Question 1

(a) • optimal growth model in sequence form.

$$\max_{\{c_t, I_t, K_{t+1}, L_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

s.t.

Feasibility or resource constraint: $c_t + I_t \leq Y_t$

law of motion $K_{t+1} = (1-\delta)K_t + I_t$.

production function $Y_t = F(K_t, L_t)$

Feasibility for labor $L_t \leq 1$

non-negativity constraints: $c_t \geq 0,\ I_t \geq 0,\ K_{t+1} \geq 0$

for $t = 0, 1, 2, \ldots$

initial condition $K_0 > 0$ given.

• by monotonicity the resource constraint in each t will hold with equality. (planner will not waste resources)

• since HHs do not value leisure here The planner will choose $L_t = 1$ as long as $MPL > 0$.

• Combining prod. funct, law of motion, resource constr:

$$c_t + K_{t+1} = F(K_t, L_t) + (1-\delta)K_t$$
$$= F(K_t, 1) + (1-\delta)K_t \equiv f(K_t)$$: per worker supply of goods in t

• The non-negativity constraints on c_t, K_{t+1} imply jointly:

$$0 \leq K_{t+1} \leq f(K_t)$$

This was the general problem we had in class.

Here $F(K_t, L_t) = A K_t^{\alpha} L_t^{1-\alpha} = A K_t^{\alpha}$

and $\delta = 1$

Thus.

$$c_t + K_{t+1} = A K_t^{\alpha}$$

And the non-negativity constraints imply

$$0 \leq K_{t+1} \leq A K_t^{\alpha}$$

further $U(c_t) = \ln(c_t)$ here.

Thus the planner's problem simplifies to:

$$\max_{\{c_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln[\underbrace{A K_t^{\alpha} - K_{t+1}}_{c_t}]$$

$$0 \leq K_{t+1} \leq A K_t^{\alpha} \quad , t = 0, 1, \ldots$$

K_0 given.

(b) $$V(K) = \max_{0 \leq K' \leq AK^{\alpha}} \left\{ \ln[AK^{\alpha} - K'] + \beta V(K') \right\}$$

or (FE)

$$V(K_t) = \max_{0 \leq K_{t+1} \leq A K_t^{\alpha}} \left\{ \ln[A K_t^{\alpha} - K_{t+1}] + \beta V(K_{t+1}) \right\}$$

(c) FONC: take derivative on RHS of (FE) with respect to K_{t+1}

$$\frac{1}{AK_t^{\alpha}-K_{t+1}} = \beta V'(K_{t+1})$$

Envelope Condition:

$$V'(K_t) = \frac{\alpha A K_t^{\alpha-1}}{AK_t^{\alpha}-K_{t+1}}$$

Forward once:

$$V'(K_{t+1}) = \frac{\alpha A K_{t+1}^{\alpha-1}}{AK_{t+1}^{\alpha}-K_{t+2}}$$

plug into FONC:

$$\frac{1}{AK_t^{\alpha}-K_{t+1}} = \beta \cdot \alpha A K_{t+1}^{\alpha-1} \cdot \frac{1}{AK_{t+1}^{\alpha}-K_{t+2}}$$

or

$$\underbrace{\frac{1}{C_t}}_{\downarrow} = \underbrace{\beta \left[\alpha A K_{t+1}^{\alpha-1}\right] \frac{1}{C_{t+1}}}$$

marginal cost in terms of utility from giving up 1 unit of cons. today

present value utility benefit from having one extra unit of capital next period in production and allocating the resulting extra output to consump

(d) • take the guess for the value function

$$V(K) = a + b\ln(K)$$

and plug it into the Bellman

• the maximization problem on the RHS of FE is:

$$\max_{0 \le K_{t+1} \le AK_t^\alpha} \left\{ \ln\left[AK_t^\alpha - K_{t+1}\right] + \beta\left[a + b\cdot\ln(K_{t+1})\right] \right\}$$

FONC: $$\frac{1}{AK_t^\alpha - K_{t+1}} = \frac{b\cdot\beta}{K_{t+1}} \Rightarrow K_{t+1} = \frac{b\beta}{1+b\beta}AK_t^\alpha$$

$$\text{or } K' = \frac{b\beta}{1+b\beta}AK^\alpha$$

that is we have a solution of the form

$$K' = g(K).$$

Substitute this solution into the Bellman (FE):

$$\underbrace{V(K)}_{a+b\cdot\ln K} = \ln\left[AK^\alpha - g(K)\right] + \beta\left[a + b\ln g(K)\right]$$

$$\Rightarrow a + b\ln K = \ln\left[AK^\alpha - \frac{b\beta AK^\alpha}{1+b\beta}\right] + \beta\left[a + b\ln\left(\left(\frac{b\beta}{1+\beta b}\right)AK^\alpha\right)\right]$$

$$\Rightarrow a + b\cdot\ln K = \ln\left[\left(A - \frac{b\beta A}{1+b\beta}\right)K^\alpha\right] + a\beta + \beta b\ln\left[\frac{b\beta}{1+\beta b}AK^\alpha\right]$$

$$\Rightarrow a + b\cdot\ln K = \ln\left[A - \frac{b\beta A}{1+b\beta}\right] + a\beta + \beta b\ln\left[\frac{b\beta}{1+\beta b}A\right] + \alpha(1+\beta b)\ln K.$$

this implies :

$$b = \alpha(1+\beta b) \Rightarrow b = \alpha + \alpha\beta b$$

$$\Rightarrow (1-\alpha\beta)b = \alpha \Rightarrow \boxed{b = \frac{\alpha}{1-\alpha\beta}}$$

and

$$a = \ln\left[A - \frac{b\beta A}{1+b\beta}\right] + \overset{a}{\cancel{\alpha}}\beta + \beta b \ln\left[\frac{b\beta A}{1+b\beta}\right]$$

$$\Rightarrow \boxed{a = \frac{1}{1-\beta}\left[\ln(A(1-\beta\alpha)) + \frac{\alpha\beta}{1-\alpha\beta}\ln[A\cdot\alpha\beta]\right]}.$$

Thus it follows that

$$\boxed{K_{t+1} = \alpha\beta A K_t^{\alpha}}$$

Question 2 [redacted]:

(a) HH budget constraint: $C_t + $ [redacted] $X_t = w_t \cdot L_t + r_t \cdot K_t$ [redacted].

Resource Constraint: $C_t + X_t = Y_t$.

(b) Transform problem in terms of units of effective workers:

Production Function

$$Y_t = K_t^{\alpha} (A_t L_t)^{1-\alpha}$$

$$\Rightarrow \frac{Y_t}{A_t L_t} = \frac{K_t^{\alpha} (A_t L_t)^{1-\alpha}}{A_t L_t} \Rightarrow \frac{Y_t}{A_t L_t} = \left(\frac{K_t}{A_t L_t}\right)^{\alpha}$$

$$\Rightarrow \boxed{y_t = k_t^{\alpha}}$$ where by definition

$$y_t \equiv \frac{Y_t}{A_t L_t}$$

$$k_t = \frac{K_t}{A_t L_t}.$$

Law of Motion

$$K_{t+1} = (1-\delta) K_t + X_t.$$

$$\Rightarrow \frac{K_{t+1}}{A_t L_t} = (1-\delta)\frac{K_t}{A_t L_t} + \frac{X_t}{A_t L_t}$$

$$\Rightarrow \frac{K_{t+1}}{A_{t+1} L_{t+1}} \cdot \frac{A_{t+1} \cdot L_{t+1}}{A_t \cdot L_t} = (1-\delta)\frac{K_t}{A_t L_t} + \frac{X_t}{A_t L_t}.$$

$$(1+g)(1+n) k_{t+1} = (1-\delta) k_t + x_t$$

Budget Constraint:

$$C_t + X_t = w_t L_t + r_t K_t$$

$$\Rightarrow \frac{C_t}{A_t \cdot L_t} + \frac{X_t}{A_t L_t} = \frac{w_t \cdot L_t}{A_t L_t} + \frac{r_t K_t}{A_t \cdot L_t}$$

$$\Rightarrow \boxed{c_t + x_t = \hat{\omega}_t + r_t k_t}$$

where $\hat{\omega}_t \equiv \frac{\omega_t}{A_t}$

Preferences:

$$\sum_{t=0}^{\infty} \beta^t \frac{(C_t/L_t)^{1-\sigma}}{1-\sigma} L_t = \sum_{t=0}^{\infty} \beta^t \frac{(C_t/A_t L_t)^{1-\sigma}}{1-\sigma} A_t^{1-\sigma} L_t$$

$$= \sum_{t=0}^{\infty} \beta^t A_0^{1-\sigma} \cdot (1+g)^{(1-\sigma)t} L_0 (1+n)^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

$$= A_0^{1-\sigma} L_0 \sum_{t=0}^{\infty} \left[\beta (1+g)^{1-\sigma} (1+n)\right]^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

$$= B \cdot \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{c_t^{1-\sigma}}{1-\sigma}.$$

where $B = A_0^{1-\sigma} \cdot L_0$

$\hat{\beta} = \beta (1+g)^{1-\sigma} (1+n).$

Firm problem (static each period).

$$\text{MAX} \left\{ K_t^{\alpha}(A_t L_t)^{1-\alpha} - w_t \cdot L_t - r_t K_t \right\}$$

FONC:

K_t: $\alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha} = r_t$.

L_t: $(1-\alpha) K_t^{\alpha} A_t^{1-\alpha} L_t^{-\alpha} = w_t$.

Re-write these conditions in terms of transformed variables:

$$r_t = \alpha \left(\frac{K_t}{A_t L_t}\right)^{\alpha-1} \Rightarrow r_t = \alpha k_t^{\alpha-1}$$

$$w_t = (1-\alpha) A_t \left(\frac{K_t}{A_t L_t}\right)^{\alpha} \Rightarrow w_t = (1-\alpha) A_t k_t^{\alpha}. \Rightarrow \hat{w}_t = (1-\alpha) k_t^{\alpha}$$

where as defined above:

$$\hat{w}_t = \frac{w_t}{A_t}.$$

We can thus write factor prices as functions of the economy's aggregate capital:

$$r(k) = \alpha k^{\alpha-1}$$

$$\hat{w}(k) = (1-\alpha) k^{\alpha}.$$

HH Problem

insequence Form: $\text{MAX} \quad \sum_{t=0}^{\infty} \beta^t \frac{(C_t/L_t)^{1-\sigma}}{1-\sigma} L_t.$

s.t.

$$C_t + [redacted] X = w_t L_t + r_t K_t \; [redacted].$$

$$K_{t+1} = (1-\delta) K_t + X_t.$$

- we can re-write this problem in dynamic programming form. in terms of the transformed variables:

$$V(k_H, k) = \max \left\{ \frac{c^{1-\sigma}}{1-\sigma} + \beta V(k_H', k') \right\}.$$

(FE)

$$c + [redacted] x = \hat{w}(k) + r(k) k_H + [redacted].$$

$$(1+g)(1+n) k_H' = (1-\delta) k_H + x$$

$$k' = \Gamma(k)$$

$$c, x \geq 0$$

- $\Gamma(k)$ is a function describing how HHs expect the aggregate capital stock k to evolve over time. (aggregate law of motion)

- k_H = individual HH capital stock per effective unit of lab

  k = aggregate economy-wide capital stock. per effective unit of labo

- in equilibrium: $k_H = k$.

Def. of RCE:

A RCE is a list of functions $v(k_t, k)$, $g^c(k_t, k)$, $g^x(k_t, k)$, $g^{k'}(k_t, k)$, $f^k(k)$, $\hat{\omega}(k)$, $r(k)$, [redacted], $\Gamma(k)$ such that.

- Given $\hat{\omega}(k)$, $r(k)$, [redacted] and the aggregate law of motion $\Gamma(k)$ the value function $v(k_t, k)$ solves the HH's (FE) where $g^c(k_t, k)$, $g^x(k_t, k)$, $g^{k'}(k_t, k)$ are the optimal decision rules.

- given $\hat{\omega}(k)$, $r(k)$ the decision function $f^k(k)$ solves the firm's problem.

- markets clear

$$g^c(k, k) + g^x(k, k) = k^\alpha.$$

$$k = f^k(k)$$

- [redacted]:

[redacted]

- indiv. and agg. laws of motion are consistent:

$$g^{k'}(k, k) = \Gamma(k)$$

(c). We can re-write the (FE) as:

$$V(k_H, k) = \max \left\{ \frac{1}{1-\sigma} \left[\hat{w}(k) + [r(k) + (1+\theta)(1-\delta)] k_H \text{ [redacted]} - \text{[redacted]} (1+g)(1+n) k_H' \right]^{1-\sigma} + \beta V(k_H', k') \right\}$$

FOC: $\quad c^{-\sigma} \{ -(1+g)(1+n) \text{[redacted]} \} + \beta V_1 (k_H', k') = 0$

EC: $\quad V_1 (k_H, k) = c^{-\sigma} [r(k) + \text{[redacted]} (1-\delta)]$

Forward (EC) once and plug into (FOC)

$$\Rightarrow \quad c^{-\sigma} (1+g)(1+n) \text{[redacted]} = \beta c'^{-\sigma} [r(k') + \text{[redacted]} (1-\delta)].$$

. For Firm FOC see part (b).

(d) A steady state equilibrium is a RCE with the property that $k' = k \; (= k_H' = k_H)$.
$c' = c$

From the Euler equation and the Firm FOC:

$$(1+g)(1+n) \text{[redacted]} = \beta [\alpha k^{\alpha-1} + \text{[redacted]} (1-\delta)].$$

$$\Rightarrow \quad \frac{(1+g)(1+n) \text{[redacted]}}{\beta} - (1-\delta) \text{[redacted]} = \alpha k^{\alpha-1}$$

$$\Rightarrow \quad \frac{(1+g)(1+n) \text{[redacted]} - \beta (1-\delta) \text{[redacted]}}{\alpha \beta} = k^{\alpha-1}$$

$$\Rightarrow \quad k^* = \left(\frac{\alpha \beta \text{ [redacted]}}{(1+g)(1+n) - \beta (1-\delta)} \right)^{\frac{1}{1-\alpha}}.$$

From the law of motion for capital in steady state:

$$k(1+n)(1+g) = (1-\delta)k + x$$

$$\Rightarrow x^* = \left[(1+n)(1+g) - (1-\delta)\right] k^*$$

From the production function:

$$y^* = k^{*\alpha}$$

Then the investment rate is:

$$\frac{x^*}{y^*} = \left[(1+n)(1+g) - (1-\delta)\right] k^{*1-\alpha}$$

$$\Rightarrow \frac{x^*}{y^*} = \left[(1+n)(1+g) - \beta(1-\delta)\right] \left[\frac{\alpha\beta}{(1+g)(1+n) - \beta(1-\delta)}\right]^{\cancel{\frac{1-\alpha}{1-\alpha}}}$$

and output is:

$$y^* = \left[\frac{\alpha\beta}{(1+g)(1+n) - \beta(1-\delta)}\right]^{\frac{\alpha}{1-\alpha}}$$

~~(e) Take 2 economies i and j. Then from (d) if they only differ in θ we have:~~

$$\cancel{\frac{(x/y)_i}{(x/y)_j} = \frac{1+\theta_j}{1+\theta_i} = \frac{1+4}{1+0} = 5}$$ ~~Factor difference in investment rates~~

$$\cancel{\frac{y_i}{y_j} = \left(\frac{1+\theta_j}{1+\theta_i}\right)^{\frac{\alpha}{1-\alpha}} = 5^{1/2} = \sqrt{5} = 2.24}$$

~~Factor difference in income levels~~

Question 3

Assume interior solution and set up the Lagrangia

ⓐ $\mathcal{L} = \sum_{t=0}^{T} \beta^t U[f(k_t) - k_{t+1}]$

FONC :

k_{t+1}: $-\beta^t U'[f(k_t) - k_{t+1}] + \beta^{t+1} U'[f(k_{t+1}) - k_{t+2}] f'(k_{t+1})$

$$\Rightarrow U'[f(k_t) - k_{t+1}] = \beta f'(k_{t+1}) U'[f(k_{t+1}) - k_{t+2}]$$

For $t = 1, 2, \ldots T$

2nd order non-linear difference equation in k

with boundary conditions:

initial condition: $k_0 > 0$ (given)

terminal condition: $k_{T+1} = 0$ (because preferences are monotonic you will eat all and not save anything in last period)

ⓑ with the given functional forms the above 2nd order difference equation becomes:

$$\frac{1}{k_t^\alpha - k_{t+1}} = \frac{\alpha\beta k_{t+1}^{\alpha-1}}{k_{t+1}^\alpha - k_{t+2}} \Rightarrow \frac{1}{k_{t-1}^\alpha - k_t} = \frac{\alpha\beta k_t^{\alpha-1}}{k_t^\alpha - k_{t+1}}$$

$$\Rightarrow \frac{k_t^\alpha - k_{t+1}}{k_t^{\alpha-1}} \frac{1}{k_t} = \alpha\beta [k_{t-1}^\alpha - k_t] \frac{1}{k_t}.$$

$$\Rightarrow 1 - \frac{k_{t+1}}{k_t^\alpha} = \alpha\beta \left[\frac{k_{t-1}^\alpha}{k_t} - 1\right]$$

Change of variable: define $Z_t \equiv \frac{k_t}{k_{t-1}^\alpha}$

Then the last equation becomes :

$$1 - Z_{t+1} = \alpha\beta [Z_t^{-1} - 1] \Rightarrow Z_{t+1} = 1 - \alpha\beta Z_t^{-1} + \alpha\beta$$

$\Rightarrow Z_{t+1} = (1+\alpha\beta) - \alpha\beta Z_t^{-1}$. Note that since $k_{T+1} = 0$, $Z_{T+1} = 0$

SOLVING THE DIFF-EQU.

$$Z_{t+1} = (1+\alpha\beta) - \alpha\beta\, Z_t^{-1}$$

$$Z_1 = (1+\alpha\beta) - \alpha\beta\, Z_0^{-1}$$

$$Z_2 = (1+\alpha\beta) - \alpha\beta\, Z_1^{-1} = (1+\alpha\beta) - \alpha\beta\left[(1+\alpha\beta) - \alpha\beta\, Z_0^{-1}\right]^{-1}$$

$$\alpha\beta\, Z_t^{-1} = (1+\alpha\beta) - Z_{t+1}$$

$$\Rightarrow \quad Z_t = \frac{\alpha\beta}{(1+\alpha\beta) - Z_{t+1}}$$

$$\boxed{Z_{T+1} = 0}$$

$$\boxed{Z_T = \frac{\alpha\beta}{(1+\alpha\beta)}}$$

$$Z_{T-1} = \frac{\alpha\beta}{(1+\alpha\beta) - Z_T} = \frac{\alpha\beta}{(1+\alpha\beta) - \left(\frac{\alpha\beta}{1+\alpha\beta}\right)}$$

$$\Rightarrow Z_{T-1} = \frac{\alpha\beta\,(1+\alpha\beta)}{(1+\alpha\beta)^2 - \alpha\beta} = \frac{\alpha\beta\,(1+\alpha\beta)(1-\alpha\beta)}{(1+(\alpha\beta)^2 + 2\alpha\beta - \alpha\beta)\,(1-\alpha\beta)}$$

$$= \alpha\beta\,\frac{(1-\alpha\beta + \alpha\beta - (\alpha\beta)^2)}{\left[1+(\alpha\beta)^2 + \alpha\beta\right](1-\alpha\beta)}$$

$$= \frac{\cancel{\alpha\beta\ (1-\alpha\beta +}}{} \quad \alpha\beta\,\frac{(1-(\alpha\beta)^2)}{1+\cancel{(\alpha\beta)^2}+\cancel{\alpha\beta}-\cancel{\alpha\beta}-(\alpha\beta)^3-\cancel{(\alpha\beta)^2}}$$

$$\boxed{Z_{T-1} = \alpha\beta\,\frac{\left[1-(\alpha\beta)^2\right]}{1-(\alpha\beta)^3}}$$

$$Z_{T-2} = \frac{\alpha\beta}{(1+\alpha\beta) - Z_{T-1}} = \frac{\alpha\beta}{(1+\alpha\beta) - \frac{\alpha\beta(1-(\alpha\beta)^2)}{1-(\alpha\beta)^3}}$$

$$\therefore Z_{T-2} = \alpha\beta \cdot \frac{1-(\alpha\beta)^3}{(1+\alpha\beta)(1-(\alpha\beta)^3) - \alpha\beta(1-(\alpha\beta)^2)}$$

$$= \alpha\beta \frac{1-(\alpha\beta)^3}{1 - \cancel{(\alpha\beta)^3} + \cancel{\alpha\beta} - (\alpha\beta)^4 - \cancel{\alpha\beta} + \cancel{(\alpha\beta)^3}}$$

$$\Rightarrow Z_{T-2} = \alpha\beta \frac{1-(\alpha\beta)^3}{1-(\alpha\beta)^4}$$

$$\vdots$$

$$Z_t = \alpha\beta \frac{1-(\alpha\beta)^{T-t+1}}{1-(\alpha\beta)^{T-t+2}}.$$

Substituting in Z_t and forwarding once: you see that

$$k_{t+1} = \frac{\alpha\beta\left[1-(\alpha\beta)^{T-t}\right]}{1-(\alpha\beta)^{T-t+1}} k_t^{\alpha},$$

$$t = 0, 1, \ldots, T.$$

$\cancel{T-(T-2)+1}$

$\cancel{T-(T-2)+2}$

$\cancel{T-t+1}$

$\cancel{T-t+2}$

(c) You can find the infinite horizon solution i.e. the savings function as $T \to \infty$ as the limit of the finite horizon solution.

$$\lim_{T\to\infty} k_{t+1} = \alpha\beta k_t^{\alpha}, \quad t = 0, 1, \ldots$$

This suggests that the solution of savings at any point in time depends only on the capital stock in that period i.e. it takes the form

$$k_{t+1} = g(k_t), \quad t = 0, 1, 2, \ldots$$

where $g(\cdot)$ is constant.

<u>Note:</u> This approach to finding the optimal policy rule as the limit of the finite horizon rule is very specific to these functional forms and will not work in general.