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Department of Economics
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Professor Tasso Adamopoulos

Practice Exercises - Set 3

1. Consider a continuous time growth model, in which output per worker (y) is produced according to the following production function, $y = f(k) = Ak$, where k is capital per worker and A is a constant efficiency parameter. Labor is constant and normalized to 1. Capital depreciates at rate δ. The representative household has preferences described by the following intertemporal utility function,

$$U = \int_{t=0}^{\infty} e^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta} dt$$

where c is consumption per worker and ρ is the rate of time preference. Each period the government levies a proportional tax on investment income τ and uses the revenues to make lump-sum rebates to households, T_t. The Euler equation from the household's maximization problem is, $\frac{\dot{c}(t)}{c(t)} = \frac{r(t)-\rho}{\theta}$. Assume that the parameters of the model are such that the right hand side is always positive.

   (a) What is the capital accumulation equation for this economy?

   (b) What is the real interest rate in a competitive equilibrium? Calculate the growth rates of consumption per worker, capital per worker, and output per worker. What is the economy's savings rate?

   (c) Show what the effect of an increase in the tax rate is on the interest rate, the savings rate and the growth rate of output per worker. Explain.

   (d) Consider two economies i and j, which are otherwise identical, except for their tax rates. In particular: $\tau_i > \tau_j$. Will the two economies converge in the long-run if economy j starts with a lower level of output per worker?

2. Consider a simple endogenous growth model of *learning by doing*. The idea is that, as individuals produce goods, they think of ways of improving the production process.

In particular, we will assume that knowledge accumulation is a side effect of capital accumulation, and thus that the stock of knowledge is a function of the stock of capital,

$$A(t) = BK(t)^{\phi}$$

where $B > 0$ and $\phi > 0$ are constants. Output is produced according to a standard Cobb-Douglas production function, $Y(t) = K(t)^{\alpha} [A(t)L(t)]^{1-\alpha}$. Capital accumulation is as in the Solow model but without depreciation, $\dot{K} = sY(t)$, where s is a constant, exogenous savings rate. Labor grows at a constant exogenous growth rate n.

(a) Derive the growth rate of capital ($g_K \equiv \frac{\dot{K}}{K}$) in terms of K and L and parameters of the model. Derive the dynamics of the growth rate of capital ($\dot{g_K}$), in terms of g_K and parameters of the model.

(b) Suppose that $\phi < 1$. Derive the economy's long run growth rate. Draw of phase diagram of $\dot{g_K}$ against g_K. Does a one time increase in the population affect the economy's growth rate? Explain.

(c) What does the phase diagram look like if $\phi > 1$? What if $\phi = 1$ and $n > 0$?

(d) Suppose now that $\phi = 1$ and $n = 0$. What is the growth rate of output per worker in this case? Does a one time increase in the population affect the economy's growth rate? Explain.

3. Consider an economy populated by a representative household with preferences over consumption streams described by,

$$\int_{t=0}^{\infty} e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} dt$$

where $C(t)$ is the households consumption and $\rho > 0$ and $\theta > 0$). There is no population growth and the household's total labor L is inelastically supplied to the market. The representative household is also endowed with the economy's initial capital stock $K(0)$. Assume that the No-Ponzi-Game condition holds for the household's problem. There are two production sectors in the economy. The consumption sector produces consumption goods according to a constant returns to scale technology,

$$Y_c(t) = BK_c(t)^{\alpha} L_c(t)^{1-\alpha}$$

where $B > 0$ is a constant efficiency parameter in the consumption sector, and $\alpha \in (0, 1)$. K_c and L_c are the amounts of capital and labor allocated to consumption production. The investment sector produces investment goods according to a linear production technology in capital K_i,

$$Y_i(t) = AK_i(t)$$

where $A > 0$ is the constant efficiency in producing investment goods. The law of motion for the economy's capital stock is,

$$\dot{K}(t) = I(t)$$

where $I(t)$ is actual investment in this economy. Let $P_c(t)$ be the price of consumption and $P_i(t)$ the price of investment. With consumption being the numeraire let $P = \frac{P_i}{P_c}$ be the relative price of investment. Firms in both sectors face the same factor prices. There is no population growth.

(a) Define a competitive equilibrium for this economy. Be sure to properly specify all of the market clearing conditions. Let $\phi(t)$ denote the share of capital devoted to the investment sector.

(b) Specify the household's flow budget constraint, and characterize its problem using a Hamiltonian function.

(c) Solve the firm problems in each production sector, and solve for the relative price of investment.

(d) Define a balanced growth path as an equilibrium in which $\phi(t)$ is constant at some $\phi \in [0, 1]$. Derive the growth rates of capital, consumption, wages, relative price of investment on the equilibrium balanced growth path.

4. Consider a discrete time version of the neoclassical growth model with the following features. Output in the economy is produced according to a constant returns to scale technology:

$$Y_t = F(K_t, L_t) = K_t^{\alpha} (A_t L_t)^{1-\alpha}$$

where A_t is a technology parameter that grows at the exogenous rate g. There is an infinitely lived representative household, with preferences:

$$\sum_{t=0}^{\infty} \beta^t \frac{(C_t/L_t)^{1-\sigma}}{1-\sigma} L_t$$

where L is the size of the working population, growing exogenously according to $L_{t+1} = (1+n)L_t$. The representative household owns the capital in the economy, rents labor and capital services to firms, and spends income on consumption and investment. There is a government in the economy that purchases part of the economy's output and taxes households. Let G be government purchases per unit of effective worker. There is a (constant) tax τ_k on the return to capital as well as a lump sum tax T_t. The law of motion for capital is:

$$K_{t+1} = (1-\delta)K_t + I_t$$

(a) Define a competitive equilibrium for this economy. Make sure to specify the problems of households and firms clearly, as well as the government budget constraint. (Hint: before defining equilibrium transform all variables in the problem in terms of effective units of labor, denoting them by lower case letters. For example $y_t = \frac{Y_t}{A_t L_t}$, and similarly define c, k, i and lump-sum taxes t. Show that the adjusted discounted factor of the transformed problem is, $\widetilde{\beta} = \beta(1+g)^{1-\sigma}(1+n))$.

(b) Derive the Euler equation for the household. Write down the first order conditions to the firm's problem.

(c) Define a steady state equilibrium. Derive the investment rate (investment-output ratio) and output per working age person in the steady state of this economy (Hint: do this in terms of the transformed variables).

(d) Consider two economies i and j, that are identical in all other parameters (including the level of government purchases) except for the tax rate τ_k. In particular assume that i funds g exclusively with lump-sum taxes, while j uses a mix of lump-sum and proportional capital income taxes. Which economy will have a higher output per worker in the long-run? Explain.