University of Toronto
Department of Economics
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Economic Theory - Macroeconomics (MA)
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Practice Exercises - Set 4

1. Consider the following infinite horizon optimal growth model. The population is fixed and normalized to 1. Household preferences are described by,

$$\sum_{t=0}^{\infty} \beta^t ln(C_t)$$

where C_t is consumption in period t and $\beta \in (0,1)$ is the subjective discount factor. Production in this economy takes places according to a Cobb-Douglas production function, $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$, where K_t is physical capital and L_t is labor. $A > 0$ is a constant efficiency parameter, and $\alpha \in (0,1)$ is the elasticity of output Y_t with respect to capital. Capital evolves according to the following law of motion, $K_{t+1} = I_t$, where I_t is investment at time t (there is full depreciation). The initial level of the capital stock is $K_0 > 0$ and is given. A planner has to choose sequences of consumption, investment, future capital, and savings $\{C_t, I_t, K_{t+1}, L_t\}_{t=0}^{\infty}$ to maximize household preferences subject to sequences of feasibility and non-negativity constraints, given initial K_0.

   (a) Write down the problem of the planner in sequence form. Make sure to explicitly state the sequences of feasibility constraints and non-negativity constraints. Simplify the problem to one of choosing sequences $\{C_t, K_{t+1}\}_{t=0}^{\infty}$.

   (b) Write down the problem of the planner in dynamic programming format (recursively), that is specify the functional equation (FE).

   (c) Using the first order condition and the envelope condition derive the planner's Euler equation. What is the interpretation of this equation?

   (d) Use the *guess and verify method* to find the value function and the policy function. In particular, guess that the value function is of the form,

$$v(K) = a + bln(K)$$

   and find the values of the undetermined coefficients a and b.

2. Consider a standard growth model with the following features. There is a single sector that produces consumption and investment goods using the constant returns to scale technology:

$$Y_t = F(K_t, L_t) = K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha}$$

where A_t is a technology parameter that grows at the exogenous rate g. There is an infinitely lived representative household, with preferences:

$$\sum_{t=0}^{\infty} \beta^t \frac{\left(C_t/L_t\right)^{1-\sigma}}{1-\sigma} L_t$$

where L is the size of the working population, growing exogenously according to $L_{t+1} = (1+n)L_t$. The representative household owns the capital in the economy, rents labor and capital services to firms, and spends income on consumption and investment. The law of motion for capital is:

$$K_{t+1} = (1-\delta)K_t + X_t$$

where X_t is investment.

(a) Write down the household's budget constraint and the economy's resource constraint.

(b) Define a recursive competitive equilibrium for this economy. Make sure to specify the problems of households and firms clearly. (Hint: before defining equilibrium transform all variables in the problem in terms of effective units of labor, denoting them by lower case letters. For example $y_t = \frac{Y_t}{A_t L_t}$, and similarly define c, k, x. Show that the adjusted discounted factor of the transformed problem is, $\widetilde{\beta} = \beta(1+g)^{1-\sigma}(1+n)$).

(c) Derive the Euler equation for the household using dynamic programming. Write down the first order conditions to the firm's problem.

(d) Define a steady state equilibrium. Derive the investment rate (investment-output ratio) and output per working age person in the steady state of this economy (Hint: do this in terms of the transformed variables).

3. Consider the following optimal growth model with a finite horizon T and a population normalized to 1. The planner faces the following problem,

$$\max_{\{k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t U\left[f(k_t) - k_{t+1}\right]$$

subject to,

$$0 \leq k_{t+1} \leq f(k_t)$$

for $t = 0, 1, 2, ...T$, and given $k_0 > 0$. The period utility function $U(.)$ and the per capita supply of goods function $f(.)$ have the standard properties. There is no technological progress.

(a) Using the Lagrange method derive the optimality condition for $t = 1, 2, ...T$ as a second order non-linear difference equation in k,

$$U'\left[f(k_t) - k_{t+1}\right] = \beta f'(k_{t+1}) U'\left[f(k_{t+1}) - k_{t+2}\right]$$

(b) Assume now that $f(k) = k^\alpha$, with $\alpha < 1$, and $U(.) = ln(.)$. Using the terminal condition that $k_{T+1} = 0$, show that the optimal policy rule for k_{t+1} in each period t is given by,

$$k_{t+1} = \alpha\beta \frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}} k_t^\alpha$$

for $t = 0, 1, ...T$ given k_0. (Hint: reduce the problem first to a first order difference equation by using the change of variable $z_t \equiv \frac{k_t}{k_{t-1}^\alpha}$).

(c) Suppose now that the horizon is infinite. Take the limit of the policy rule in (b) to find a policy rule that is independent of time.