University of Toronto
Department of Economics
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Economic Theory - Macroeconomics (MA)
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Professor Tasso Adamopoulos

Answers to Practice Set 3

1. (a) The capital accumulation equation is,

$$\dot{k}(t) = Ak(t) - c(t) - \delta k(t)$$

$$\dot{k}(t) = (A - \delta)\, k(t) - c(t)$$

(b) Real interest rate:

$$r(t) = (1 - \tau)\, A - \delta$$

Growth of consumption per worker:

$$\frac{\dot{c}(t)}{c(t)} = \frac{(1-\tau)\, A - (\delta + \rho)}{\theta}$$

Consider the capital accumulation equation from part (a). Divide both sides by $k(t)$, and use the fact that $\frac{\dot{k}(t)}{k(t)}$ will be constant. Let $\frac{\dot{k}(t)}{k(t)} \equiv b$. Then we have,

$$b = (A - \delta) - \frac{c(t)}{k(t)}$$

This implies that $\frac{\dot{k}(t)}{k(t)} = \frac{\dot{c}(t)}{c(t)}$, i.e., the growth rate of capital per worker is the same as that of consumption per worker. From the production function $y(t) = Ak(t)$, you can see that also $\frac{\dot{y}(t)}{y(t)} = \frac{\dot{k}(t)}{k(t)}$. The savings rate $s = \frac{y-c}{y}$, can be shown to be,

$$s = \frac{Ak - c}{Ak} = 1 - \frac{1}{A}\frac{c}{k} = 1 - \frac{1}{A}\left[A - \delta - \frac{(1-\tau)\, A - (\delta + \rho)}{\theta}\right]$$

$$= 1 - \left[\frac{(\theta - 1 + \tau)\, A + \rho + \delta\,(1 - \theta)}{\theta A}\right]$$

(c) You can see that the interest rate, the savings rate, and the growth rate are all decreasing in the tax, by calculating the derivatives: $\frac{\partial r}{\partial \tau} = -A < 0$, $\frac{\partial s}{\partial \tau} = -\frac{1}{\theta} < 0$, $\frac{\partial (\frac{\dot{y}}{y})}{\partial \tau} = -\frac{A}{\theta} < 0$.

(d) The economy with the higher tax rate will have the lower growth rate as the result from (c) shows. Consequently the two economies will diverge eventually. Economy j with the low tax rate, will overpass economy i (with the high tax rate), even though initially it starts off with a lower income level.

2. (a) Substitute $A(t)$ into the production function and then the resulting $Y(t)$ into the the capital accumulation equation to get,

$$\dot{K}(t) = sB^{1-\alpha}K(t)^{\alpha+(1-\alpha)\phi}L(t)^{1-\alpha}$$

To get an expression for the growth rate $\frac{\dot{K}(t)}{K(t)}$, which I will denote $g_K(t)$, divide both sides of the last equation by $K(t)$,

$$g_K(t) \equiv \frac{\dot{K}(t)}{K(t)} = sB^{1-\alpha}K(t)^{(1-\alpha)(\phi-1)}L(t)^{1-\alpha}$$

Take logs and then the time derivative to get an expression for movements in g_K (i.e. to calculate the growth rate of the growth rate of K),

$$\frac{\dot{g_K}(t)}{g_K(t)} = (1-\alpha)(\phi-1)g_K(t) + (1-\alpha)n$$

or (after multiplying both sides by $g_K(t)$),

$$\dot{g_K}(t) = 1-\alpha)(\phi-1)\left[g_K(t)\right]^2 + (1-\alpha)ng_K(t)$$

(b) Using the last equation, if $\phi < 1$, we can solve for the BGP level of g_K,

$$g_K^* = \frac{n}{1-\phi}$$

See Fig. 1 for the phase diagram. A one-time increase in L will have no effect on the g_K^* as you can see from the equation. Intuitively: diminishing returns to g_K imply there is a limited contribution of additional capital to the production of new capital, so a one time increase in L will raise g_K at the time of the change but in the long-run the effect will die off.

(c) Look at Fig.2 and Fig.3 respectively. When $\phi > 1$, $\dot{g}_K$ is a convex function of g_K. When $\phi = 1$ and $n > 0$ then $\dot{g}_K$ is proportional to g_K.

(d) If $\phi = 1$ and $n = 0$, then we have that $Y(t) = B^{1-\alpha}L^{1-\alpha}K(t) \equiv bK(t)$, and thus $\dot{K}(t) = sbK(t)$. This implies that $\frac{\dot{K}(t)}{K(t)} = sb$. This is essentially the AK model. Here a one-time increase in L increases b and thus the economy's growth rate.

3. A competitive equilibrium is a sequence of prices $\{w(t), r(t), P(t)\}_{t=0}^{\infty}$ and a sequence of allocations $\{C(t), I(t), Y_c(t), Y_i(t), K_c(t), K_i(t), L_c(t)\}_{t=0}^{\infty}$ such that: (1) given sequences of prices $\{w(t), r(t), P(t)\}_{t=0}^{\infty}$, households maximize their lifetime utility subject to their flow budget constraint, the no-ponzi game condition and their initial level of capital $k(0)$; (2) firms in consumption and investment sectors maximize their profits taking prices as given $\{w(t), r(t), P(t)\}_{t=0}^{\infty}$; (3) all markets clear. The market clearing conditions for investment goods, consumption goods, capital and labor respectively are:

$$Y_c(t) = C(t)$$

$$Y_i(t) = I(t)$$

$$K_c(t) + K_i(t) = K(t)$$

$$L_c(t) = L$$

With $\phi(t)$ denoting the share of capital devoted to the investment sector we can re-write, $K_i(t) = \phi(t)K(t)$ and $K_c(t) = [1 - \phi(t)]\, K(t)$.

The flow budget constraint for the household is,

$$P(t)\dot{k}(t) + C(t) = r(t)k(t) + w(t)$$

Set up the current value Hamiltonian for the household's problem:

$$H = \frac{C(t)^{1-\theta}}{1-\theta} + \frac{\pi(t)}{P(t)} \{r(t)k(t) + w(t) - C(t)\}$$

The conditions for optimality are:

$$H_c = 0 \Rightarrow C(t)^{-\theta} = \frac{\pi(t)}{P(t)}$$

$$H_k = -\dot{\pi} + \rho\pi \Rightarrow \frac{\pi(t)}{P(t)} r(t) = -\dot{\pi} + \rho\pi$$

and the transversality condition $lim_{t\to\infty} \{e^{-\rho t}\pi(t)k(t)\} = 0$. The first two optimality conditions imply the following Euler equation,

$$\frac{\dot{C(t)}}{C(t)} = \frac{\frac{r(t)}{P(t)} + \frac{\dot{P}(t)}{P(t)} - \rho}{\theta}$$

Intuitively, $\frac{r(t)}{P(t)} + \frac{\dot{P}(t)}{P(t)}$ is the interest rate on consumption denominated loans: if you give up one unit of consumption today you can buy $1/P(t)$ units of capital which will give you an instantaneous return $r(t)$. Thus you will get $\frac{r(t)}{P(t)}$ plus the change in the relative price of investment that could occur $\frac{\dot{P}(t)}{P(t)}$.

(b) The problem of the firm in the investment sector is:

$$max\left\{P(t)AK_i(t) - r(t)K_i(t)\right\}$$

and the FOC is:

$$r(t) = P(t)A$$

The problem of the firm in the investment sector is:

$$max\left\{BK_c^{\alpha}(t)L_c^{1-\alpha}(t) - r(t)K_c(t) - w(t)L_c(t)\right\}$$

The FOC are:

$$r(t) = \alpha BK_c^{\alpha-1}(t)L_c^{1-\alpha}(t)$$

$$w(t) = (1-\alpha)BK_c^{\alpha}(t)L_c^{-\alpha}(t)$$

Return equalization in $r(t)$ across sectors implies that the relative price of investment is,

$$P(t) = \alpha\frac{B}{A}K_c^{\alpha-1}(t)L_c^{1-\alpha}(t)$$

(c) Now define a BGP as an equilibrium path along which $\phi(t)$ is constant at ϕ. To proceed use that in equilibrium $L_c(t) = L$. Plug this in the relative price of investment. Take logs and differentiate with respect to time to get,

$$\frac{\dot{P}(t)}{P(t)} = (\alpha-1)g_K$$

where $g_K = \frac{\dot{K}(t)}{K(t)}$ is the steady-state (BGP) growth rate of capital. We also know from the FOC of the investment firm that $\frac{r}{P} = A$. Then the consumption based interest rate faced by the household is $\frac{r(t)}{P(t)} + \frac{\dot{P}(t)}{P(t)} = A - (1-\alpha)g_K$. Then from the Euler equation the consumption growth rate is,

$$\frac{\dot{C}(t)}{C(t)} = \frac{A - (1-\alpha)g_K - \rho}{\theta}$$

Use the market clearing condition for consumption goods along with the consumption production function and that $K_c = (1-\phi)K$ and $L_c = L$. Take logs and differentiate with respect to time to get,

$$\frac{\dot{C}(t)}{C(t)} = \alpha g_K$$

This along with the Euler imply that,

$$g_K^* = \frac{A-\rho}{1-\alpha(1-\theta)}$$

Then $\frac{\dot{C}}{C} = \alpha g_K^*$. Take the FOC with respect to labor in the consumption sector and take logs and differentiate with respect to time to find that,

$$\frac{\dot{w}}{w} = \alpha g_K^*$$

4. (a) The budget constraint of the representative household is,

$$C_t + I_t = W_t L_t + (1 - \tau_{kt}) R_t K_t - T_t$$

Before defining and characterizing equilibrium it is useful to re-write the model economy in terms of variables normalized in units of effective labor: for example if the aggregate variable is X the same variable per unit of effective labor is $x \equiv \frac{X}{AL}$. Then the household budget constraint and the law of motion for capital can be re-written,

$$c_t + i_t = w_t + R_t(1 - \tau_{kt})k_t - t_t$$

$$(1+n)(1+g)k_{t+1} = i_t + (1-\delta)k_t$$

The production function can be re-written as $y_t = k_t^\alpha$. Household lifetime utility can be expressed as,

$$\sum_{t=0}^{\infty} \beta^t \frac{(C_t/L_t)^{1-\sigma}}{1-\sigma} L_t = \sum_{t=0}^{\infty} \beta^t \frac{(c_t)^{1-\sigma}}{1-\sigma} L_0 (1+n)^t A(0)^{1-\theta} (1+g)^{(1-\theta)t} =$$

$$= B \sum_{t=0}^{\infty} \widetilde{\beta}^t \frac{(c_t)^{1-\sigma}}{1-\sigma}$$

where $\widetilde{\beta} = \beta(1+g)^{1-\sigma}(1+n)$ and $B = A_0^{1-\theta} L_0$.

A competitive equilibrium is a sequence of policies $\{t_t, \tau_{kt}, G\}_{t=0}^\infty$, a sequence of prices $\{w_t, R_t\}$, and a sequence of allocations $\{c_t, k_t, i_t, y_t\}_{t=0}^\infty$ such that: (1) given sequences of policies and prices $\{t_t, \tau_{kt}, G, w_t, R_t\}_{t=0}^\infty$ the household solves,

$$max B \sum_{t=0}^{\infty} \widetilde{\beta}^t \frac{(c_t)^{1-\sigma}}{1-\sigma}$$

s.t.

$$c_t + i_t = w_t + R_t(1 - \tau_{kt})k_t - t_t$$

$$(1+n)(1+g)k_{t+1} = i_t + (1-\delta)k_t$$

and given $k_0 > 0$. (2) Firms maximize profits given sequences of prices and policies $\{t_t, \tau_{kt}, G, w_t, R_t\}_{t=0}^\infty$, i.e., factor prices are competitive,

$$w_t = (1-\alpha)k_t^\alpha$$

$$R_t = \alpha k_t^{\alpha-1}$$

(3) all markets clear - in particular the goods market, $c_t + i_t + G = y_t$. (4) The government satisfies its budget constraint in each period,

$$G = \tau_{kt} R_t k_t + t_t$$

(b) The first order conditions to the firm's problem are those in (a) under statement (2) in the definition of equilibrium. To solve the household's problem set up the Lagrangian:

$$L = B \sum_{t=0}^{\infty} \widetilde{\beta}^t \frac{(c_t)^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \lambda_t \left\{ w_t + R_t(1-\tau_{kt})k_t - t_t + (1-\delta)k_t - c_t - (1+n)(1+g)k_{t+1} \right\}$$

Take FOCs with respect to $\{c_t, k_{t+1}, \lambda_t\}$ and combine them to get the Euler equation,

$$c_t^{-\sigma}(1+n)(1+g) = \widetilde{\beta} c_{t+1}^{-\theta} \left[(1-\tau_{kt+1})R_{t+1} + (1-\delta) \right]$$

(c) A steady-state equilibrium is a competitive equilibrium in which variables in units of effective workers (c, k, y, i) are constant, the rental rate of capital R is constant, and the real wage rate W grows at a constant rate g. In steady-state the Euler implies,

$$(1+n)(1+g) = \widetilde{\beta} \left[(1-\tau_k)R + (1-\delta) \right]$$

Using the firm FOC for capital this implies that the steady-state capital stock is,

$$k^* = \left[\frac{\alpha \widetilde{\beta}(1-\tau_k)}{(1+g)(1+n) - \widetilde{\beta}(1-\delta)} \right]^{\frac{1}{1-\alpha}}$$

Steady-state investment is $i^* = \left[(1+n)(1+g) - (1-\delta) \right] k^*$. Output per effective worker in steady state is

$$y^* = \{k^*\}^{\alpha} = \left[\frac{\alpha \widetilde{\beta}(1-\tau_k)}{(1+g)(1+n) - \widetilde{\beta}(1-\delta)} \right]^{\frac{\alpha}{1-\alpha}}$$

The investment output ratio is then, $\frac{i^*}{y^*} = \left[(1+n)(1+g) - (1-\delta) \right] \{k^*\}^{1-\alpha}$

(d) The proportional tax on capital income is distortionary and reduces output. If you have two economies with different tax rates the implied income ratio will be,

$$\frac{y^i}{y^j} = \left[\frac{1-\tau_k^i}{1-\tau_k^j} \right]^{\frac{\alpha}{1-\alpha}}$$

Since $\tau_k^i = 0$ this economy will have higher income in the long-run. The reason is that a higher tax rate on capital income deters capital accumulation and thus reduces capital in the long-run.

Fig.1: $\phi < 1$

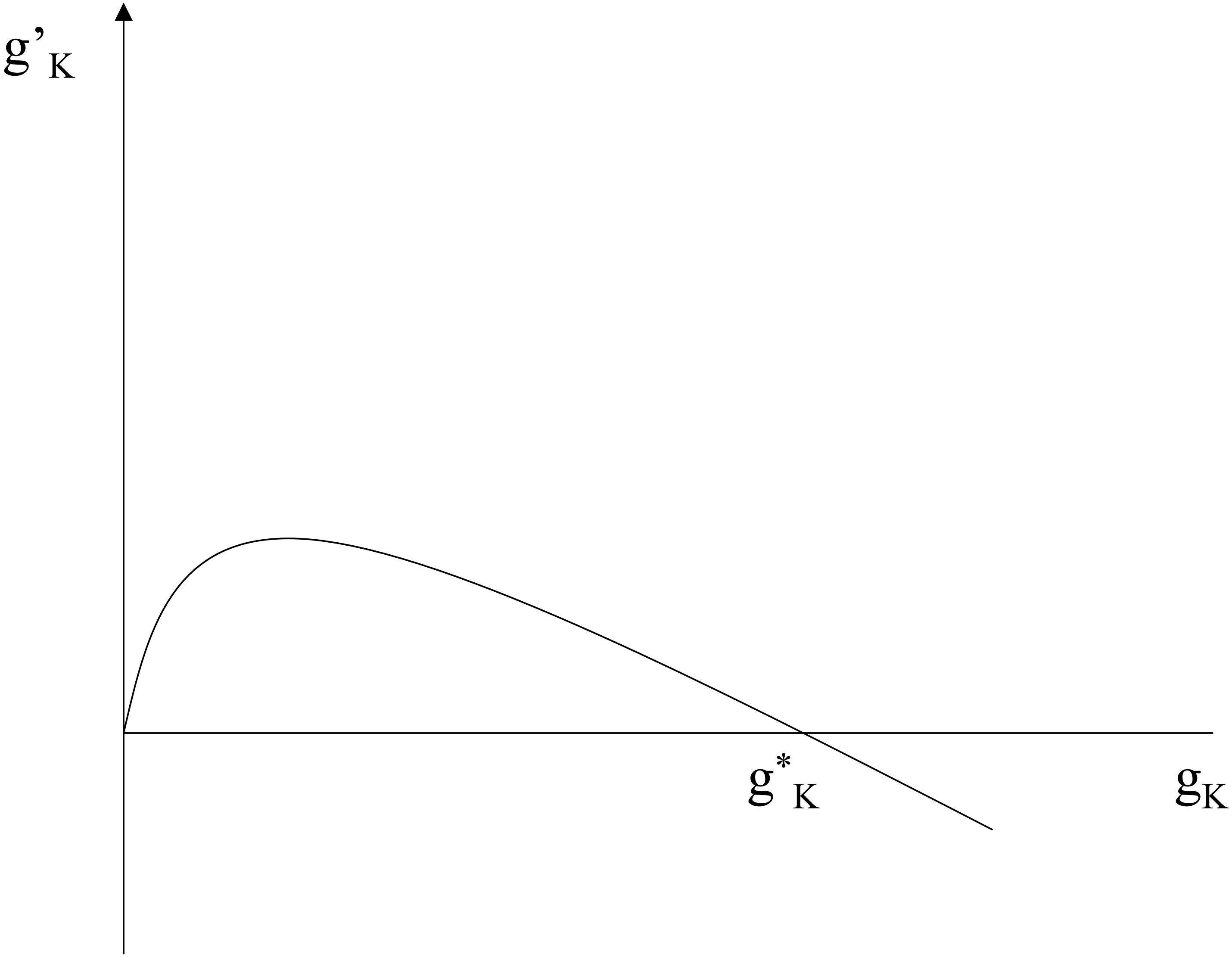

Fig.2: $\phi > 1$

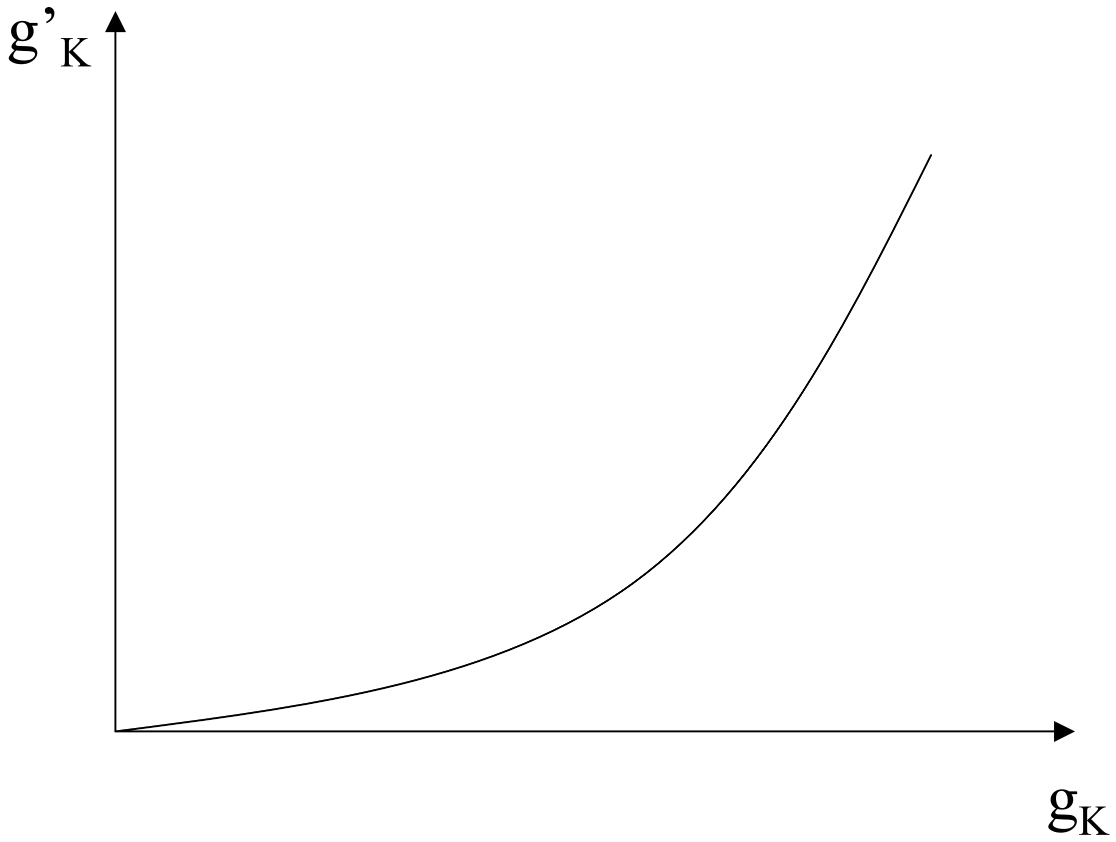

Fig.3: $\phi = 1$ and $n > 0$