Note: The numbering of the answers in what follows is 4.4, 4.5, 4.6, 4.8 (this was the numbering in the previous edition of the book). These answers correspond to exercises 5.4, 5.5, 5.6, 5.8 that I have assigned from the current edition of the book.

(4.4)

$$u_t = \ln c_t + b\frac{(1-\ell_t)^{1-\gamma}}{1-\gamma}.$$

) one-period problem

$$\max \quad \ln c + b\frac{(1-\ell)^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \qquad c = w\ell.$$

$$\mathcal{L} = \ln c + b\frac{(1-\ell)^{1-\gamma}}{1-\gamma} + \lambda\left[w\ell - c\right].$$

FONC:

c: $\quad \frac{1}{c} = \lambda. \qquad (1)$

ℓ: $\quad -b(1-\ell)^{-\gamma} + \lambda w = 0. \qquad (2)$

λ: $\quad w\ell = c. \qquad (3)$

from (1), (2):

$$b(1-\ell)^{-\gamma} = \frac{w}{c}.$$

using (3):

$$b(1-\ell)^{-\gamma} = \frac{\cancel{w}}{\cancel{w}\ell}.$$

$$\Rightarrow \quad b(1-\ell)^{-\gamma} = \frac{1}{\ell}.$$

this is independent of the wage w.

(b) Two-period Problem

$$\max\left\{ \ln c_1 + \frac{b(1-\ell_1)^{1-\gamma}}{1-\gamma} + e^{-\rho}\left[\ln c_2 + \frac{b(1-\ell_2)^{1-\gamma}}{1-\gamma}\right]\right\}.$$

s.t. $$w_1\ell_1 + \frac{w_2\ell_2}{1+r} = c_1 + \frac{c_2}{1+r}.$$

$$\mathcal{L} = \ln c_1 + \frac{b(1-\ell_1)^{1-\gamma}}{1-\gamma} + e^{-\rho}\left[\ln c_2 + \frac{b(1-\ell_2)^{1-\gamma}}{1-\gamma}\right]$$
$$+\lambda\left\{ w_1\ell_1 + \frac{w_2\ell_2}{1+r} - c_1 - \frac{c_2}{1+r}\right\}.$$

FONC:

c_1: $\quad \frac{1}{c_1} = \lambda.$ (1)

ℓ_1: $\quad -b(1-\ell_1)^{-\gamma} + \lambda w_1 = 0.$ (2)

c_2: $\quad \frac{e^{-\rho}}{c_2} = \frac{\lambda}{1+r}.$ (3)

ℓ_2: $\quad -e^{-\rho} b(1-\ell_2)^{-\gamma} + \lambda\frac{w_2}{1+r} = 0.$ (4)

λ: $\quad w_1\ell_1 + \frac{w_2\ell_2}{1+r} = c_1 + \frac{c_2}{1+r}$ (5)

From (2), (4):

$$\cancel{b}(1-\ell_1)^{-\gamma} = \frac{w_1}{w_2}\, e^{-\rho}(1+r)\,\cancel{b}(1-\ell_2)^{-\gamma}.$$

$$\Rightarrow \left(\frac{1-\ell_1}{1-\ell_2}\right)^{\gamma} = \frac{w_2}{w_1}\cdot\frac{1}{1+r}\cdot\frac{1}{e^{-\rho}}.$$

$$\Rightarrow \frac{1-\ell_1}{1-\ell_2} = \left(\frac{w_2}{w_1}\right)^{\frac{1}{\gamma}}\left(\frac{1}{1+r}\right)^{\frac{1}{\gamma}}\cdot\left(\frac{1}{e^{-\rho}}\right)^{\frac{1}{\gamma}}.$$

- how does the relative demand for leisure depend on the relative wage?

$$\frac{d\left(\frac{1-\ell_1}{1-\ell_2}\right)}{d\left(\frac{w_2}{w_1}\right)} = \frac{1}{\gamma}\left(\frac{w_2}{w_1}\right)^{\frac{1}{\gamma}-1}\left(\frac{1}{1+r}\right)^{\frac{1}{\gamma}}\left(\frac{1}{e^{\rho}}\right)^{\frac{1}{\gamma}} > 0.$$

so a rise in w_1 relative to w_2 (i.e. a ~~rise~~ $\downarrow$ in $\frac{w_2}{w_1}$) means a ~~rise~~ decrease in $\frac{1-\ell_1}{1-\ell_2}$ or a rise in ℓ_1 relative to ℓ_2.

<u>Intuition:</u> if you expect to make more today relative to tomorrow you work a bit harder today. to take advantage of the good opportunity.

- how does the relative demand for leisure depend on the interest rate?

$$\frac{\partial\left(\frac{1-\ell_1}{1-\ell_2}\right)}{\partial r} = \left(\frac{w_2}{w_1}\right)^{\frac{1}{\gamma}}\left(\frac{1}{e^{\rho}}\right)^{\frac{1}{\gamma}} \cdot \frac{1}{\gamma}\left(\frac{1}{1+r}\right)^{\frac{1}{\gamma}-1}\left(-\frac{1}{(1+r)^2}\right) < 0$$

so a rise in the interest rate implies a decrease in $\frac{1-\ell_1}{1-\ell_2}$ and thus a rise in ℓ_1 relative to ℓ_2.

<u>Intuition:</u> as r rises it becomes more attractive for you to save b/c next period you will have a higher payoff.
thus you will work harder today to make some extra cash to take advantage of the better savings opportunities.

- low $\gamma \Rightarrow$ high $\frac{1}{\gamma}$ and thus higher effect of $\frac{w_2}{w_1}$ or r on the relative demand for leisure.

- low γ means that utility is not very sharply curved in $\ell \Rightarrow$ people are more willing to tolerate movement (change) in ℓ in response to a change in wages or the interest rate.

- $$\varepsilon_w = \frac{\partial\left(\frac{1-\ell_1}{1-\ell_2}\right)}{\partial\left(\frac{w_2}{w_1}\right)} \cdot \frac{w_2/w_1}{(1-\ell_1)/(1-\ell_2)} =$$

$$= \frac{1}{\gamma}\left(\frac{w_2}{w_1}\right)^{\frac{1}{\gamma}-1}\left(\frac{1}{1+r}\right)^{\frac{1}{\gamma}}\left(\frac{1}{e^{\rho}}\right)^{\frac{1}{\gamma}}\frac{w_2}{w_1}\frac{1-\ell_2}{1-\ell_1}$$

$$= \frac{1}{\gamma}\left(\frac{w_2}{w_1}\right)^{\frac{1}{\gamma}}\left(\frac{1}{1+r}\right)^{\frac{1}{\gamma}}\left(\frac{1}{e^{\rho}}\right)^{\frac{1}{\gamma}}\frac{1-\ell_2}{1-\ell_1} = \frac{1}{\gamma}.$$

- $$\varepsilon_r = \frac{\partial\left(\frac{1-\ell_1}{1-\ell_2}\right)}{\partial\left(\frac{1}{1+r}\right)} \cdot \frac{1+r}{(1-\ell_1)/(1-\ell_2)} =$$

$$= \left(\frac{w_2}{w_1}\right)^{\frac{1}{\gamma}}\left(\frac{1}{e^{\rho}}\right)^{\frac{1}{\gamma}}\frac{1}{\gamma}\left(\frac{1}{1+r}\right)^{\frac{1}{\gamma}-1}\left(-\frac{1}{(1+r)^2}\right)\frac{(1+r)(1-\ell_2)}{1-\ell_1}$$

$$= -\frac{1}{\gamma}\left(\frac{w_2}{w_1}\right)^{\frac{1}{\gamma}}\left(\frac{1}{e^{\rho}}\right)^{\frac{1}{\gamma}}\left(\frac{1}{1+r}\right)^{\frac{1}{\gamma}}\frac{(1-\ell_2)}{(1-\ell_1)} = -\frac{1}{\gamma}.$$

4.5

$$\max\left\{ \ln c_1 + b\ln(1-\ell_1) + e^{-\rho}\left[\ln c_2 + b\ln(1-\ell_2)\right]\right\}.$$

s.t.

$$w_1\ell_1 + \frac{w_2\ell_2}{1+r} = c_1 + \frac{c_2}{1+r}.$$

$$\mathcal{L} = \ln c_1 + b\ln(1-\ell_1) + e^{-\rho}\left[\ln c_2 + b\ln(1-\ell_2)\right]$$
$$+ \lambda\left[w_1\ell_1 + \frac{w_2\ell_2}{1+r} - c_1 - \frac{c_2}{1+r}\right]$$

FONC:

c_1: $\quad \frac{1}{c_1} = \lambda \qquad (1)$

ℓ_1: $\quad \frac{-b}{1-\ell_1} + \lambda w_1 = 0. \qquad (2)$

c_2: $\quad \frac{e^{-\rho}}{c_2} = \frac{\lambda}{1+r}. \qquad (3)$

ℓ_2: $\quad \frac{-e^{-\rho}b}{1-\ell_2} = \lambda\frac{w_2}{1+r}. \qquad (4)$

λ: $\quad w_1\ell_1 + \frac{w_2\ell_2}{1+r} = c_1 + \frac{c_2}{1+r}. \qquad (5)$

From (1): $\quad c_1 = \frac{1}{\lambda} \qquad (6)$

From (2): $\quad \ell_1 = 1 - \frac{b}{\lambda w_1} \qquad (7)$

From (3): $\quad c_2 = \frac{e^{-\rho}(1+r)}{\lambda}. \qquad (8)$

From (4): $\quad \ell_2 = 1 - \frac{e^{-\rho}b(1+r)}{\lambda w_2} \qquad (9)$

sub. (6) - (9) into (5) :

$$w_1\left(1-\frac{b}{\lambda w_1}\right) + \frac{\omega_2}{1+r}\left(1-\frac{e^{-\rho}b(1+r)}{\lambda\omega_2}\right)$$

$$= \frac{1}{\lambda} + \frac{e^{-\rho}(1+r)}{\lambda}.$$

$$\Rightarrow \omega_1 - \frac{b}{\lambda} + \frac{\omega_2}{1+r} - \frac{e^{-\rho}b}{\lambda} = \frac{1}{\lambda} + \frac{e^{-\rho}\,b}{\lambda}.$$

$$\Rightarrow \omega_1 + \frac{\omega_2}{1+r} = \frac{1+b}{\lambda} + \frac{e^{-\rho}(1+b)}{\lambda}.$$

$$\Rightarrow w_1 + \frac{\omega_2}{1+r} = \frac{1+b}{\lambda}(1+e^{-\rho})$$

$$\Rightarrow \boxed{\lambda = \frac{(1+b)(1+e^{-\rho})}{w_1 + \frac{w_2}{1+r}}.} \quad (10)$$

To solve for l_1, l_2 sub. (10) into (7) and (9)

$$l_1 = 1 - \frac{b}{\omega_1\left(\frac{(1+b)(1+e^{-\rho})}{\omega_1 + \frac{\omega_2}{1+r}}\right)}$$

$$= 1 - \frac{b}{(1+b)}\frac{\left(1+\frac{\omega_2}{\omega_1}\cdot\frac{1}{1+r}\right)}{(1+e^{-\rho})}$$

$$l_2 = 1 - \frac{b\bar{e}^{\rho}(1+r)}{\lambda w_2} = 1 - \frac{b}{1+b}\,\frac{\bar{e}^{\rho}(1+r)}{1+\bar{e}^{\rho}}\left(\frac{w_1}{w_2}+\frac{1}{1+r}\right)$$

From the solutions to l_1 and l_2 we see that they depend only on the relative wage

thus a change in w_1 that is offset by a change in w_2 will leave the rel. wage uneffected and thus the labor demands uneffected.

(b) (i) 4.23 continues to hold b/c z does not afect FOCs.

(ii) does the result above continue to hold? No.

inter. b.c. now:

$$w_1 l_1 + \frac{w_2 l_2}{1+r} + z = c_1 + \frac{c_2}{1+r}.$$

Follow same steps to get.

$$\lambda = \frac{(1+\bar{e}^{\rho})(1+b)}{z + w_1 + \frac{w_2}{1+r}}.$$

Follow same steps to get:

$$\ell_1 = 1 - \frac{b\left(\frac{z}{\omega_1} + 1 + \frac{\omega_2}{\omega_1}\frac{1}{1+r}\right)}{(1+\bar{e}^{\rho})(1+b)}.$$

$$\ell_2 = 1 - \frac{b(1+r)\,\bar{e}^{\rho}\left[\frac{z}{\omega_2} + \frac{\omega_1}{\omega_2} + \frac{1}{1+r}\right]}{(1+\bar{e}^{\rho})(1+b)}$$

so changes in ω_1 or ω_2 even if they leave ~~$\ell_1 \ell_2$~~ $\frac{\omega_2}{\omega_1}$ unchanged will affect ℓ_1, ℓ_2.

in particular: $\uparrow \omega_2$ by $d\omega_2 = d\omega_1$

will lead to a rise in ℓ_1, ℓ_2.

(46.) 2 period model with preferences $\ln c_1 + \ln c_2$.

Let. $r = E(r) + \varepsilon$
$E(r)$: mean; ε: mean zero random disturbance.

(a)

(i) $Y_1 =$ First period labor income.

$Y_2 = 0$.

b.c. $c_1 = Y_1 - S$

$c_2 = (1+r)S = (1+r)(Y_1 - c_1)$

$$\Rightarrow \quad c_1 + \frac{c_2}{1+r} = Y_1$$

HH solves

$$\max \quad \ln c_1 + E \ln c_2$$
$$\text{s.t.} \quad c_2 = (1+r)(Y_1 - c_1)$$

or $$\max \quad \ln c_1 + E \ln (1+r)(Y_1 - c_1)$$

$$\Rightarrow \max \quad \ln c_1 + \ln (Y_1 - c_1) + E \ln (1+r).$$

$$c_1 : \quad \frac{1}{c_1} + \frac{(-1)}{Y_1 - c_1} = 0$$

$$\Rightarrow \quad Y_1 - c_1 = c_1 \Rightarrow c_1 = Y_1/2$$

c_1 is indep. of r.

(ii) c_1 indep. of whether $r = E(r)$ or $r = E(r) + \varepsilon$.
uncertainty doesn't affect the choice of c_1

(b) $Y_1 = 0$
$Y_2 > 0$

b.c.

$$\left.\begin{aligned} C_1 &= B \\ C_2 &= Y_2 - (1+r)B \end{aligned}\right\} \quad C_2 = Y_2 - (1+r)C_1.$$

~~HH~~ Problem

$$\max \quad \ln C_1 + E \ln C_2$$

$$\text{s.t.} \quad C_2 = Y_2 - (1+r)C_1$$

$$\text{or} \quad \max \quad \ln C_1 + E \ln\left[Y_2 - (1+r)C_1\right]$$

FONC:

$$C_1: \quad \frac{1}{C_1} - E\left(\frac{1+r}{Y_2 - (1+r)C_1}\right) = 0.$$

$$\Rightarrow \frac{1}{C_1} = E\left((1+r) \cdot \frac{1}{C_2}\right)$$

$$\Rightarrow \frac{1}{C_1} = E(1+r) \cdot E\left(\frac{1}{C_2}\right) + \operatorname{cov}\left(1+r, \frac{1}{C_2}\right).$$

certainty: $E(1+r) = 1+r.$

$$\text{then } E\left(\frac{1}{Y_2 - C_1(1+r)}\right) = \frac{1}{Y_2 - C_1(1+r)}$$

$$\operatorname{Cov}\left(1+r, \frac{1}{C_2}\right) = 0.$$

$$\text{So} \quad \frac{1}{C_1} = \frac{1+r}{Y_2 - C_1(1+r)} \quad \Rightarrow \quad C_1 = \frac{Y_2}{\underbrace{(1+r)}_{E(r)}2}.$$

uncertainty

$$\text{cov}\left(1+r, \frac{1}{c_2}\right) > 0.$$

since higher ε $\Rightarrow$ higher $1+r$ and higher $\frac{1}{Y_2-(1+r)c_1}$

Jensen's inequality: $E\left(\frac{1}{c_2}\right) > \frac{1}{E(c_2)}$.

since $\frac{1}{c_2}$ is a convex function of c_2.

$$\frac{1}{c_1} = E(1+r)\,E\left(\frac{1}{c_2}\right) + \text{cov}\left(1+r, \frac{1}{c_2}\right) > E(1+r)\,E\left(\frac{1}{c_2}\right) > \frac{E(1+r)}{E(c_2)}$$

$$= \frac{1+E(r)}{Y_2 - c_1(1+E(r))}$$

$$\Rightarrow \quad Y_2 - c_1(1+E(r)) > c_1(1+E(r))$$

$$\Rightarrow \quad c_1 < \frac{Y_2}{2(1+E(r))}$$

uncertainty reduces the amount of first period cons. optimally chosen.

4.8

- constant pop. of infinitely lived individuals.
- pref. $\sum_{t=0}^{\infty} \frac{u(C_t)}{(1+\rho)^t}$, $u(C_t) = C_t - \theta C_t^2$
- $Y_t = AK_t + e_t$ — linear PF.
- $K_{t+1} = K_t + Y_t - C_t$.
- $r = A$. From linear PF.
- Assume $A = \rho$.
- Assume disturbance follows AR(1):
  $e_t = \phi e_{t-1} + \varepsilon_t$. , $\phi \in (-1, 1)$.
  $\varepsilon_t \sim$ iid with mean zero.

<u>Note:</u> Here we solve the Planner's problem. This is equivalent to solving the decentr. problem b/c FWT holds.

(a) Find FOC relating C_t to $E_t C_{t+1}$.

<u>HH Problem:</u>

$$\max \; E_0 \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} \left(C_t - \theta C_t^2 \right).$$

s.t.

$$K_{t+1} + C_t = (1+A)K_t + e_t.$$

Lagrangian:

$$\mathcal{L} = E_0 \left\{ \sum_{t=0}^{\infty} \left[\frac{1}{(1+\rho)^t} (c_t - \theta c_t^2) \quad + \lambda_t \left[(1+A)k_t + e_t - c_t - k_{t+1} \right] \right] \right\}$$

FONC:

c_t: $$E_t\left\{ \frac{1}{(1+\rho)^t}(1 - 2\theta c_t) - \lambda_t \right\} = 0. \quad ①$$

k_{t+1}: $$E_t\left\{ -\lambda_t + \lambda_{t+1}(1+A) \right\} = 0. \quad ②$$

λ_t: $$E_t\left\{ (1+A)k_t + e_t - c_t - k_{t+1} \right\} = 0. \quad ③$$

From ①, ②:

$$\lambda_t = \frac{1}{(1+\rho)^t}(1-2\theta c_t).$$

$$\frac{1}{(1+\rho)^t}(1-2\theta c_t) = (1+A)E_t\left[\frac{1}{(1+\rho)^{t+1}}(1-2\theta c_{t+1}) \right].$$

$\Rightarrow$

$$1-2\theta c_t = \left(\frac{1+A}{1+\rho}\right) E_t(1-2\theta c_{t+1})$$

since $A=\rho$ by assumption.

$\Rightarrow$

$$\boxed{c_t = E_t(c_{t+1}).}$$

(b) Guess : $c_t = \alpha + \beta k_t + \gamma e_t$.

then $k_{t+1} = (1+A) k_t - c_t + e_t$.

$$\Rightarrow \quad k_{t+1} = (1+A) k_t - \alpha - \beta k_t - \gamma e_t + e_t.$$

$$\Rightarrow \quad \boxed{k_{t+1} = (1+A-\beta) k_t + (1-\gamma) e_t - \alpha.} \quad (*)$$

(c) what values must α, β, γ have for the foc in (a) to be satisfied for all (k_t, e_t)

we have guessed

$$c_t = \alpha + \beta k_t + \gamma e_t.$$

then

$$c_{t+1} = \alpha + \beta k_{t+1} + \gamma e_{t+1}.$$

sub. these in foc

$$c_t = E_t(c_{t+1})$$

$$\alpha + \beta k_t + \gamma e_t = E_t(\alpha + \beta k_{t+1} + \gamma e_{t+1})$$

$$\Rightarrow \quad \alpha + \beta k_t + \gamma e_t = \alpha + \beta k_{t+1} + \gamma E(e_{t+1})$$

but

$$e_{t+1} = \phi e_t + \varepsilon_{t+1}$$

with $E(\varepsilon_{t+1}) = 0$.

so

$$\cancel{\alpha} + \beta k_t + \gamma e_t = \cancel{\alpha} + \beta k_{t+1} + \gamma \phi e_t .$$

$$\Rightarrow \quad k_{t+1} = \frac{\cancel{\beta}}{\cancel{\beta}} k_t + \frac{\gamma(1-\phi) e_t}{\beta} .$$

$$\Rightarrow \quad \boxed{k_{t+1} = k_t + \frac{\gamma(1-\phi)}{\beta} e_t .} \quad (**)$$

on (*), (**) it must be that

$$1 + A - \beta = 1 \quad \Rightarrow \quad \boxed{A = \beta}$$

$$\frac{\gamma(1-\phi)}{\beta} = (1-\gamma) \quad \Rightarrow \quad \frac{\gamma(1-\phi)}{A} = 1-\gamma$$

$$\boxed{\alpha = 0.}$$

$$\Rightarrow \quad \gamma(1-\phi) = A - \gamma A .$$

$$\Rightarrow \quad \gamma(1-\phi+A) = A .$$

$$\boxed{\gamma = \frac{A}{1+A-\phi}}$$

(d) effects of a one time shock to ε on the paths of y, k, c?

<u>one time shock</u>

$$\varepsilon_t = 1$$
$$\varepsilon_{t+j} = 0 \qquad \text{for } j = 1, 2, 3, \ldots$$

Assume economy is initially on its BGP.

$$e_t = 1$$
$$e_{t+1} = \phi e_t = \phi.$$
$$e_{t+2} = \phi e_{t+1} = \phi^2$$
$$\vdots$$
$$e_{t+j} = \phi^j$$

We have that

$$\cancel{K_{t+1} = K_t + \gamma(1-\phi)}$$

$$K_{t+1} = K_t + (1-\gamma) e_t.$$

on BGP:

$$K_t = K^{SS}$$
$$K_{t+1} = K_t + (1-\gamma) \cdot e_t.$$
$$= K^{SS} + (1-\gamma) \cdot 1$$

$$K_{t+2} = K_{t+1} + (1-\gamma) e_{t+1}$$
$$= K^{SS} + (1-\gamma) + (1-\gamma)\phi.$$
$$= K^{SS} + (1-\gamma)(1+\phi).$$

$$k_{t+3} = k_{t+2} + (1-\gamma)e_{t+2}$$

$$= k^{ss} + (1-\gamma)(1+\phi) + (1-\gamma)\phi^2.$$

$$= k^{ss} + (1-\gamma)(1+\phi+\phi^2)$$

$$\vdots$$

$$k_{t+j} = k^{ss} + (1-\gamma)(1+\phi+\phi^2+\dots\phi^{j-1})$$

As $j \to \infty$

$$\lim_{j\to\infty} k_{t+j} = k^{ss} + \frac{1-\gamma}{1-\phi}.$$

<u>then For cons.</u>

$$c_t = Ak_t + \gamma e_t.$$

$$= Ak_t + \gamma$$

$$c_{t+1} = Ak_{t+1} + \gamma e_{t+1}$$

$$= Ak_{t+1} + \gamma\phi$$

$$c_{t+2} = Ak_{t+2} + \gamma e_{t+2}$$

$$= Ak_{t+2} + \gamma\phi^2.$$

$$\vdots$$

$$c_{t+j} = Ak_{t+j} + \gamma\phi^j$$

$$c_\infty = Ak_\infty.$$

<u>Then For. output.</u>

$$y_t = Ak_t + e_t.$$

$$= Ak_t + 1$$

$$y_{t+1} = Ak_{t+1} + \phi, \quad y_{t+2} = Ak_{t+2} + \phi^2.$$

$$y_{t+j} = Ak_{t+j} + \phi^j, \quad y_\infty = Ak_\infty$$

Thus at time $t+n$ we have

$$\begin{cases} k_{t+n} - k^{ss} = (1-\gamma)(1+\phi+\phi^2+\cdots+\phi^{n-1}). \\ c_{t+n} - c^{ss} = Ak_{t+n} + \gamma\phi^n - Ak^{ss}. \\ y_{t+n} - y^* = Ak_{t+n} + \phi^n - Ak^{ss}. \end{cases}$$