 - \eta \left(\hat{\theta}_1 \right) \frac{\hat{\theta}_t}{\hat{\theta}_1}; 0 \right\} + \hat{\theta}_1 - \theta_1; 0 \right\} \right] [\theta_t - \hat{\theta}_t] \geq 0$$

which holds because, by assumption, $\eta \left(\hat{\theta}_1 \right) \leq \hat{\theta}_1$ for each $\hat{\theta}_1$. ■

Proof of Proposition 5. We prove the result by backward induction, starting from $t = T$. Using the characterization of the necessary conditions for incentive compatibility in the main text, we have that (to ease the exposition, hereafter we drop the qualification “for *almost every truthful* history”):

$$V^{\hat{\Omega}} \left(\theta_1, \varepsilon^T \right) = V^{\hat{\Omega}} \left(\theta_1, \varepsilon^{T-1}, \underline{\varepsilon}_T \right) + \int_{\underline{\varepsilon}_T}^{\varepsilon^T} D_{T,T} \left(\theta_1, \varepsilon^{T-1}, x \right) \psi' \left(\hat{e}_T \left(\theta_1, \varepsilon^{T-1}, x \right) \right) dx$$

Also,

$$\begin{aligned} V^{\hat{\Omega}} \left(\theta_1, \varepsilon^{T-1} \right) &= V^{\hat{\Omega}} \left(\theta_1, \varepsilon^{T-2}, \underline{\varepsilon}_{T-1} \right) \\ &+ \mathbb{E}_{\tilde{\varepsilon}_T} \left[\int_{\underline{\varepsilon}_{T-1}}^{\varepsilon^{T-1}} \left(\begin{array}{l} D_{T-1,T-1} \left(\theta_1, \varepsilon^{T-2}, x \right) \psi' \left(\hat{e}_{T-1} \left(\theta_1, \varepsilon^{T-2}, x \right) \right) \\ + D_{T-1,T} \left(\theta_1, \varepsilon^{T-2}, x, \tilde{\varepsilon}_T \right) \psi' \left(\hat{e}_T \left(\theta_1, \varepsilon^{T-2}, x, \tilde{\varepsilon}_T \right) \right) \end{array} \right) dx \right] \end{aligned}$$

Using the fact that

$$V^{\hat{\Omega}} \left(\theta_1, \varepsilon^{T-1} \right) = \mathbb{E}_{\tilde{\varepsilon}_T} \left[V^{\hat{\Omega}} \left(\theta_1, \varepsilon^{T-1}, \tilde{\varepsilon}_T \right) \right]$$

we then have that

$$\begin{aligned} V^{\hat{\Omega}} \left(\theta_1, \varepsilon^{T-1}, \underline{\varepsilon}_T \right) &= \mathbb{E}_{\tilde{\varepsilon}_T} \left[V^{\hat{\Omega}} \left(\theta_1, \varepsilon^{T-1}, \tilde{\varepsilon}_T \right) - \int_{\underline{\varepsilon}_T}^{\tilde{\varepsilon}_T} D_{T,T} \left(\theta_1, \varepsilon^{T-1}, x \right) \psi' \left(\hat{e}_T \left(\theta_1, \varepsilon^{T-1}, x \right) \right) dx \right] \\ &= V^{\hat{\Omega}} \left(\theta_1, \varepsilon^{T-1} \right) - \mathbb{E}_{\tilde{\varepsilon}_T} \left[\int_{\underline{\varepsilon}_T}^{\tilde{\varepsilon}_T} D_{T,T} \left(\theta_1, \varepsilon^{T-1}, x \right) \psi' \left(\hat{e}_T \left(\theta_1, \varepsilon^{T-1}, x \right) \right) dx \right] \\ &= V^{\hat{\Omega}} \left(\theta_1, \varepsilon^{T-2}, \underline{\varepsilon}_{T-1} \right) \\ &+ \mathbb{E}_{\tilde{\varepsilon}_T} \left[\int_{\underline{\varepsilon}_{T-1}}^{\varepsilon^{T-1}} \left(\begin{array}{l} D_{T-1,T-1} \left(\theta_1, \varepsilon^{T-2}, x \right) \psi' \left(\hat{e}_{T-1} \left(\theta_1, \varepsilon^{T-2}, x \right) \right) \\ + D_{T-1,T} \left(\theta_1, \varepsilon^{T-2}, x, \tilde{\varepsilon}_T \right) \psi' \left(\hat{e}_T \left(\theta_1, \varepsilon^{T-2}, x, \tilde{\varepsilon}_T \right) \right) \end{array} \right) dx \right] \\ &- \mathbb{E}_{\tilde{\varepsilon}_T} \left[\int_{\underline{\varepsilon}_T}^{\tilde{\varepsilon}_T} D_{T,T} \left(\theta_1, \varepsilon^{T-1}, x \right) \psi' \left(\hat{e}_T \left(\theta_1, \varepsilon^{T-1}, x \right) \right) dx \right] \end{aligned}$$

Therefore,

$$\begin{aligned}
V^{\hat{\Omega}}(\theta_1, \varepsilon^T) &= V^{\hat{\Omega}}(\theta_1, \varepsilon^{T-2}, \underline{\varepsilon}_{T-1}) \\
&+ \mathbb{E}_{\tilde{\varepsilon}_T} \left[\int_{\underline{\varepsilon}_{T-1}}^{\varepsilon^{T-1}} \left(D_{T-1, T-1}(\theta_1, \varepsilon^{T-2}, x) \psi'(\hat{e}_{T-1}(\theta_1, \varepsilon^{T-2}, x)) \right. \right. \\
&\quad \left. \left. + D_{T-1, T}(\theta_1, \varepsilon^{T-2}, x, \tilde{\varepsilon}_T) \psi'(\hat{e}_T(\theta_1, \varepsilon^{T-2}, x, \tilde{\varepsilon}_T)) \right) dx \right] \\
&+ \hat{H}_T(\theta_1, \varepsilon^T)
\end{aligned}$$

This establishes the first step of the induction. Now suppose that there exists a $t \leq T - 1$ such that the following representation holds for all periods $s, t \leq s < T$:

$$\begin{aligned}
V^{\hat{\Omega}}(\theta_1, \varepsilon^T) &= V^{\hat{\Omega}}(\theta_1, \varepsilon^{s-1}, \underline{\varepsilon}_s) \\
&+ \mathbb{E}_{(\tilde{\varepsilon}_{s+1}, \dots, \tilde{\varepsilon}_T)} \left[\int_{\underline{\varepsilon}_s}^{\varepsilon_s} \sum_{\tau=s}^T D_{s, \tau}(\theta_1, \varepsilon^{s-1}, x, \tilde{\varepsilon}_{s+1}, \dots, \tilde{\varepsilon}_\tau) \psi'(\hat{e}_\tau(\theta_1, \varepsilon^{s-1}, x, \tilde{\varepsilon}_{s+1}, \dots, \tilde{\varepsilon}_\tau)) dx \right] \\
&+ \sum_{\tau=s+1}^T \hat{H}_\tau(\theta_1, \varepsilon^\tau)
\end{aligned} \tag{34}$$

We then want to show that it holds also for $s = t - 1$. Note that, by incentive compatibility,

$$\begin{aligned}
V^{\hat{\Omega}}(\theta_1, \varepsilon^{t-1}, \underline{\varepsilon}_t) &= \mathbb{E}_{\tilde{\varepsilon}_t} \left[V^{\hat{\Omega}}(\theta_1, \varepsilon^{t-1}, \tilde{\varepsilon}_t) \right] \\
&- \mathbb{E}_{(\tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_T)} \left[\int_{\underline{\varepsilon}_t}^{\tilde{\varepsilon}_t} \sum_{\tau=t}^T D_{t, \tau}(\theta_1, \varepsilon^{t-1}, x, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_\tau) \psi'(\hat{e}_\tau(\theta_1, \varepsilon^{t-1}, x, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_\tau)) dx \right] \\
&= V^{\hat{\Omega}}(\theta_1, \varepsilon^{t-1}) \\
&- \mathbb{E}_{(\tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_T)} \left[\int_{\underline{\varepsilon}_t}^{\tilde{\varepsilon}_t} \sum_{\tau=t}^T D_{t, \tau}(\theta_1, \varepsilon^{t-1}, x, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_\tau) \psi'(\hat{e}_\tau(\theta_1, \varepsilon^{t-1}, x, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_\tau)) dx \right]
\end{aligned}$$

and that, again by incentive compatibility,

$$\begin{aligned}
V^{\hat{\Omega}}(\theta_1, \varepsilon^{t-1}) &= V^{\hat{\Omega}}(\theta_1, \varepsilon^{t-2}, \underline{\varepsilon}_{t-1}) \\
&+ \mathbb{E}_{(\tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_T)} \left[\int_{\underline{\varepsilon}_{t-1}}^{\varepsilon_{t-1}} \sum_{\tau=t-1}^T D_{t-1, \tau}(\theta_1, \varepsilon^{t-2}, x, \tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_\tau) \psi'(\hat{e}_\tau(\theta_1, \varepsilon^{t-2}, x, \tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_\tau)) dx \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
V^{\hat{\Omega}}(\theta_1, \varepsilon^{t-1}, \underline{\varepsilon}_t) &= V^{\hat{\Omega}}(\theta_1, \varepsilon^{t-2}, \underline{\varepsilon}_{t-1}) \\
&+ \mathbb{E}_{(\tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_T)} \left[\int_{\underline{\varepsilon}_{t-1}}^{\varepsilon_{t-1}} \sum_{\tau=t-1}^T D_{t-1, \tau}(\theta_1, \varepsilon^{t-2}, x, \tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_\tau) \psi'(\hat{e}_\tau(\theta_1, \varepsilon^{t-2}, x, \tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_\tau)) dx \right] \\
&- \mathbb{E}_{(\tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_T)} \left[\int_{\underline{\varepsilon}_t}^{\tilde{\varepsilon}_t} \sum_{\tau=t}^T D_{t, \tau}(\theta_1, \varepsilon^{t-1}, x, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_\tau) \psi'(\hat{e}_\tau(\theta_1, \varepsilon^{t-1}, x, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_\tau)) dx \right]
\end{aligned} \tag{35}$$

Using (34) for $s = t$ and combining it with (35), we then have that

$$\begin{aligned}
V^{\hat{\Omega}}(\theta_1, \varepsilon^T) &= V^{\hat{\Omega}}(\theta_1, \varepsilon^{t-2}, \underline{\varepsilon}_{t-1}) \\
&+ \mathbb{E}_{(\tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_T)} \left[\int_{\underline{\varepsilon}_{t-1}}^{\varepsilon_{t-1}} \sum_{\tau=t-1}^T D_{t-1, \tau}(\theta_1, \varepsilon^{t-2}, x, \tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_\tau) \psi'(\hat{e}_\tau(\theta_1, \varepsilon^{t-2}, x, \tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_\tau)) ds \right] \\
&- \mathbb{E}_{(\tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_T)} \left[\int_{\underline{\varepsilon}_t}^{\tilde{\varepsilon}_t} \sum_{\tau=t}^T D_{t, \tau}(\theta_1, \varepsilon^{t-1}, x, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_\tau) \psi'(\hat{e}_\tau(\theta_1, \varepsilon^{t-1}, x, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_\tau)) dx \right] \\
&+ \mathbb{E}_{(\tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_T)} \left[\int_{\underline{\varepsilon}_t}^{\varepsilon_t} \sum_{\tau=t}^T D_{t, \tau}(\theta_1, \varepsilon^{t-1}, x, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_\tau) \psi'(\hat{e}_\tau(\theta_1, \varepsilon^{t-1}, x, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_\tau)) dx \right] \\
&+ \sum_{\tau=t+1}^T \hat{H}_\tau(\theta_1, \varepsilon^\tau) \\
&= V^{\hat{\Omega}}(\theta_1, \varepsilon^{t-2}, \underline{\varepsilon}_{t-1}) \\
&+ \mathbb{E}_{(\tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_T)} \left[\int_{\underline{\varepsilon}_{t-1}}^{\varepsilon_{t-1}} \sum_{\tau=t-1}^T D_{t-1, \tau}(\theta_1, \varepsilon^{t-2}, x, \tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_\tau) \psi'(\hat{e}_\tau(\theta_1, \varepsilon^{t-2}, x, \tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_\tau)) ds \right] \\
&+ \sum_{\tau=t}^T \hat{H}_\tau(\theta_1, \varepsilon^\tau)
\end{aligned}$$

which proves that the representation in (34) holds also for $s = t - 1$. The result then follows directly from the fact that the agent's payoff under truth-telling must coincide with the value function almost surely. ■

Proof of Example 5. Using (25), the principal's payoff can be written as

$$\begin{aligned}
\mathbb{E}[\hat{U}^P] &= \mathbb{E}[\tilde{\theta}_1 + \hat{e}_1(\tilde{\theta}_1) + \gamma \tilde{\theta}_1 + \tilde{\varepsilon}_2 + \hat{e}_2(\tilde{\theta}_1)] \tag{36} \\
&- \mathbb{E} \left[\begin{aligned} &\frac{\alpha}{2} \left(\begin{aligned} &\psi(\hat{e}_1(\tilde{\theta}_1)) + \psi(\hat{e}_2(\tilde{\theta}_1)) + V^{\hat{\Omega}}(\underline{\theta}_1) \\ &+ \int_0^{\tilde{\theta}_1} \psi'(\hat{e}_1(s)) ds + \int_0^{\tilde{\theta}_1} \gamma \psi'(\hat{e}_2(s)) ds + \psi'(\hat{e}_2(\tilde{\theta}_1)) (\tilde{\varepsilon}_2 - \mathbb{E}(\tilde{\varepsilon}_2)) \end{aligned} \right)^2 \\ &+ \beta \left(\begin{aligned} &\psi(\hat{e}_1(\tilde{\theta}_1)) + \psi(\hat{e}_2(\tilde{\theta}_1)) + V^{\hat{\Omega}}(\underline{\theta}_1) \\ &+ \int_0^{\tilde{\theta}_1} \psi'(\hat{e}_1(s)) ds + \int_0^{\tilde{\theta}_1} \gamma \psi'(\hat{e}_2(s)) ds + \psi'(\hat{e}_2(\tilde{\theta}_1)) (\tilde{\varepsilon}_2 - \mathbb{E}(\tilde{\varepsilon}_2)) \end{aligned} \right) \end{aligned} \right]
\end{aligned}$$

To maximize $\mathbb{E}[\hat{U}^P]$, it is then clearly optimal to set $V^{\hat{\Omega}}(\underline{\theta}_1) = 0$. Furthermore, it is clearly optimal to restrict attention to positive effort policies (if \hat{e}_t were negative over some non-empty subset $Q \subset \Theta_1$, the principal could increase her gross profits by asking the agent to exert a higher (negative) effort over Q , without any effect on the agent's compensation).

Next note that, because $\tilde{\theta}_1$ and $\tilde{\varepsilon}_2$ are independent, the expected payment to the agent [i.e. the second term in (36)] can be conveniently rewritten as

$$\mathbb{E} \left[\begin{aligned} & \frac{\alpha}{2} \left(\psi \left(\hat{e}_1 \left(\tilde{\theta}_1 \right) \right) + \psi \left(\hat{e}_2 \left(\tilde{\theta}_1 \right) \right) + \int_0^{\tilde{\theta}_1} \psi' \left(\hat{e}_1 \left(s \right) \right) ds + \int_0^{\tilde{\theta}_1} \gamma \psi' \left(\hat{e}_2 \left(s \right) \right) ds \right)^2 \\ & + \beta \left(\psi \left(\hat{e}_1 \left(\tilde{\theta}_1 \right) \right) + \psi \left(\hat{e}_2 \left(\tilde{\theta}_1 \right) \right) + \int_0^{\tilde{\theta}_1} \psi' \left(\hat{e}_1 \left(s \right) \right) ds + \int_0^{\tilde{\theta}_1} \gamma \psi' \left(\hat{e}_2 \left(s \right) \right) ds \right) \\ & + \frac{\alpha}{2} \left[\psi' \left(\hat{e}_2 \left(\tilde{\theta}_1 \right) \right) \right]^2 Var(\tilde{\varepsilon}_2) \end{aligned} \right]$$

Finally note that it is optimal to restrict attention to policies such that $\hat{e}_t(\theta_1) \leq K$ for all θ_1 , $t = 1, 2$. To see this, suppose there exists a non-empty subset $Q \subset \Theta_1$ such that $\hat{e}_t(\theta_1) > K$ for all $\theta_1 \in Q$, $t \in \{1, 2\}$. Then, the principal could reduce $\hat{e}_t(\theta_1)$ by $\epsilon > 0$ for all $\theta_1 \in Q$ without affecting $\psi'(\hat{e}_t(\theta_1))$. This would increase her expected payoff because $\bar{e} > \frac{1}{k\beta}$ implies that, conditional on any $\theta_1 \in Q$, her expected payoff is locally decreasing over $\hat{e}_t(\theta_1)$ for any $\hat{e}_t(\theta_1) \in (K, +\infty)$, $t = 1, 2$.

Given any pair of policies $\hat{e}_t : \Theta_1 \rightarrow [0, K]$, $t = 1, 2$, we then have that

$$\mathbb{E}[\hat{U}^P] = \mathbb{E}[\tilde{\varepsilon}_2] + (1 + \gamma)\mathbb{E}[\tilde{\theta}_1] + \mathbb{E} \left[L \left(\hat{e}_1 \left(\tilde{\theta}_1 \right), \hat{e}_2 \left(\tilde{\theta}_1 \right), x_1 \left(\tilde{\theta}_1 \right), x_2 \left(\tilde{\theta}_1 \right) \right) \right]$$

where

$$\begin{aligned} L(\hat{e}_1, \hat{e}_2, x_1, x_2) \equiv & \hat{e}_1 + \hat{e}_2 - \frac{\alpha}{2} \left(\frac{k\hat{e}_1^2}{2} + \frac{k\hat{e}_2^2}{2} + k(x_1 + \gamma x_2) \right)^2 \\ & - \frac{\alpha}{2} Var(\tilde{\varepsilon}_2) k^2 \hat{e}_2^2 - \beta \left(\frac{k\hat{e}_1^2}{2} + \frac{k\hat{e}_2^2}{2} + k(x_1 + \gamma x_2) \right). \end{aligned}$$

with

$$x_t(\theta_1) = \int_0^{\theta_1} \hat{e}_t(s) ds, \quad \forall \theta_1 \in \Theta_1, \quad t = 1, 2. \quad (37)$$

The principal's problem thus consists of choosing a pair of functions $\hat{e}_t : \Theta_1 \rightarrow [0, K]$ together with a pair of functions $x_t : \Theta_1 \rightarrow \mathbb{R}$ so as to maximize

$$\mathbb{E} \left[L \left(\hat{e}_1 \left(\tilde{\theta}_1 \right), \hat{e}_2 \left(\tilde{\theta}_1 \right), x_1 \left(\tilde{\theta}_1 \right), x_2 \left(\tilde{\theta}_1 \right) \right) \right]$$

subject to (37).

We will solve this problem with optimal control by treating \hat{e}_1 and \hat{e}_2 as the control variables and x_1 and x_2 as the state variables. To satisfy (37), we will thus restrict the policies $\hat{e}_t(\cdot)$ to be integrable.

Before turning to the Euler-Lagrange equations, we first verify that a solution to this optimal control problem exists and that the Pontryagin principle applies. To this aim, note that the function $L(\cdot)$ is concave with respect to $(\hat{e}_1, \hat{e}_2, x_1, x_2) \in \mathbb{R}_+^4$. Moreover, there exists a function $\tau(r) \equiv a - \frac{k^2\alpha}{2}r^4$, with a chosen sufficiently large, that satisfies $\lim_{r \rightarrow \infty} \frac{\tau(r)}{r} = -\infty$ and $\tau\left(\sqrt{\frac{\hat{e}_1^2}{2} + \frac{\hat{e}_2^2}{2}}\right) \geq L(\hat{e}_1, \hat{e}_2, x_1, x_2)$ for all $(\hat{e}_1, \hat{e}_2, x_1, x_2) \in \mathbb{R}_+^4$.²⁸ Tonelli's theorem then implies existence of a solution to the optimal control problem. That $\hat{e}_t(\cdot)$ is bounded and integrable, then guarantees that the solution satisfies the Pontryagin principle.²⁹

Assuming an interior solution, the Hamiltonian function is given by

$$\begin{aligned} \mathcal{H} = & \hat{e}_1(\theta_1) + \hat{e}_2(\theta_1) - \frac{\alpha}{2} \left(\frac{k\hat{e}_1(\theta_1)^2}{2} + \frac{k\hat{e}_2(\theta_1)^2}{2} + k(x_1(\theta_1) + \gamma x_2(\theta_1)) \right)^2 \\ & - \frac{\alpha}{2} k^2 \hat{e}_2(\theta_1)^2 \text{Var}(\varepsilon_2) - \beta \left(\frac{k\hat{e}_1(\theta_1)^2}{2} + \frac{k\hat{e}_2(\theta_1)^2}{2} + k(x_1(\theta_1) + \gamma x_2(\theta_1)) \right) \\ & + \mu_1(\theta_1)\hat{e}_1(\theta_1) + \mu_2(\theta_1)\hat{e}_2(\theta_1). \end{aligned}$$

where μ_1 and μ_2 are the two co-state variables associated with the constraints given by (37). The following Euler-Lagrange equations must hold for almost every $\theta_1 \in \Theta_1$:

$$\begin{aligned} x'_t(\theta_1) &= \hat{e}_t(\theta_1) \quad t = 1, 2 \\ \mu'_1(\theta_1) &= -\alpha \left(\frac{k\hat{e}_1(\theta_1)^2}{2} + \frac{k\hat{e}_2(\theta_1)^2}{2} + k(x_1(\theta_1) + \gamma x_2(\theta_1)) \right) - \beta \\ \mu'_2(\theta_1) &= -\gamma \left(\alpha \left(\frac{k\hat{e}_1(\theta_1)^2}{2} + \frac{k\hat{e}_2(\theta_1)^2}{2} + k(x_1(\theta_1) + \gamma x_2(\theta_1)) \right) - \beta \right) \\ k\mu_1(\theta_1) &= - \left(1 - \left(\alpha \left(\frac{k\hat{e}_1(\theta_1)^2}{2} + \frac{k\hat{e}_2(\theta_1)^2}{2} + k(x_1(\theta_1) + \gamma x_2(\theta_1)) \right) - \beta \right) k\hat{e}_1(\theta_1) \right) \end{aligned}$$

$$k(\mu_2(\theta_1) + k\alpha \text{Var}(\varepsilon_2)\hat{e}_2(\theta_1)) = - \left(1 - \left(\alpha \left(\frac{k\hat{e}_1(\theta_1)^2}{2} + \frac{k\hat{e}_2(\theta_1)^2}{2} + k(x_1(\theta_1) + \gamma x_2(\theta_1)) \right) - \beta \right) k\hat{e}_2(\theta_1) \right)$$

along with the boundary conditions $\mu_t(\bar{\theta}_1) = x_t(\underline{\theta}_1) = 0$, $t = 1, 2$. Together, these conditions imply (28) and (29) hold almost everywhere. Moreover, one may choose an optimal effort policy \hat{e}^* such that they hold everywhere without changing the principal's payoff. To see this, let $\theta_1^* \in [0, 1]$ be a point at which they do not hold and let (θ_1^n) be a sequence converging to θ_1^* for which (28)

²⁸This condition is referred to as ‘‘coercivity’’ in the literature on optimal control.

²⁹See, for example, Torres (2000).

and (29) are defined. Since $\hat{e}_1(\theta_1^n), \hat{e}_2(\theta_1^n) \in [0, \bar{e}]$ for all n , there exists a subsequence $(\theta_1^{n_k})$ with $(\hat{e}_1(\theta_1^{n_k}), \hat{e}_2(\theta_1^{n_k})) \rightarrow (\bar{e}_1, \bar{e}_2)$. Next note that the functions

$$h_1(d_1, d_2, \theta_1) = kd_1 \left(\alpha \left[\frac{kd_1^2}{2} + \frac{kd_2^2}{2} + k \int_0^{\theta_1} [\hat{e}_1(s) + \gamma \hat{e}_2(s)] ds \right] + \beta \right) \\ - 1 + k \int_{\theta_1}^1 \left(\alpha \left[\frac{k\hat{e}_1(q)^2}{2} + \frac{k\hat{e}_2(q)^2}{2} + k \int_0^q [\hat{e}_1(s) + \gamma \hat{e}_2(s)] ds \right] + \beta \right) dq$$

and

$$h_2(d_1, d_2, \theta_1) = kd_2 \left(\alpha \left[\frac{kd_1^2}{2} + \frac{kd_2^2}{2} + k \int_0^{\theta_1} [\hat{e}_1(s) + \gamma \hat{e}_2(s)] ds \right] + \beta \right) + \alpha k^2 \text{Var}(\varepsilon_2) d_2 \\ - 1 + \gamma k \int_{\theta_1}^1 \left(\alpha \left[\frac{k\hat{e}_1(q)^2}{2} + \frac{k\hat{e}_2(q)^2}{2} + k \int_0^q [\hat{e}_1^*(s) + \gamma \hat{e}_2(s)] ds \right] + \beta \right) dq$$

are continuous in (d_1, d_2, θ_1) and equal to zero at $(d_1, d_2, \theta_1) = (\hat{e}_1^{n_k}, \hat{e}_2^{n_k}, \theta_1^{n_k})$ for every n_k . It follows (since limits of continuous functions commute) that they are equal to zero at $(d_1, d_2, \theta_1) = (\bar{e}_1, \bar{e}_2, \theta_1^*)$ as well. It therefore suffices to put $\hat{e}_1^*(\theta_1^*) = \bar{e}_1$ and $\hat{e}_2^*(\theta_1^*) = \bar{e}_2$. Since this change in the policy is only on a set of zero measure, it does not affect the principal's payoff.

To justify our consideration of interior solutions, we show that for $\alpha > 0$ sufficiently small, \hat{e}_1^* and \hat{e}_2^* must be strictly positive on $[0, 1]$. To see this, re-write (28) and (29) as

$$\hat{e}_1^*(\theta_1) = \frac{1 - k \int_{\theta_1}^1 \left(\alpha \left[\frac{k\hat{e}_1^*(q)^2}{2} + \frac{k\hat{e}_2^*(q)^2}{2} + k \int_0^q [\hat{e}_1^*(s) + \gamma \hat{e}_2^*(s)] ds \right] + \beta \right) dq}{k \left(\alpha \left[\frac{k\hat{e}_1^*(\theta_1)^2}{2} + \frac{k\hat{e}_2^*(\theta_1)^2}{2} + k \int_0^{\theta_1} [\hat{e}_1^*(s) + \gamma \hat{e}_2^*(s)] ds \right] + \beta \right)}$$

and

$$\hat{e}_2^*(\theta_1) = \frac{1 - \gamma k \int_{\theta_1}^1 \left(\alpha \left[\frac{k\hat{e}_1^*(q)^2}{2} + \frac{k\hat{e}_2^*(q)^2}{2} + k \int_0^q [\hat{e}_1^*(s) + \gamma \hat{e}_2^*(s)] ds \right] + \beta \right) dq}{k \left(\alpha \left[\frac{k\hat{e}_1^*(\theta_1)^2}{2} + \frac{k\hat{e}_2^*(\theta_1)^2}{2} + k \int_0^{\theta_1} [\hat{e}_1^*(s) + \gamma \hat{e}_2^*(s)] ds \right] + \beta \right) + \alpha k^2 \text{Var}(\varepsilon_2)}.$$

The right-hand sides converge uniformly to $\frac{1}{\beta k} - (1 - \theta_1)$ and $\frac{1}{\beta k} - \gamma(1 - \theta_1)$ over $\theta_1 \in [0, 1]$ and any (integrable) functions $\hat{e}_1^*, \hat{e}_2^* : [0, 1] \rightarrow [0, \bar{e}]$ as $\alpha \rightarrow 0$. The assumption that $\beta k, \gamma < 1$ implies the result.

Finally, we show that for $\alpha > 0$ sufficiently small, the effort functions $\hat{e}_1^*(\cdot)$ and $\hat{e}_2^*(\cdot)$ are increasing and so satisfy the single-crossing condition (26), guaranteeing that they can be implemented with the “quasi-linear” schemes of Proposition 6. Let $\theta_1' > \theta_1''$ and suppose with a view to contra-

diction that $\hat{e}_1^*(\theta'_1) < \hat{e}_1^*(\theta''_1)$. We show that this implies $\hat{e}_2^*(\theta'_1)$ must be greater than $\hat{e}_2^*(\theta''_1)$ by a sufficiently large margin to violate (29).

Using the fact that $\hat{e}_1^*(\theta_1), \hat{e}_2^*(\theta_1) \in [0, \bar{e}]$ for all $\theta_1 \in [0, 1]$ and equation (28), we have

$$\begin{aligned}
& \max\{\alpha k^2 \bar{e} K (\hat{e}_2^*(\theta'_1) - \hat{e}_2^*(\theta''_1)), 0\} + \alpha k^2 \bar{e} (1 + \gamma) (\theta'_1 - \theta''_1) \\
\geq & k\beta (\hat{e}_1^*(\theta'_1) - \hat{e}_1^*(\theta''_1)) + \alpha k \left(\begin{aligned} & \hat{e}_1^*(\theta'_1) \left[\frac{k\hat{e}_1^*(\theta'_1)^2}{2} + \frac{k\hat{e}_2^*(\theta'_1)^2}{2} + k \int_0^{\theta'_1} \hat{e}_1^*(s) + \gamma \hat{e}_2^*(s) ds \right] \\ & - \hat{e}_1^*(\theta''_1) \left[\frac{k\hat{e}_1^*(\theta''_1)^2}{2} + \frac{k\hat{e}_2^*(\theta''_1)^2}{2} + k \int_0^{\theta''_1} \hat{e}_1^*(s) + \gamma \hat{e}_2^*(s) ds \right] \end{aligned} \right) \\
= & k \int_{\theta''_1}^{\theta'_1} \left(\alpha \left[\frac{k\hat{e}_1^*(q)^2}{2} + \frac{k\hat{e}_2^*(q)^2}{2} + k \int_0^q [\hat{e}_1^*(s) + \gamma \hat{e}_2^*(s)] ds \right] + \beta \right) dq \\
\geq & k\beta (\theta'_1 - \theta''_1).
\end{aligned}$$

This shows firstly that, if $\alpha < \frac{\beta}{k\bar{e}(1+\gamma)}$, we must have $\hat{e}_2^*(\theta'_1) > \hat{e}_2^*(\theta''_1)$. In this case, $\hat{e}_2^*(\theta'_1) - \hat{e}_2^*(\theta''_1) \geq M(\alpha) (\theta'_1 - \theta''_1)$, with $M(\alpha) = \frac{\beta - \alpha k \bar{e} (1 + \gamma)}{\alpha k \bar{e} K} \rightarrow \infty$ as $\alpha \rightarrow 0$. On the other hand, again using the fact that $\hat{e}_1^*(\theta_1), \hat{e}_2^*(\theta_1) \in [0, \bar{e}]$ for all $\theta_1 \in [0, 1]$, equation (29) implies³⁰

$$\begin{aligned}
\gamma k (\theta'_1 - \theta''_1) (\alpha [k\bar{e}^2 + k(1 + \gamma)\bar{e}] + \beta) & \geq \gamma k \left[\int_{\theta''_1}^{\theta'_1} \left(\alpha \left[\frac{k\hat{e}_1^*(q)^2}{2} + \frac{k\hat{e}_2^*(q)^2}{2} \right. \right. \right. \\ & \left. \left. \left. + k \int_0^q [\hat{e}_1^*(s) + \gamma \hat{e}_2^*(s)] ds \right] + \beta \right) dq \right] \\ & = \alpha k^2 \text{Var}(\varepsilon_2) (\hat{e}_2^*(\theta'_1) - \hat{e}_2^*(\theta''_1)) \\ & \quad + k\hat{e}_2^*(\theta'_1) \left(\alpha \left[\frac{k\hat{e}_1^*(\theta'_1)^2}{2} + \frac{k\hat{e}_2^*(\theta'_1)^2}{2} \right. \right. \\ & \quad \left. \left. + k \int_0^{\theta'_1} [\hat{e}_1^*(s) + \gamma \hat{e}_2^*(s)] ds \right] + \beta \right) \\ & \quad - k\hat{e}_2^*(\theta''_1) \left(\alpha \left[\frac{k\hat{e}_1^*(\theta''_1)^2}{2} + \frac{k\hat{e}_2^*(\theta''_1)^2}{2} \right. \right. \\ & \quad \left. \left. + k \int_0^{\theta''_1} [\hat{e}_1^*(s) + \gamma \hat{e}_2^*(s)] ds \right] + \beta \right) \\ & \geq (\hat{e}_2^*(\theta'_1) - \hat{e}_2^*(\theta''_1)) (\alpha k^2 \text{Var}(\varepsilon_2) + k\beta).
\end{aligned}$$

This shows that $(\hat{e}_2^*(\theta'_1) - \hat{e}_2^*(\theta''_1)) \leq N(\alpha) (\theta'_1 - \theta''_1)$ for $N(\alpha) = \frac{\gamma(\alpha[k\bar{e}^2 + k(1+\gamma)\bar{e}] + \beta)}{\alpha k \text{Var}(\varepsilon_2) + \beta} \rightarrow \gamma < \infty$ as $\alpha \rightarrow 0$. Thus, for any α sufficiently small, we obtain a contradiction. The function \hat{e}_2^* can be shown to be non-decreasing for sufficiently small α in the same manner. ■

Proof of Proposition 9. First note that if s implements e , then e can also be implemented by

³⁰The final inequality makes use of the fact that the right-hand side of (28) is increasing. This implies, under the hypothesis $\hat{e}_1^*(\theta'_1) < \hat{e}_1^*(\theta''_1)$, that $\frac{k\hat{e}_1^*(\theta'_1)^2}{2} + \frac{k\hat{e}_2^*(\theta'_1)^2}{2} + k \int_0^{\theta'_1} [\hat{e}_1^*(s) + \gamma \hat{e}_2^*(s)] ds$ is greater than $\frac{k\hat{e}_1^*(\theta''_1)^2}{2} + \frac{k\hat{e}_2^*(\theta''_1)^2}{2} + k \int_0^{\theta''_1} [\hat{e}_1^*(s) + \gamma \hat{e}_2^*(s)] ds$.

the following “bonus” scheme:

$$\hat{s}_t(\theta^t, \pi^t) = \begin{cases} c_t(\theta^t) & \text{if } \pi_t = \theta_t + e_t(\theta^t) \\ -L & \text{otherwise} \end{cases} \quad (38)$$

with $L > 0$ arbitrarily large. So, without loss, assume s itself satisfies condition (38).

Now, suppose there exists a period t and a (positive measure) set $Q \subset \Theta^t$ such that, for any $\theta^t \in \Theta^t$,

$$\frac{1}{v'(c_t(\theta^t))} > \mathbb{E}_{\tilde{\theta}_{t+1}|\theta^t} \left[\frac{\delta}{v'(c_{t+1}(\theta^t, \tilde{\theta}_{t+1}))} \right].$$

The argument for the case where the inequality is reversed is symmetric. Then consider the following alternative scheme $s^\#$. For any $\tau \neq t, t+1$ and any (θ^τ, π^τ) , $s_\tau^\#(\theta^\tau, \pi^\tau) = s_\tau(\theta^\tau, \pi^\tau)$; in period t ,

$$s_t^\#(\theta^t, \pi^t) = \begin{cases} s_t(\theta^t, \pi^t) & \text{if } \theta^t \notin Q \\ v^{-1}(v(c_t(\theta^t)) + k) & \text{if } \theta^t \in Q \text{ and } \pi_t = \theta_t + e_t(\theta^t) \ ; \\ -L & \text{if } \theta^t \in Q \text{ and } \pi_t \neq \theta_t + e_t(\theta^t) \end{cases}$$

and in period $t+1$,

$$s_{t+1}^\#(\theta^{t+1}, \pi^{t+1}) = \begin{cases} s_{t+1}(\theta^{t+1}, \pi^{t+1}) & \text{if } \theta^t \notin Q \\ v^{-1}(v(c_{t+1}(\theta^{t+1})) - k/\delta) & \text{if } \theta^t \in Q \text{ and } \pi_{t+1} = \theta_{t+1} + e_{t+1}(\theta^{t+1}) \ . \\ -L & \text{if } \theta^t \in Q \text{ and } \pi_{t+1} \neq \theta_{t+1} + e_{t+1}(\theta^{t+1}) \end{cases}$$

Clearly, this scheme preserves incentives for both truthful revelation and obedience and, in equilibrium, gives the agent the same payoff as the original scheme s .³¹ The difference between the ex-ante expected cost to the principal under this scheme and under the original scheme s is given by

$$\Delta(k) \equiv F(\tilde{\theta}^t \in Q) \mathbb{E}_{\{\tilde{\theta}^t: \tilde{\theta}^t \in Q\}} \left[\begin{aligned} & \delta^{t-1} \left(v^{-1}(v(c_t(\tilde{\theta}^t)) + k) - c_t(\tilde{\theta}^t) \right) \\ & + \delta^t \mathbb{E}_{\tilde{\theta}_{t+1}|\tilde{\theta}^t} [v^{-1}(v(c_{t+1}(\tilde{\theta}^{t+1})) - \frac{k}{\delta}) - c_{t+1}(\tilde{\theta}^{t+1})] \end{aligned} \right]$$

where $F(\tilde{\theta}^t \in Q)$ denotes the probability that $\tilde{\theta}^t \in Q$ and $\mathbb{E}_{\{\tilde{\theta}^t: \tilde{\theta}^t \in Q\}}[\cdot]$ denotes the conditional expectation over $\tilde{\Theta}^t$ given the sigma-algebra generated by the event that $\theta^t \in Q$.

Clearly, $\Delta(0) = 0$ and

$$\frac{\partial \Delta(0)}{\partial k} = F(\tilde{\theta}^t \in Q) \mathbb{E}_{\{\tilde{\theta}^t: \tilde{\theta}^t \in Q\}} \delta^{t-1} \left[\frac{1}{v'(c_t(\tilde{\theta}^t))} + \mathbb{E}_{\tilde{\theta}_{t+1}|\theta^t} \left(\frac{\delta}{v'(c_{t+1}(\theta^t, \tilde{\theta}_{t+1}))} \right) \right] > 0$$

The principal can then reduce her expected payment to the agent by switching to a scheme $s^\#$ with $k < 0$ arbitrarily small, contradicting the assumption that s is cost-minimizing. ■

³¹Since the choice of L in the scheme s was arbitrary, it may be chosen large enough that incentives are still preserved in $s^\#$.

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