University of Toronto
Department of Economics
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Economic Theory - Macroeconomics (MA)
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Answers to Practice Set 1

1. (a) The assumptions about the initial conditions imply that the two economies start off from the same level of capital per unit of effective labor,

$$k_i(0) = \frac{K_i(0)}{A_i(0)L_i(0)} = \frac{2K_j(0)}{A_j(0)2L_j(0)} = \frac{K_j(0)}{A_j(0)L_j(0)} = k_j(0)$$

However, these two economies will converge to different steady states because they do not share the same characteristics. In particular,

$$k_i^* = \left(\frac{s_i}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} > \left(\frac{s_j}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} = k_j^*$$

$$y_i^* = \left(\frac{s_i}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} > \left(\frac{s_j}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} = y_j^*$$

In the long run once they reach their steady states the economy with the higher savings rate will have a higher k. By the property of diminishing returns this implies that the rate of return to capital will be higher in the economy with the low savings rate (economy j). Thus one would expect that if these two economies were to open up their capital markets and permit capital flows, capital would flow from i to j where the return is higher. This process would continue until the net returns are equalized and there are no more arbitrage opportunities to be exploited.

(b) If $s_i = s_j$ then the two economies not only have the same starting point but they will also have the same long run equilibrium. This means that they will converge to the steady state at exactly the same rate, and thus they will have the same k at every instant. In fact, these two economies would be observationally equivalent in terms of k, y both on and off the steady state. Thus the return to capital will be the same at every instant. Consequently there will be no incentive for capital to flow in either direction, in the short run or the long run.

2. The production function in intensive form is, $y(t) = k(t)^{\alpha}$. The inverse function of this (k in terms of y) is $k(t) = y(t)^{1/\alpha}$. As $\dot{y} = f'(k) \cdot \dot{k}$ and we can write k as a function of y, we can re-write $\dot{y}$ as a function of the level of y: $\dot{y} \equiv \dot{y}(y)$. Take a first order Taylor series approximation of $\dot{y}(y)$ around the steady-state value $y = y^*$,

$$\dot{y}(y) \simeq \dot{y}(y^*) + \frac{\partial \dot{y}}{\partial y}|_{y=y^*} \cdot (y - y^*)$$

Since $\dot{y} = 0$ at $y = y^*$ and denoting $\lambda = -\frac{\partial \dot{y}}{\partial y}|_{y=y^*}$, we have,

$$\dot{y}(t) \simeq -\lambda \cdot (y(t) - y^*)$$

This implies in turn that,

$$(y(t) - y^*) \simeq e^{-\lambda t} \cdot (y(0) - y^*)$$

where $y(0)$ is the initial value of y.

We still have to determine λ. Take the time derivative on both sides of the production function, $\dot{y} = \alpha k^{\alpha-1}\dot{k}$. Replace $\dot{k}$ with its equal from the law of motion for k ($\dot{k} = sk^{\alpha} - (n + g + \delta)k$),

$$\dot{y} = \alpha k^{\alpha-1} \cdot (sk^{\alpha} - (n + g + \delta)k) = \alpha \cdot s \cdot k^{2\alpha-1} - \alpha(n + g + \delta)k^{\alpha}$$

To get $\dot{y}$ as a function of y, use the inverse function $k = y^{1/\alpha}$,

$$\dot{y} = \alpha \cdot s \cdot y^{\frac{2\alpha-1}{\alpha}} - \alpha(n + g + \delta)y$$

Take the derivative of $\dot{y}$ with respect to y,

$$\frac{\partial \dot{y}}{\partial y} = (2\alpha - 1)sy^{\frac{\alpha-1}{\alpha}} - \alpha(n + g + \delta)$$

Evaluate at the steady state $y = y^*$,

$$\frac{\partial \dot{y}}{\partial y}|_{y=y^*} = (2\alpha - 1)s\,(y^*)^{\frac{\alpha-1}{\alpha}} - \alpha(n + g + \delta)$$

We know that the steady state value of y is $y^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$. Plug this in λ to get,

$$\lambda = -\frac{\partial \dot{y}}{\partial y}|_{y=y^*} = (1 - \alpha)(n + g + \delta)$$

According to the numbers in the question $n + g + \delta = 0.05$ and $\alpha = 1 - \frac{2}{3} = \frac{1}{3}$. Plugging these into λ we find that $\lambda = 0.033$. Plug these numbers in,

$$\frac{(y(t) - y^*)}{(y(0) - y^*)} \simeq e^{-\lambda t}$$

along with $1/2$ on the LHS to solve for $t^* \simeq 21$ years.

Romer 1.3

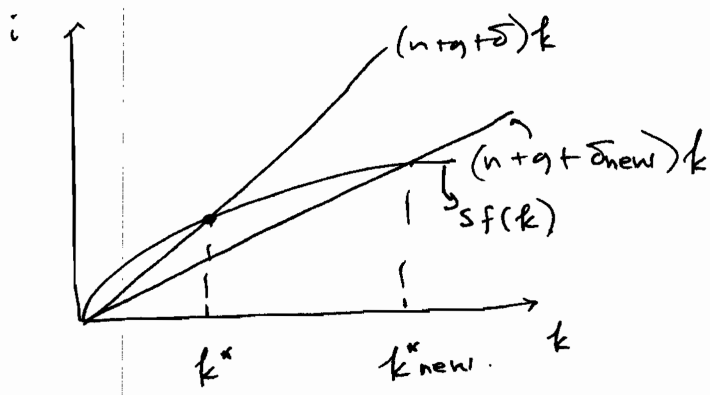
a) · slope of break even inv. line = $n+g+\delta$

$\downarrow \delta$ $\Rightarrow$ decreases in slope of break-even inv. line

$\Rightarrow$ shifts (rotates) downwards.

· actual inv. schedule is unaffected

· the steady-state (BGP) of k rises from k^* to k^*_{new}

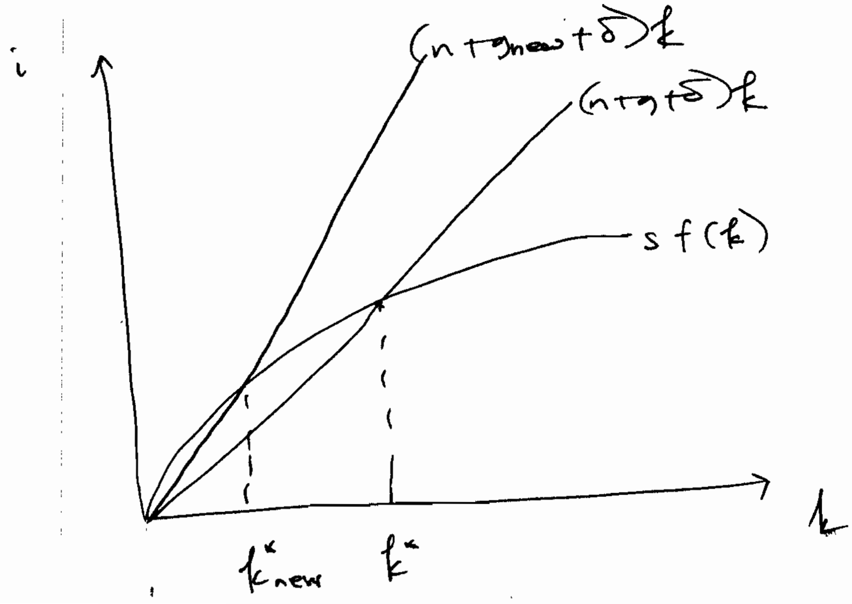


b) · $\uparrow g$ $\Rightarrow$ break-even inv. schedule becomes steeper

$\Rightarrow$ rotates upwards

· actual inv. schedule is unaffected.

· steady-state (BGP) level of k falls from k^* to k^*_{new}



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- break-even inv. line is unaffected by ↑ in α.
- ↑ α will affect the actual inv-schedule. sk^{α}.

to see the effect calculate the derivative:

$$\frac{\partial (sk^{\alpha})}{\partial \alpha} = s \cdot k^{\alpha} \cdot \ln k$$

For $\alpha \in (0,1)$ and for $k > 0$ the sign of this derivative is determined by the sign of $\ln k$.

For $\ln k > 0$ (or $k > 1$) we have that $\frac{\partial s k^{\alpha}}{\partial \alpha} > 0$

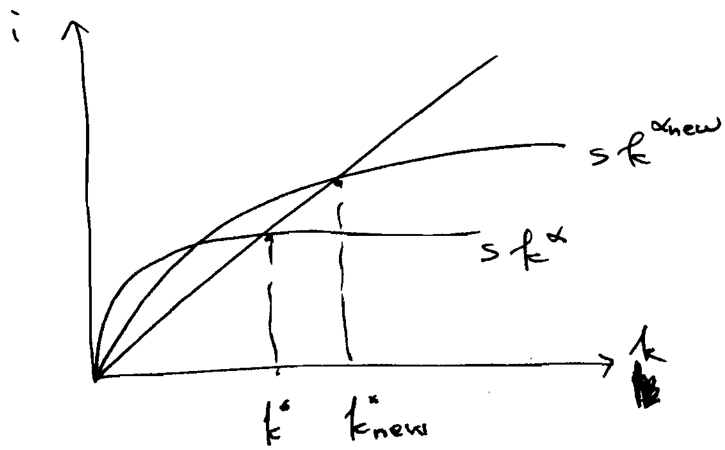
$\Rightarrow$ actual inv. schedule lies above old one

For $\ln k < 0$ (or $k < 1$) we have that $\frac{\partial s k^{\alpha}}{\partial \alpha} < 0$

$\Rightarrow$ actual inv-schedule lies below old one.

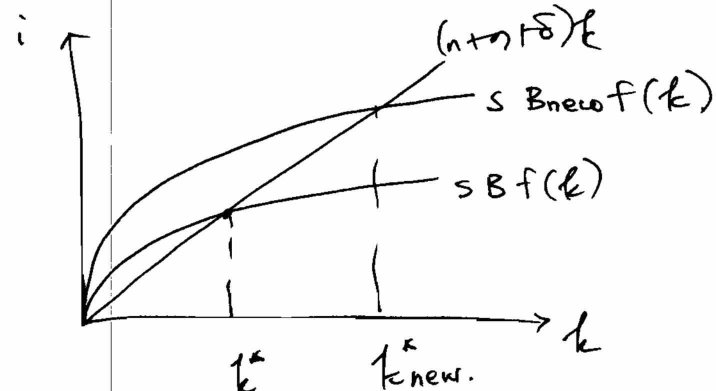
At $\ln k = 0$ (or $k = 1$) the new actual inv. schedule intersects the old one.

In addition the effect of a rise in α on k^* is ambiguous and depends on the rel. magnitudes of s and $(n+g+\delta)$. It is possible to show that a rise in α will cause k^* to rise if $s > (n+g+\delta)$. See Fig. below for such a case.



d)

- the production function in intensive form is now $Bf(k)$., $B>0$
- the actual inv. schedule is then $sBf(k)$
- workers exerting more effort (so that output per unit of effective labor is higher than before can be modeled as an $\uparrow B$
- $\uparrow B \Rightarrow$ shifts actual inv. schedule up.
- break-even inv. schedule is unaffected.
- steady-state (BGP) level of k increases from k^* to k^*_{new}



Romer. 1.4

a) • at some time t_0 there is a discrete upward jump in the number of workers

- by definition $k \equiv \frac{K}{AL}$.
- so an $\uparrow L$ reduces the amount of capital per unit of effective labor from k^* to $\frac{K}{AL_{new}} = k_{new}$ (this is because K, A do not exhibit a similar jump).

- since $f'(k) > 0$ the fall in capital per unit of effective labor from k^* to k_{new} will also reduce output per unit of effective labor from y^* to y_{new}.

b) Now at $k_{new} < k^*$ we have that

actual inv. per unit of effective labor > break-even invest. per unit of effective labor.

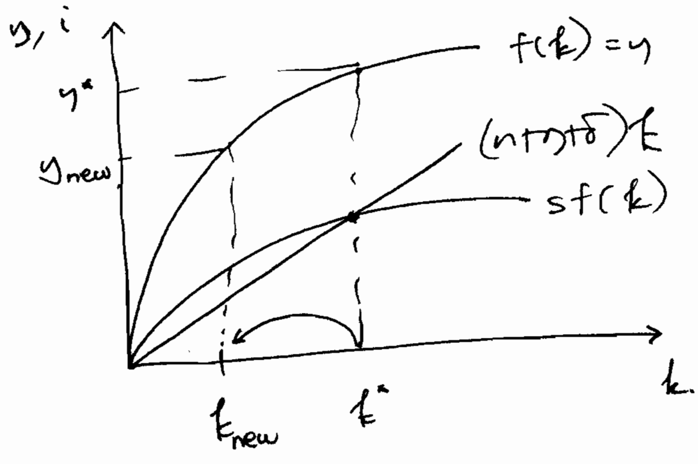
or $sf(k_{new}) > (g+\delta)k_{new}$

$\Rightarrow$ From the law of motion for capital this implies $\dot{k} > 0$

$\Rightarrow$ k begins to rise back to the original k^*

(reason: no determinant of the steady-state has changed)

- since $\dot{y} = f'(k)\cdot\dot{k}$, y will also increase back to y^*_{new}.



Romer 1.5

a)

- law of motion for k:

$$\dot{k} = sf(k) - (n+g+\delta)k. \quad (1)$$

- Cobb-Douglas production function in intensive form

$$y = k^{\alpha} \quad (2)$$

- Substitute (2) into (1)

$$\dot{k} = sk^{\alpha} - (n+g+\delta)k.$$

- on BGP: $\dot{k} = 0 \Rightarrow sk^{*\alpha} = (n+g+\delta)k^{*}$.

$$\Rightarrow k^{*} = \left[\frac{s}{n+g+\delta}\right]^{\frac{1}{1-\alpha}}. \quad (3)$$

- since $y = k^{\alpha}$ we have that

$$y^{*} = \left[\frac{s}{n+g+\delta}\right]^{\frac{\alpha}{1-\alpha}} \quad (4)$$

- since $c = (1-s)y$ we have that

$$c^{*} = (1-s)\left[\frac{s}{n+g+\delta}\right]^{\frac{\alpha}{1-\alpha}} \quad (5)$$

b) By definition the golden rule level of capital per unit of effective labor is that which maximizes steady-state consumption per unit of effective labor.

To derive this level of k, first re-arrange (3) to solve for s:

$$s = (n+g+\delta)\, k^{*\,1-\alpha} \quad (6)$$

Romer 1.6 :

a) • since $g=0$ we carry out the analysis in terms of capital and output per worker.

Define now: $y \equiv \frac{Y}{L}$, $k \equiv \frac{K}{L}$

(Note: with constant A capital and output per worker behave the same as capital and output per unit of effective labor).

• $\downarrow n \Rightarrow$ break-even inv. schedule becomes flatter (rotates down).

• with $g=0$ the law of motion for capital per worker is:

$$\dot{k} = sf(k) - (n+\delta)k.$$

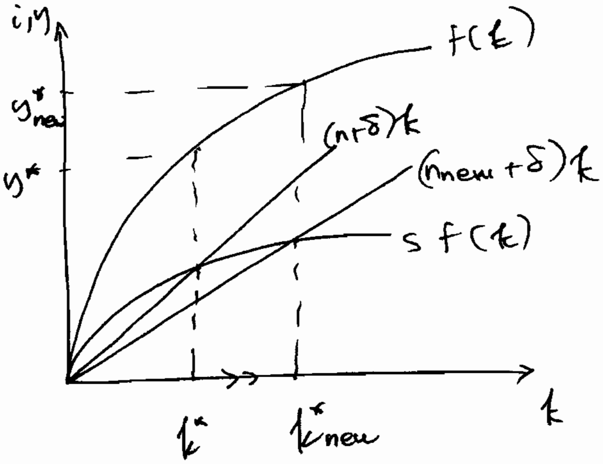
• at time of $\downarrow n$. we have that

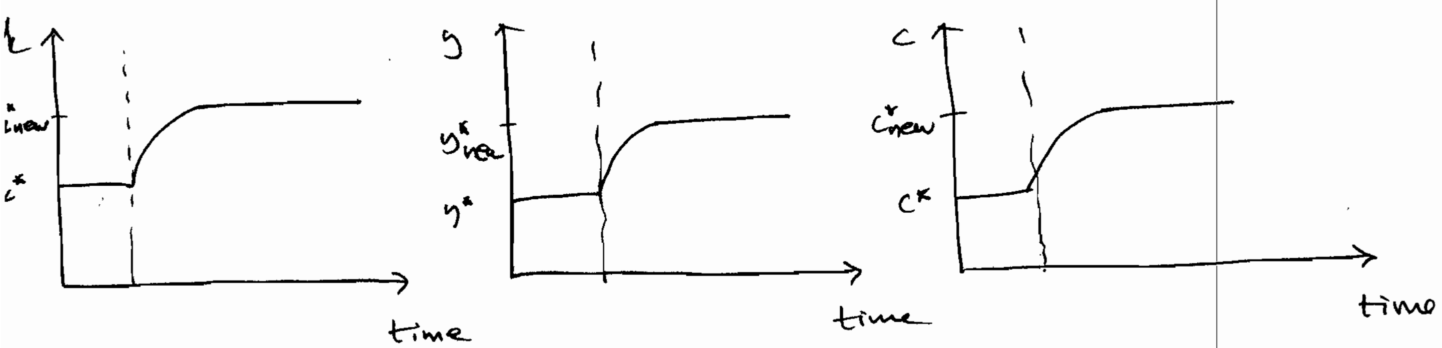
$$sf(k^*) > (n_{new} + \delta) k^*$$

where k^* is the original steady state.

• Thus we have that $\dot{k} > 0$. $\Rightarrow k$ will gradually move to the new higher steady-state k^*_{new}.

• As k rises y (output per worker) will also rise since $\dot{y} = f'(k)\dot{k}$. Since $\dot{c} = (1-s)\dot{y}$, consumption per worker also rises.





b). By definition total output is : $Y = L \cdot y$

The growth rate of total output is thus

$$\frac{\dot{Y}}{Y} = \frac{\dot{L}}{L} + \frac{\dot{y}}{y}.$$

on the original BGP: $\frac{\dot{y}}{y} = 0$

and thus $\frac{\dot{Y}}{Y} = \frac{\dot{L}}{L} = n.$

on the new BGP : $\frac{\dot{y}}{y} = 0$

and thus $\frac{\dot{Y}}{Y} = \frac{\dot{L}}{L} = n_{new} < n$

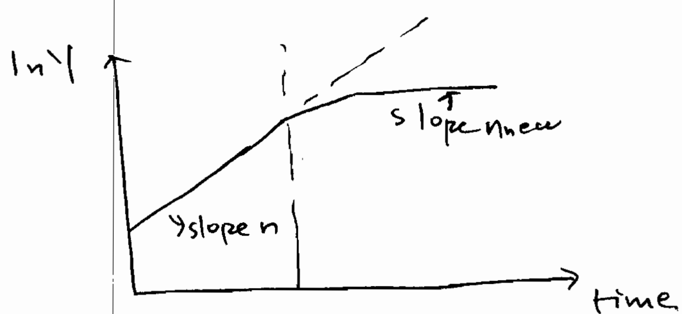
i.e. output grows at a permanently lower rate.

In transition?

$\frac{\dot{y}}{y} > 0$ since $\frac{\dot{k}}{k} > 0$

thus $\frac{\dot{Y}}{Y} = \frac{\dot{L}}{L} + \frac{\dot{y}}{y} = n_{new} + \frac{\dot{y}}{y} > n_{new}$

so in transition, output grows faster than in the new BGP.



Romer 1.9:

a) Define the MPL to be $w \equiv \frac{\partial F(K, AL)}{\partial L}$

Then write the production function as:

$Y = ALf(k) = ALf\left(\frac{K}{AL}\right)$.

Taking the partial derivative with respect to L:

$w = \frac{\partial Y}{\partial L} = ALf'(k)\left(-\frac{K}{AL^2}\right) + Af(k) = A\left(f(k) - kf'(k)\right)$

b) Define the MPK as $r \equiv \frac{\partial F(K, AL)}{\partial K} - \delta$.

Write the production function as $Y = ALf(k) = ALf\left(\frac{K}{AL}\right)$

Then we have:

$r = \frac{\partial Y}{\partial K} - \delta = ALf'(k)\left(\frac{1}{AL}\right) - \delta = f'(k) - \delta$.

then

$$wL + rK = A\left[f(k) - k f'(k)\right]L + \left[f'(k) - \delta\right]K$$

$$= ALf(k) - f'(k)\left(\frac{K}{AL}\right)AL + f'(k)K - \delta K.$$

$$= ALf(k) - f'(k)K + f'(k)K - \delta K$$

$$= AL\,F\left(\frac{K}{AL}, 1\right) - \delta K$$

$$= F(K, AL) - \delta K.$$

c) $r = f'(k) - \delta$ as shown.

Since δ constant and k constant on BGP. so is $f'(k)$ and thus r. $\Rightarrow \frac{\dot r}{r} = 0$ on BGP.

$\Rightarrow$ the Solow Model exhibits the property that the return to capital is constant over time.

Since capital is paid its MP the share of output going to capital is $\frac{rK}{Y}$. On the BGP:

$$\frac{\dot{(rK/Y)}}{rK/Y} = \frac{\dot r}{r} + \frac{\dot K}{K} - \frac{\dot Y}{Y} = 0 + (n+g) - (n+g) = 0$$

this implies that the share of output going to labor is also 0. on the BGP.

we know that $w = A\left[f(k) - k f'(k)\right]$.

take logs and time derivatives:

$$\frac{\dot w}{w} = \frac{\dot A}{A} + \frac{\dot{\left[f(k) - k f'(k)\right]}}{\left[f(k) - k f'(k)\right]} = g + \frac{\left[f'(k)\dot k - \dot k f'(k) - k f''(k)\dot k\right]}{f(k) - k f'(k)}$$

$$\Rightarrow \quad \frac{\dot{w}}{w} = g + \frac{-k f''(k)\dot{k}}{f(k) - k f'(k)}$$

On the BGP. $\dot{k} = 0 \Rightarrow \frac{\dot{w}}{w} = g$.

d). Since $\frac{\dot{w}}{w} = g + \frac{-k f''(k)\cdot \dot{k}}{f(k) - k f'(k)}$

if $k < k^*$ thereas $k \to k^*$ we have that

$\frac{\dot{w}}{w} > g$ (because $\dot{k} > 0$).

Since $\frac{\dot{r}}{r} = \frac{\dot{[f'(k)]}}{f'(k)} = \frac{f''(k)\cdot \dot{k}}{f'(k)}$

As $k \to k^*$, $\frac{\dot{r}}{r} < 0$ since $\dot{k} > 0$.