**University of Toronto**
**Department of Economics**
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**Answers to Assignment 2**

1. The household's problem is,

$$\max_{c_t,\ell_t,I_t} E_0 \sum_{t=0}^{\infty} e^{-\rho t}\left[\ln(c_t) + \frac{(1-\ell_t)^{\theta}}{\theta}\right]$$

s.t.

$$c_t + I_t = w_t\ell_t(1-\tau_t) + R_tK_t + T_t$$

$$K_{t+1} = (1-\delta)K_t + I_t$$

where $T_t$ are lump-sum taxes, and $R_t = r_t + \delta$. Set-up the Lagrangian,

$$\pounds = E_0\left\{\sum_{t=0}^{\infty} e^{-\rho t}\left[\ln(c_t) + \frac{(1-\ell_t)^{\theta}}{\theta}\right] + \sum_{t=0}^{\infty}\lambda_t\left[w_t\ell_t(1-\tau_t) + R_tK_t + T_t + (1-\delta)K_t - c_t - K_{t+1}\right]\right\}$$

(a) The first order conditions are,

$$E_t\left\{e^{-\rho t}\frac{1}{c_t} - \lambda_t\right\} = 0$$

$$E_t\left\{-e^{-\rho t}(1-\ell_t)^{\theta-1} + \lambda_t w_t(1-\tau_t)\right\} = 0$$

$$E_t\left\{-\lambda_t + \lambda_{t+1}[R_{t+1} + 1 - \delta]\right\} = 0$$

(b) Combining the first and the third, the second and the third and the first and the second you get respectively,

$$\frac{1}{c_t} = e^{-\rho}E_t\left\{\frac{1}{c_{t+1}}(1+r_{t+1})\right\}$$

$$e^{-\rho}E_t\left\{(1+r_{t+1})\left(\frac{1-\ell_{t+1}}{1-\ell_t}\right)^{\theta-1}\frac{w_t}{w_{t+1}}\frac{(1-\tau_t)}{(1-\tau_{t+1})}\right\} = 1$$

$$(1-\ell_t)^{\theta-1} = \frac{w_t(1-\tau_t)}{c_t}$$

where I have used that $R_t = r_t + \delta$. The first of these equations reflects the trade-off between consumption today and consumption tomorrow. The second reflects the trade-off between leisure (labor supply) today and leisure (labor supply) tomorrow. The third reflects the trade-off between leisure and consumption today. See class notes for the intuition.

(c) We can rewrite the Euler equation as,

$$\frac{1}{c_t} = e^{-\rho}\left\{E_t\left(\frac{1}{c_{t+1}}\right)E_t(1+r_{t+1}) + Cov\left(\frac{1}{c_{t+1}}, 1+r_{t+1}\right)\right\}$$

Denote by $c_{t+1}^{+}$ the level of consumption for which $Cov\left(c_{t+1}^{+}, 1+r_{t+1}\right) > 0$ and thus $Cov\left(\frac{1}{c_{t+1}^{+}}, 1+r_{t+1}\right) < 0$. Denote by $c_{t+1}^{-}$ the level of consumption for which $Cov\left(c_{t+1}^{-}, 1+r_{t+1}\right) < 0$ and thus $Cov\left(\frac{1}{c_{t+1}^{-}}, 1+r_{t+1}\right) > 0$. From the right hand side of the above equation this implies that $\frac{1}{c_t^{+}} < \frac{1}{c_t^{-}}$, and consequently $c_t^{-} < c_t^{+}$. Intuitively: when the covariance between consumption and the interest rate is positive (you get a good return in good times) then the marginal utility of consumption and the interest rate is negative. Consequently the expected marginal benefit from giving up more consumption today (and thus saving) is lower; consequently you will save less and consume more today.

(a) HH budget constraint $C_t + (1+\tau_x) X_t = w_t L_t + r_t \cdot K_t + T_t.$

Resource Constraint $C_t + X_t = Y_t.$

(b) Transform problem in terms of units of effective workers.

Production Function

$$Y_t = K_t^{\alpha} (A_t L_t)^{1-\alpha}$$

$$\Rightarrow \frac{Y_t}{A_t L_t} = \frac{K_t^{\alpha} (A_t L_t)^{1-\alpha}}{A_t L_t} \Rightarrow \frac{Y_t}{A_t L_t} = \left(\frac{K_t}{A_t L_t}\right)^{\alpha}.$$

$$\Rightarrow \boxed{y_t = k_t^{\alpha}}$$ where by definition

$$y_t \equiv \frac{Y_t}{A_t L_t}$$

$$k_t = \frac{K_t}{A_t L_t}.$$

Law of Motion

$$K_{t+1} = (1-\delta) K_t + X_t.$$

$$\Rightarrow \frac{K_{t+1}}{A_t L_t} = (1-\delta) \frac{K_t}{A_t L_t} + \frac{X_t}{A_t L_t}$$

$$\Rightarrow \frac{K_{t+1}}{A_{t+1} L_{t+1}} \cdot \frac{A_{t+1} \cdot L_{t+1}}{A_t \cdot L_t} = (1-\delta) \frac{K_t}{A_t L_t} + \frac{X_t}{A_t L_t}.$$

$$\Rightarrow \boxed{(1+g)(1+n) k_{t+1} = (1-\delta) k_t + x_t}$$

Budget Constraint:

$$C_t + (1+\theta) X_t = w_t L_t + r_t K_t + T_t$$

$$\Rightarrow \frac{C_t}{A_t \cdot L_t} + (1+\theta)\frac{X_t}{A_t L_t} = \frac{w_t \cdot L_t}{A_t L_t} + \frac{r_t K_t}{A_t \cdot L_t} + \frac{T_t}{A_t L_t}.$$

$$\Rightarrow \boxed{c_t + (1+\theta) x_t = \hat{\omega}_t + r_t k_t + \tau_t}$$

where $\hat{\omega}_t \equiv \dfrac{\omega_t}{A_t}$

$\tau_t = \dfrac{T_t}{A_t \cdot L_t}$.

Preferences:

$$\sum_{t=0}^{\infty} \beta^t \frac{(C_t/L_t)^{1-\sigma}}{1-\sigma} \cdot L_t = \sum_{t=0}^{\infty} \beta^t \frac{(C_t/A_t L_t)^{1-\sigma}}{1-\sigma} A_t^{1-\sigma} L_t.$$

$$= \sum_{t=0}^{\infty} \beta^t A_0^{1-\sigma} (1+g)^{(1-\sigma)t} L_0 (1+n)^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

$$= A_0^{1-\sigma} \cdot L_0 \sum_{t=0}^{\infty} \left[\beta (1+g)^{1-\sigma} (1+n)\right]^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

$$= B \cdot \sum_{t=0}^{\infty} \hat{\beta}^t \frac{c_t^{1-\sigma}}{1-\sigma}.$$

where $B = A_0^{1-\sigma} \cdot L_0$

$\hat{\beta} = \beta (1+g)^{1-\sigma} (1+n)$.

Firm Problem (static each period).

$$\text{MAX} \left\{ K_t^{\alpha}(A_tL_t)^{1-\alpha} - w_t \cdot L_t - r_t K_t \right\}.$$

FONC:

$K_t$: $\quad \alpha K_t^{\alpha-1}(A_tL_t)^{1-\alpha} = r_t.$

$L_t$: $\quad (1-\alpha) K_t^{\alpha} A_t^{1-\alpha} L_t^{-\alpha} = w_t.$

Re-write these conditions in terms of transformed variables:

$$r_t = \alpha \left(\frac{K_t}{A_tL_t}\right)^{\alpha-1} \Rightarrow r_t = \alpha k_t^{\alpha-1}$$

$$w_t = (1-\alpha) A_t \left(\frac{K_t}{A_tL_t}\right)^{\alpha} \Rightarrow w_t = (1-\alpha) A_t k_t^{\alpha}. \Rightarrow \hat{w}_t = (1-\alpha) k_t^{\alpha}$$

where as defined above:

$$\hat{w}_t \equiv \frac{w_t}{A_t}.$$

We can thus write factor prices as functions of the economy's aggregate capital:

$$r(k) = \alpha k^{\alpha-1}$$

$$\hat{\omega}(k) = (1-\alpha) k^{\alpha}.$$

## HH Problem

<u>in sequence Form</u>: MAX $\sum_{t=0}^{\infty} \beta^t \frac{(c_t/L_t)^{1-\sigma}}{1-\sigma} L_t$.

s.t.

$$C_t + (1+\theta) X_t = w_t L_t + r_t K_t + T_t.$$

$$K_{t+1} = (1-\delta) K_t + X_t.$$

- we can re-write this problem in <u>dynamic programming</u> form. in terms of the <u>transformed</u> variables:

$$V(k_H, k) = \max \left\{ \frac{c^{1-\sigma}}{1-\sigma} + \beta V(k_H', k') \right\}.$$

(FE)

$$c + (1+\theta) x = \hat{w}(k) + r(k) k_H + \tau(k)$$

$$(1+g)(1+n) k_H' = (1-\delta) k_H + x$$

$$k' = \Gamma(k)$$

$$c, x \geq 0$$

- $\Gamma(k)$ is a function describing how HHs expect the aggregate capital stock $k$ to evolve over time. (aggregate law of motion)

- $k_H$ = individual HH capital stock per effective unit of labor
  $k$ = aggregate economy-wide capital stock. per effective unit of labor

- in equilibrium: $k_H = k$.

## Def. of RCE:

A RCE is a list of functions $v(k_H, K)$, $g^c(k_H, K)$, $g^x(k_H, K)$, $g^{k'}(k_H, K)$, $f^k(K)$, $\hat{\omega}(K)$, $r(K)$, $\tau(K)$, $\Gamma(K)$ such that.

- Given $\hat{\omega}(K)$, $r(K)$, $\tau(K)$ and the aggregate law of motion $\Gamma(K)$ the value function $v(k_H, K)$ solves the HH's (FE): where $g^c(k_H, K)$, $g^x(k_H, K)$, $g^{k'}(k_H, K)$ are the optimal decision rules.

- given $\hat{\omega}(K)$, $r(K)$ the decision function $f^k(K)$ solves the firm's problem.

- markets clear

$$g^c(K, K) + g^x(K, K) = K^\alpha.$$

$$K = f^k(K)$$

- the government's budget constraint is satisfied:

$$\theta \cdot g^x(K, K) = \tau(K)$$

- indiv. and agg. laws of motion are consistent:

$$g^{k'}(K, K) = \Gamma(K)$$

(c). We can re-write the (FE) as:

$$V(k_H, k) = \max \left\{ \frac{1}{1-\sigma} \Big[ \hat{w}(k) + [r(k) + (1+\theta)(1-\delta)] k_H + \tau(k) - (1+\theta)(1+g)(1+n) k_H' \Big]^{1-\sigma} + \beta V(k_H', k') \right\}$$

FOC: $$c^{-\sigma} \{ -(1+g)(1+n)(1+\theta) \} + \beta V_1 (k_H', k') = 0$$

EC: $$V_1 (k_H, k) = c^{-\sigma} [ r(k) + (1+\theta)(1-\delta) ]$$

$$\Rightarrow \quad c^{-\sigma} (1+g)(1+n)(1+\theta) = \beta c'^{-\sigma} [ r(k') + (1+\theta)(1-\delta) ].$$

. for firm FOC see part (b).

(d) A steady state equilibrium is a RCE with the property that $k' = k \; (= k_H' = k_H)$.
$c' = c$

From the Euler equation and the Firm FOC:

$$(1+g)(1+n)(1+\theta) = \beta [ \alpha k^{\alpha - 1} + (1+\theta)(1-\delta) ].$$

$$\Rightarrow \quad \frac{(1+g)(1+n)(1+\theta)}{\beta} - (1-\delta)(1+\theta) = \alpha k^{\alpha-1}$$

$$\Rightarrow \quad \frac{(1+g)(1+n)(1+\theta) - \beta(1-\delta)(1+\theta)}{\alpha\beta} = k^{\alpha-1}$$

$$\Rightarrow \quad k^* = \left( \frac{\alpha\beta / (1+\theta)}{(1+g)(1+n) - \beta(1-\delta)} \right)^{\frac{1}{1-\alpha}}.$$

From the law of motion for capital in steady state:

$$k(1+n)(1+g) = (1-\delta)k + x$$

$$\Rightarrow \quad x^* = \left[(1+n)(1+g) - (1-\delta)\right] k^*$$

From the production function:

$$y^* = k^{*\alpha}$$

then the investment rate is:

$$\frac{x^*}{y^*} = \left[(1+n)(1+g) - (1-\delta)\right] k^{*1-\alpha}$$

$$\Rightarrow \quad \frac{x^*}{y^*} = \left[(1+n)(1+g) - \beta(1-\delta)\right] \left[\frac{\alpha\beta/(1+\theta)}{(1+g)(1+n) - \beta(1-\delta)}\right]^{\cancel{\frac{1-\alpha}{1-\alpha}}}$$

and output is:

$$y^* = \left[\frac{\alpha\beta/(1+\theta)}{(1+g)(1+n) - \beta(1-\delta)}\right]^{\frac{\alpha}{1-\alpha}}$$