Note: Exercises denoted as 2.15 and 2.16 in the present solution are those corresponding to 2.16 and 2.17 respectively of the current 4th edition of Romer. Similarly exercise 10.5 in the solution corresponds to 11.5 from the current edition of the book.

Problem 2.15

(a) The individual's optimization problem is not affected by δ which means that $r_t = f'(k_t) - \delta$. The HH's problem is still:

$$\max \quad U_t = \frac{c_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\,\frac{c_{2t+1}^{1-\theta}}{1-\theta}$$

$$\text{s.t.} \qquad c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = A_t \cdot w_t .$$

As in class the savings rate (fraction of labor income saved) can be calculated:

$$s(r_{t+1}) = \frac{1}{1 + (1+\rho)^{1/\theta}\,(1+r_{t+1})^{\frac{(\theta-1)}{\theta}}}$$

Thus the way the savings rate depends on the real interest rate is unchanged. The only difference is that the real interest rate itself is now $f'(k_{t+1}) - \delta$, rather than just $f'(k_{t+1})$.

The capital stock in period t+1 equals the amount saved by the young individuals at time t. Thus,

$K_{t+1} = \hat{S}_t \cdot L_t$

where $\hat{S}_t \equiv s(r_{t+1}) A_t w_t$ is the total amount saved by one young individual in period t.

We can then write K_{t+1} as:

$K_{t+1} = s(r_{t+1}) \cdot A_t \cdot w_t \cdot L_t$

To express this in terms of t+1 units of effective labor divide through by $A_{t+1} \cdot L_{t+1}$:

$$\frac{K_{t+1}}{A_{t+1} \cdot L_{t+1}} = s(r_{t+1})\,\frac{A_t}{A_{t+1}} \cdot \frac{L_t}{L_{t+1}} \cdot w_t .$$

or

$$k_{t+1} = \frac{s(r_{t+1}) \cdot w_t}{(1+n)(1+g)}$$

where I have used that $A_{t+1} = (1+g) A_t$ and $L_{t+1} = (1+n) L_t$.

Now substitute in the factor prices:

$$\begin{cases} r_{t+1} = f'(k_{t+1}) - \delta . \\ w_t = f(k_t) - k_t f'(k_t) \end{cases}$$

Then you have:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \left[s\left(f'(k_{t+1}) - \delta\right) \right] \left[f(k_t) - k_t f'(k_t) \right].$$

In the book compare this to equation (2.59)
Thus adding depreciation does alter the relationship between k_{t+1} and k_t. Whether k_{t+1} will be higher or lower for a given k_t depends on the way in which savings vary with r_{t+1}.

(b) With log utility the fraction of income saved does not depend on r_{t+1}. In this case the saving rate is:

$$s(r_{t+1}) = \frac{1}{2+\rho}$$

In addition with Cobb-Douglas production $y_t = k_t^{\alpha}$ and the real wage is $w_t = k_t^{\alpha} - k_t \alpha \cdot k_t^{\alpha-1} = (1-\alpha) k_t^{\alpha}$.
Then the law of motion for capital per unit of effective worker becomes:

$$(*) \quad k_{t+1} = \frac{(1-\alpha) k_t^{\alpha}}{(1+n)(1+g)(2+\rho)}$$

We need to compare this with the law of motion for capital in discrete time version of the Solow model with $\delta = 1$:

$$k_{t+1} = \frac{s}{(1+n)(1+g)} f(k_t) \qquad \left[\text{Hint: } K_{t+1} = s \cdot Y_t + (1-\delta) K_t \right]$$

The savings rate in this economy $\frac{\text{total saving}}{\text{total output}}$. Note this is not the same as $s(r_{t+1})$, which is simply the fraction of their labor income that the young save. Denote the economy's total saving rate as $\hat{s}$. Then $\hat{s}$ will equal the saving of the young plus the dissaving of the old, all divided by total output and in addition all variables are measured in units of effective labor.
The saving of the young is $\frac{1}{2+\rho} \cdot (1-\alpha) k_t^{\alpha}$. Since there is 100% depreciation the old do not get to dissave by the amount of the capital stock. ⇒ there is no dissaving by the old.

Thus;

$$\hat{s} = \frac{\frac{(1-\alpha)k_t^\alpha}{2+\rho}}{k_t^\alpha} = \frac{1}{2+\rho}(1-\alpha).$$

Thus equation (*) can be re-written as:

$$k_{t+1} = \frac{1}{(1+n)(1+g)}\hat{s}k_t^\alpha = \frac{\hat{s}}{(1+n)(1+g)}f(k_t).$$

Note that this is exactly the same as the expression for k_{t+1} as a function of k_t in the discrete-time Solow model with $\delta=1$. Thus that version of the Solow model does have some micro-foundations, although the assumption of 100% depreciation is quite unrealistic.

Problem 2.16

(a) (i) The utility function is:

$$U_t = \ln c_{1t} + \frac{1}{1+\rho}\ln c_{2t+1}$$

With the social security tax of T per person, the individual faces the following constraints (Note with $g=0$, $A_t=A$ for all t):

$$c_{1t} + S_t = A\cdot W_t - T$$

$$c_{2t+1} = (1+r_{t+1})\cdot S_t + (1+n)T$$

where S_t is the individual's total savings in the first period.
As far as the indiv. is concerned the rate of return on social security is $(1+n)$, which in general will not be equal to the return on private saving $(1+r_{t+1})$.
From the individual's second period budget constraint.

$$S_t = \frac{c_{2,t+1}}{1+r_{t+1}} - \frac{(1+n)}{(1+r_{t+1})}\cdot T$$

Substitute this into the first period budget constraint:

$$c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = AW_t - T + \left(\frac{1+n}{1+r_{t+1}}\right)\cdot T$$

$$\Rightarrow \quad c_{1t} + \frac{c_{2t+1}}{1+r_{t+1}} = AW_t - \frac{(r_{t+1}-n)}{(1+r_{t+1})}\cdot T$$

We know that with log utility the indiv. will consume a fraction $\frac{1+\rho}{2+\rho}$ of his lifetime income:

$$C_{1t} = \left(\frac{1+\rho}{2+\rho}\right)\left[A\cdot w_t - \left(\frac{r_{t+1}-n}{1+r_{t+1}}\right)T\right]$$

Then the savings per young individual is:

$$S_t = A\cdot w_t - \left(\frac{1+\rho}{2+\rho}\right)\left[A w_t - \left(\frac{r_{t+1}-n}{1+r_{t+1}}\right)T\right] - T$$

$$\Rightarrow S_t = \left[1 - \left(\frac{1+\rho}{2+\rho}\right)\right]A w_t - \left[1 - \left(\frac{1+\rho}{2+\rho}\right)\left(\frac{r_{t+1}-n}{1+r_{t+1}}\right)\right]T$$

$$\Rightarrow S_t = \frac{1}{2+\rho}\, A w_t - \left[\frac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})}\right]\cdot T$$

Note here that if $r_{t+1} = n$, saving is reduced one-for-one by the social security tax. If $r_{t+1} > n$, saving falls less than one-for-one. Finally if $r_{t+1} < n$ saving falls more than one-for-one.

Denote: $$Z_t \equiv \frac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})}$$

Then we can re-write S_t as:

$$S_t = \frac{1}{2+\rho}\, A\cdot w_t - Z_t\cdot T$$

It is still the case that the total capital stock in period $t+1$ will be equal to the total saving of the young in period t:

$$K_{t+1} = S_t\cdot L_t$$

To convert this in units of effective labor divide both sides by $A\cdot L_{t+1}$:

$$k_{t+1} = \frac{1}{1+n}\left[\frac{1}{2+\rho}\,\omega_t - Z_t\cdot\frac{T}{A}\right]$$

With a Cobb-Douglas production function the real wage is:

$$w_t = (1-\alpha)k_t^{\alpha}$$

Substitute w_t into k_{t+1}:

$$k_{t+1} = \frac{1}{1+n}\left[\frac{1}{2+\rho}(1-\alpha)k_t^{\alpha} - Z_t\cdot\frac{T}{A}\right]$$

(ii) To see what the effect of introducing social security is on the BGP value of k_t we must determine the sign of Z_t. If positive then the introduction of the tax T shifts down the k_{t+1} curve and reduces the BGP value of k:

$$Z_t = \frac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})} = \frac{(1+1+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})}$$

$$= \frac{(1+r_{t+1}) + (1+\rho)\left[(1+r_{t+1}) - (r_{t+1}-n)\right]}{(2+\rho)(1+r_{t+1})} = \frac{(1+r_{t+1}) + (1+\rho)(1+n)}{(2+\rho)(1+r_{t+1})} > 0$$

So the k_{t+1} curve shifts down. relative to the case without social security, and k^* is reduced.

(iii) If the economy was initially dynamically efficient $(r>n)$ a marginal increase in T would result in a gain to the old generation that would receive the extra benefits. However it would reduce k^* further below k_{GR} and thus leave future generations worse off, with lower consumption possibilities. If the economy was initially dynamically inefficient $(r<n)$ so that $k^* > k_{GR}$, the old generation would gain due to the extra benefits. In this case the reduction in k^* would actually allow for higher cons. for future generations and would be welfare improving. The introduction of the tax in this case would reduce/eliminate the dynamic inefficiency caused by the overaccumulation of capital.

(b) (i) In this case the individual's second period budget constraint becomes:

$$C_{2t+1} = (1+r_{t+1}) S_t + (1+r_{t+1}) \cdot T.$$

As far as the indiv. is concerned the rate of return on social security is the same as that on private saving. Solving for S_t:

$$S_t = \frac{C_{2t+1}}{1+r_{t+1}} - T.$$

substitute into the individual's first period budget constraints

$$C_{1t} + \frac{C_{2t+1}}{1+r_{t+1}} = Aw_t - T + T$$

or

$$C_{1t} + \frac{C_{2t+1}}{1+r_{t+1}} = A\cdot w_t .$$

This is the standard intertemporal budget constraint in the Diamond model. Solving the individual's maximization problem yields the usual Euler equation:

$$C_{2t+1} = \frac{1+r_{t+1}}{1+\rho}\cdot C_{1t} .$$

Substituting into the lifetime budget constraint yields:

$$C_{1t} = \left(\frac{1+\rho}{2+\rho}\right)\cdot A\cdot w_t$$

To get saving per person substitute C_{1t} into the first period budget constraint:

$$S_t = A\cdot w_t - \left[\frac{(1+\rho)}{(2+\rho)}\right]\cdot A\cdot w_t - T$$

$$\Rightarrow S_t = \frac{1}{2+\rho}\cdot A\cdot w_t - T$$

The social security tax causes a one-for-one reduction in private saving.

The capital stock in period $t+1$ will be equal to the sum of total private saving of the young plus the total amount invested by the government:

$$K_{t+1} = S_t\cdot L_t + T\cdot L_t .$$

To convert into units of effective labor divide both sides by $A\cdot L_{t+1}$:

$$k_{t+1} = \left(\frac{1}{1+n}\right)\left[\left(\frac{1}{2+\rho}\right)w_t - \frac{T}{A}\right] + \left(\frac{1}{1+n}\right)\cdot\frac{T}{A}$$

$$\Rightarrow k_{t+1} = \left(\frac{1}{1+n}\right)\left(\frac{1}{2+\rho}\right)\cdot w_t .$$

With Cobb-Douglas production function: $w_t = (1-\alpha) k_t^{\alpha}$.
So we have:

$$k_{t+1} = \frac{(1-\alpha) k_t^{\alpha}}{(1+n)(2+\rho)}$$

Thus the fully-funded social security system has no effect on the relationship between the capital stock in successive periods.

(ii) Since there is no effect on the relationship between k_{t+1} and k_t, the BGP value of k is the same as before the introduction of the fully funded social security system. The basic idea is that total investment and saving is still the same each period; the government is simply doing some of the saving for the young. Since social security pays the same rate of return as private saving, individuals are indifferent as to who does the saving. Thus the individuals offset one-for-one any saving the government does for them.

Problem 10.5

(a) We have $\pi_t = p_t - p_{t-1}$ and $\pi_t^e = p_t^e - p_{t-1}$. Thus $\pi_t - \pi_t^e = (p_t - p_{t-1}) - (p_t^e - p_{t-1})$
$\Rightarrow \pi_t - \pi_t^e = p_t - p_t^e$. So we can re-write the Lucas supply curve as

$$y_t = \bar{y} + b(p_t - p_t^e)$$

Set aggregate supply equal to AD (given by $y_t = m_t - p_t$),

$$m_t - p_t = \bar{y} + b(p_t - p_t^e)$$

Solve this for p_t:

$$p_t = \frac{1}{1+b} \cdot m_t + \frac{b}{1+b} p_t^e - \frac{1}{1+b} \cdot \bar{y}$$

With rational expectations the expected value of both sides must be equal

$$p_t^e = \frac{1}{1+b}(m_{t-1} + a) + \frac{b}{1+b} p_t^e - \frac{1}{1+b} \bar{y}$$

where I have used that $E(m_t) = E(m_{t-1} + a) + E(\varepsilon_t) = m_{t-1} + a$

Subtract p_t^e from p_t:

$$p_t - p_t^e = \frac{1}{1+b} \cdot m_t - \frac{1}{1+b}(m_{t-1} + a) = \frac{1}{1+b}(m_t - m_{t-1} - a)$$

Substitute $p_t - p_t^e$ into the Lucas supply curve:

$$y_t = \bar{y} + \frac{b}{1+b}(m_t - m_{t-1} - a)$$

(b) From the last equation we can see that we also need to know a, as well as m_t and m_{t-1}, in order to determine the current level of output. Intuitively, it says that only unexpected money affects output since $m_t - (m_{t-1} + a)$ is the random shock ε_t. However, if we do not know a, we cannot know how much of the change in the nominal money stock from $t-1$ to t was due to a (expected) and how much was due to ε (unexpected).

(c) Now $E(m_t) = m_{t-1} + p(0) + (1-p)a = m_{t-1} + (1-p)a$, since private agents believe that the probability that $a=0$ is p.

Thus: $p_t^e = \frac{1}{1+b}[m_{t-1} + (1-p)a] + \frac{b}{1+b} p_t^e - \frac{1}{1+b}\bar{y}$

Now $p_t - p_t^e = \frac{1}{1+b}[m_t - m_{t-1} - (1-p)a]$

Substitute into the Lucas supply curve:

$$y_t = \bar{y} + \frac{b}{1+b}[m_t - m_{t-1} - (1-p)a]$$

(d) $y_t = \bar{y} + \frac{b}{1+b}(m_t - m_{t-1} - a)$ holds in any period in which there is no regime shift. So if there is no regime shift in period $t-1$:

$y_{t-1} = \bar{y} + \frac{b}{1+b}(m_{t-1} - m_{t-2} - a)$. Subtract $y_t - y_{t-1}$:

$$y_t - y_{t-1} = \frac{b}{1+b}\left[(m_t - m_{t-1}) - (m_{t-1} - m_{t-2})\right].$$

Define: $\Delta y_t = y_t - y_{t-1}$ and $\Delta m_t = m_t - m_{t-1}$. So then:

(*) $\Delta y_t = \frac{b}{1+b}[\Delta m_t - \Delta m_{t-1}]$ which says that in the absence of regime shifts output growth is determined by the change in money growth.

If there is a regime shift then: $y_t = \bar{y} + \frac{b}{1+b}[m_t - m_{t-1} - (1-p)a]$

Then: $y_t - y_{t-1} = \frac{b}{1+b}\left[(m_t - m_{t-1}) - (m_{t-1} - m_{t-2})\right] + \frac{b}{1+b}[a - (1-p)a]$

$$\Rightarrow \Delta y_t = \frac{pab}{1+b} + \frac{b}{1+b}[\Delta m_t - \Delta m_{t-1}] \quad (**)$$

Under the null hypothesis of no credibility of announcement of regime shift, $p=0$ and thus $\frac{pab}{1+b} = 0 \Rightarrow$ (*) and (**) are identical.

So we can regress Δy_t on $[\Delta m_t - \Delta m_{t-1}]$ and a dummy variable that equals one in the period of a regime shift. In fact since we will have an estimate of $\frac{b}{1+b}$ and can determine a, we can calculate an estimate of p from the coefficient on the dummy variable.