University of Toronto
Department of Economics
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Economic Theory - Macroeconomics (MA)
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Answers to Assignment 1

1. In principle, this is not necessarily true. The Solow model might still be correct but incomes may differ for two reasons: (a) countries today are not all in their steady states. So even if in the long-run they were to converge to the same steady state it could be the case that today they are at different distances from that steady state. (b) they may not have the same steady states, i.e., their characteristics may differ. Thus even if countries were in their steady states today those may be different from one another. Certainly a combination of these two alternatives could also be true. However the overwhelming quantitative and empirical evidence taking these issues into account, suggests that the basic Solow model is not the end of the story in accounting for international income differences.

2. In the Ramsey-Cass-Koopmans model the dynamics of the economy are described by the Euler equation and the law of motion for the capital stock (per unit of effective labor),

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta} = \frac{f'(k(t)) - \rho}{\theta} - g$$

$$\dot{k}(t) = f(k(t)) - c(t) - (n+g)\,k(t)$$

The Euler equation is derived from household optimization. In re-writing it as above I have used that in a competitive equilibrium capital is paid its marginal product (from the firm's problem), $r(t) = f'(k(t))$. The law of motion for capital per unit of effective labor tells you how k evolves over time. In particular, when actual investment $(y(t) - c(t))$ is greater than required investment $((n+g)\,k(t))$, capital per unit of effective labor will increase. These two equations can be used to discuss both the long-run equilibrium (BGP) and the transitional dynamics of this economy.

Suppose the economy is initially in long-run equilibrium, i.e., along the BGP. In this case $\dot{c}(t) = \dot{k}(t) = 0$. This means that (c, k) are equal to their constant steady state values

(c^*, k^*), which are given by the following pair of equations (by setting $\dot{c}(t) = \dot{k}(t) = 0$ above),

$$f'(k^*) = \rho + \theta g$$

$$c^* = f(k^*) - (n + g)\,k^*$$

This BGP equilibrium is depicted as point E in Fig.1.

Consider now an increase in θ. Since θ is a preference parameter it will have no effect on the $\dot{k} = 0$ locus. It will affect the $\dot{c} = 0$ locus however. Intuitively, a higher θ implies a lower intertemporal elasticity of substitution between consumption today and tomorrow $(1/\theta)$. In other words, households are less willing to give up consumption today for consumption tomorrow, and thus consumption growth (savings) will be lower. Now the level of capital per unit of effective labor in the BGP is given by $\dot{c}_{new} = 0$ or $f'(k^*_{new}) = \rho + \theta_{new} g$. We have that, $\theta_{new} > \theta$ or $(\rho + \theta_{new} g) > (\rho + \theta g)$ or $f'(k^*_{new}) > f'(k^*)$. Since we have a diminishing marginal product of capital this implies that $k^*_{new} < k^*$, i.e., in the long-run there will be a lower capital stock per unit of effective labor. The new BGP is depicted in Fig.2 as point E'. The transitional dynamics of this economy are the following. At the time of the change in θ the level of capital per unit of effective labor is pre-determined in the economy and equal to k^*. Therefore, k cannot change at the time the shock. This implies that the burden of adjustment falls on consumption per unit of effective labor, which is a "control" variable. At the time of the shock c will jump up to place the economy on the new saddle path (point A) that will deliver the economy to the new steady state. See the phase diagram in Fig.3 for a graphical illustration.

3. (a)

$$\frac{\partial Y(t)}{\partial K(t)} = \frac{\partial \left(K(t)^{\alpha}[A(t)L(t)]^{1-\alpha}\right)}{\partial K(t)} = \alpha K(t)^{\alpha-1}[A(t)L(t)]^{1-\alpha} = \alpha \left(\frac{K(t)}{A(t)L(t)}\right)^{\alpha-1} =$$

$$= \alpha k(t)^{\alpha-1}$$

$$\frac{\partial Y(t)}{\partial L(t)} = \frac{\partial \left(K(t)^{\alpha}[A(t)L(t)]^{1-\alpha}\right)}{\partial L(t)} = (1-\alpha)K(t)^{\alpha}A(t)^{1-\alpha}L(t)^{-\alpha} =$$

$$= (1-\alpha)A(t)\left(\frac{K(t)}{A(t)L(t)}\right)^{\alpha} = A(t)(1-\alpha)k(t)^{\alpha}$$

(b) Following the steps we did in class you should derive the following law of motion for capital per unit of effective labor

$$\dot{k}(t) = sk(t)^{\alpha} - (n + g + \delta)\,k(t)$$

(c) The values of k and y on the steady state are respectively,

$$k^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

$$y^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Growth rates of k, y on the balanced growth path: $\frac{\dot{k}}{k} = \frac{\dot{y}}{y} = 0$.

Growth rates of K, Y on the balanced growth path: $\frac{\dot{K}}{K} = \frac{\dot{Y}}{Y} = n + g$.

Growth rates of $K/L, Y/L$ on the balanced growth path: $\frac{\dot{\left(\frac{K}{L}\right)}}{\left(\frac{K}{L}\right)} = \frac{\dot{\left(\frac{Y}{L}\right)}}{\left(\frac{Y}{L}\right)} = g$.

(d) Look at Figs.1-2. They convey the same intuition. At $k(0) > k^*$, actual investment is below required investment. From the law of motion for capital per unit of effective labor this means that k will start falling (negative growth):

$$\frac{\dot{k}(t)}{k(t)} = s\frac{y(t)}{k(t)} - (n+g+\delta) < 0$$

The red line in Fig.1 and Fig.2 indicate the negative growth of k when the economy starts off above the balanced growth path. Since output per unit of effective labor is monotonically related to capital per unit of effective labor, output will fall in transition to the steady state as well:

$$\dot{y}(t) = \alpha k(t)^{\alpha-1}\dot{k}(t) \rightarrow \frac{\dot{y}(t)}{y(t)} = \alpha\frac{\dot{k}(t)}{k(t)} < 0$$

Then the growth rates of K, Y, in transition to the balanced growth path, are:

$$\frac{\dot{K}(t)}{K(t)} = \frac{\dot{k}(t)}{k(t)} + n + g < n + g$$

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{y}(t)}{y(t)} + n + g < n + g$$

The growth rates of K/L, Y/L, in transition to the balanced growth path, are:

$$\frac{\dot{\left(\frac{K(t)}{L(t)}\right)}}{\left(\frac{K(t)}{L(t)}\right)} = \frac{\dot{k}(t)}{k(t)} + g < g$$

$$\frac{\dot{\left(\frac{Y(t)}{L(t)}\right)}}{\left(\frac{Y(t)}{L(t)}\right)} = \frac{\dot{y}(t)}{y(t)} + g < g$$

(e) The golden rule level of capital per unit of effective labor k_G is defined as the level of k that maximizes steady state consumption. In the Solow model consumption is given by $c = (1-s)f(k) = f(k) - sf(k)$. On the steady state however we know that $sf(k) = (n+g+\delta)k$. Thus steady state consumption in the Cobb-Douglas case is given by,

$$c = k^\alpha - (n+g+\delta)k$$

The golden rule level of capital per unit of effective labor is then the solution to the following maximization problem,

$$\max_k \{k^\alpha - (n+g+\delta)k\}$$

Re-arranging the first order condition from this problem with respect to k, gives the golden rule,

$$k_G = \left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

(f) Let's call the new level of efficiency growth g_{new}. Then the new steady state levels of k, y are,

$$k^*_{new} = \left(\frac{s}{n+g_{new}+\delta}\right)^{\frac{1}{1-\alpha}} < \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} = k^*$$

$$y^*_{new} = \left(\frac{s}{n+g_{new}+\delta}\right)^{\frac{\alpha}{1-\alpha}} < \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} = y^*$$

Both will be lower relative to the original steady states as a result of the increase in g. This can also be seen in Fig.3, where the required investment schedule rotates upwards as a result of this change. Intuitively, capital is now being "diffused" across a larger base of units of effective labor. The growth rates of k, y are zero in both steady states (remember these variables have already been normalized by units of effective labor). So this change has no long term effects on the normalized variables. It would however change the growth rates of Y and Y/L (why?). In transition to the new lower steady state both k and y will fall until they reach their new levels. See Fig.3, the law of motion for k, and the production function. The reason is that at the original steady state after the increase in g, we have that required investment exceeds actual investment: $s\,(k^*)^\alpha < (n+g_{new}+\delta)k^*$.

4. Consider the augmented Solow model in which the economy's total output is produced according to the following production function

$$Y(t) = K(t)^{\alpha_K} H(t)^{\alpha_H} [A(t)L(t)]^{1-\alpha_K-\alpha_H} \tag{1}$$

where Y is total output, K is the stock of physical capital, H is the stock of human capital, L is raw labor, and A is the level of efficiency. We assume that $0 < \alpha_K, \alpha_H < 1$. Further assume that efficiency and labor grow at exogenous and constant rates g and n respectively.

(a) For some positive number b, we have that:

$$(b(K(t))^{\alpha_K}\,(bH(t))^{\alpha_H}\,[A(t)bL(t)]^{1-\alpha_K-\alpha_H} = bK(t)^{\alpha_K}H(t)^{\alpha_H}[A(t)L(t)]^{1-\alpha_K-\alpha_H} = bY(t)$$

(b) Yes it does, since $\alpha_K + \alpha_H < 1$. Then this economy will reach a steady state (BGP).

(c) Take logs on both sides of the above production function,

$$\ln Y(t) = \alpha_K \ln K(t) + \alpha_H \ln H(t) + (1 - \alpha_K - \alpha_H) \ln A(t) + (1 - \alpha_K - \alpha_H) \ln L(t)$$

Take the derivative with respect to time on both sides,

$$\frac{\dot{Y}(t)}{Y(t)} = \alpha_K \frac{\dot{K}(t)}{K(t)} + \alpha_H \frac{\dot{H}(t)}{H(t)} + (1 - \alpha_K - \alpha_H)\frac{\dot{L}(t)}{L(t)} + (1 - \alpha_K - \alpha_H)\frac{\dot{A}(t)}{A(t)}$$

Re-arranging terms and defining $SR(t) \equiv (1 - \alpha_K - \alpha_H)\frac{\dot{A}(t)}{A(t)}$ we have,

$$\left[\frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{L}(t)}{L(t)}\right] = \alpha_K \left[\frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}\right] + \alpha_H \left[\frac{\dot{H}(t)}{H(t)} - \frac{\dot{L}(t)}{L(t)}\right]$$

(d) The augmented production function in intensive form is,

$$\frac{Y(t)}{A(t)L(t)} = \left(\frac{K(t)}{A(t)L(t)}\right)^{\alpha_K}\left(\frac{H(t)}{A(t)L(t)}\right)^{\alpha_H}$$

or

$$y(t) = k(t)^{\alpha_K} h(t)^{\alpha_H}$$

(e) Following the same procedure as we did in class for the simple Solow model you can show that the laws of motion for the physical and human capital stocks are,

$$\dot{k}(t) = s_K k(t)^{\alpha_K} h(t)^{\alpha_H} - (n + g + \delta)\, k(t)$$

$$\dot{h}(t) = s_H k(t)^{\alpha_K} h(t)^{\alpha_H} - (n + g + \delta)\, h(t)$$

(f) In steady state we have that $\dot{k}(t) = \dot{h}(t) = 0$. From the physical and human capital accumulation equations we have that,

$$s_K k^{\alpha_K} h^{\alpha_H} = (n + g + \delta)\, k$$

$$s_K k^{\alpha_K} h^{\alpha_H} = (n + g + \delta)\, h$$

This is a system of two equations in two unknowns, the steady state levels of k, h. One way to solve this system is to divide the two equations to get, $h = \frac{s_H}{s_K} k$. Using this equation substitute out h from the first equation and. Then you have one equation in one unknown. Doing this you should find the following solution for steady state k,

$$k^* = \left(\frac{s_H^{\alpha_H} s_K^{1-\alpha_H}}{n + g + \delta} \right)^{\frac{1}{1-\alpha_K-\alpha_H}}$$

Using k^* along with $h = \frac{s_H}{s_K} k$ you can find the value of h in the steady state to be,

$$h^* = \left(\frac{s_K^{\alpha_K} s_H^{1-\alpha_K}}{n + g + \delta} \right)^{\frac{1}{1-\alpha_K-\alpha_H}}$$

As in the standard Solow model the growth rates of Y and Y/L, in the augmented Solow model are $n+g$ and g respectively. In other words, diminishing returns to all accumulable factors of production ensure that in the long run any growth coming from accumulation will cease.

(g) Divide both sides of the production function by $Y^{\alpha_K+\alpha_H}$ and re-arrange terms to get the following equation,

$$\frac{Y}{L} = \left(\frac{K}{Y}\right)^{\frac{\alpha_K}{1-\alpha_K-\alpha_H}} \left(\frac{H}{Y}\right)^{\frac{\alpha_H}{1-\alpha_K-\alpha_H}} A$$

Consider now 2 countries i and j, then the model would predict an income disparity of,

$$\frac{\left(\frac{Y}{L}\right)_i}{\left(\frac{Y}{L}\right)_j} = \left[\frac{\left(\frac{K}{Y}\right)_i}{\left(\frac{K}{Y}\right)_j}\right]^{\frac{\alpha_K}{1-\alpha_K-\alpha_H}} \left[\frac{\left(\frac{H}{Y}\right)_i}{\left(\frac{H}{Y}\right)_j}\right]^{\frac{\alpha_H}{1-\alpha_K-\alpha_H}} \frac{A_i}{A_j}$$

Substituting in the numbers provided we have that,

$$\frac{\left(\frac{Y}{L}\right)_i}{\left(\frac{Y}{L}\right)_j} = 3^{\frac{1/3}{1-1/3-1/3}} 2^{\frac{1/3}{1-1/3-1/3}} = 6$$

Country i should be 6 times richer than country j.

5. (a) The representative firm's problem is to choose capital and labor services to maximize profits in every period t. In other words, the firm faces a static profit maximization problem each period. The firm's problem is,

$$\max_{K(t),L(t)} \left\{ K(t)^{\alpha}[A(t)L(t)]^{1-\alpha} - W(t)L(t) - r(t)K(t) \right\}$$

where $r(t)$ is the rental price of capital in period t, and $W(t)$ is the real wage rate at time t. The first order conditions to this problem imply,

$$r(t) = \alpha K(t)^{\alpha-1}[A(t)L(t)]^{1-\alpha} = \alpha \left(\frac{K(t)}{A(t)L(t)}\right)^{\alpha-1} = \alpha k(t)^{\alpha-1}$$

$$W(t) = (1-\alpha)K(t)^{\alpha}A(t)^{1-\alpha}L(t)^{-\alpha} = (1-\alpha)A(t)\left(\frac{K(t)}{A(t)L(t)}\right)^{\alpha} = A(t)(1-\alpha)k(t)^{\alpha}$$

Define the wage rate per unit of effective labor as $w(t) \equiv \frac{W(t)}{A(t)} = (1-\alpha)k(t)^{\alpha}$.

(b) The household budget constraint is,

$$\int_{t=0}^{\infty} e^{-R(t)}C(t)\frac{L(t)}{H}dt \leq \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}W(t)\frac{L(t)}{H}dt$$

This says that the household's present discounted value of future consumption stream cannot exceed the present discounted value of future labor earnings stream plus initial wealth.

(c) The household maximizes lifetime utility subject to the lifetime budget constraint. We need to convert both of these into units of effective labor. First consider the lifetime budget constraint. Let $c = \frac{C}{A}$ be consumption per unit of effective labor. Then we can write consumption per household member as $C(t) = c(t)A(t)$. We know that $A(t) = A(0)e^{gt}$ and $L(t) = L(0)e^{nt}$. Then we have that $C(t)\frac{L(t)}{H} = c(t)e^{(n+g)t}\frac{A(0)L(0)}{H}$ and $W(t)\frac{L(t)}{H} = w(t)e^{(n+g)t}\frac{A(0)L(0)}{H}$. Also note that we can re-write initial wealth as $\frac{K(0)}{H} = \frac{K(0)}{A(0)L(0)}\frac{A(0)L(0)}{H}$. Substitute these into the lifetime budget constraint above and divide through by $\frac{A(0)L(0)}{H}$ to get the budget in units of effective labor,

$$\int_{t=0}^{\infty} e^{-R(t)}e^{(n+g)t}c(t)dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)}e^{(n+g)t}w(t)dt$$

In equilibrium, because the marginal utility of consumption is positive, this budget constraint will hold with equality.

Next consider the lifetime utility,

$$U = \int_{t=0}^{\infty} e^{-\rho t}\ln\left(C(t)\right)\frac{L(t)}{H}dt = \int_{t=0}^{\infty} e^{-\rho t}\ln\left(c(t)A(t)\right)\frac{L(t)}{H}dt =$$

$$= \int_{t=0}^{\infty} e^{-\rho t}\ln\left(c(t)A(0)e^{gt}\right)\frac{L(0)e^{nt}}{H}dt = \int_{t=0}^{\infty} e^{-(\rho-n)t}\left\{\ln\left(c(t)\right) + \ln\left(A(0)\right) + gt\right\}\frac{L(0)}{H}dt =$$

$$= \frac{L(0)}{H}\int_{t=0}^{\infty} e^{-(\rho-n)t}\ln\left(c(t)\right)dt + \int_{t=0}^{\infty} e^{-(\rho-n)t}gt\frac{L(0)}{H}dt + \int_{t=0}^{\infty} e^{-(\rho-n)t}\ln(A(0))\frac{L(0)}{H}dt =$$

$$= B_0 + B_1\int_{t=0}^{\infty} e^{-(\rho-n)t}\ln(c(t))dt$$

To solve the household's problem we set up the Lagrangian,

$$\pounds = B_0 + B_1 \int_{t=0}^{\infty} e^{-(\rho-n)t} \ln(c(t))dt$$

$$+\lambda \left\{ k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t)} w(t)dt - \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} c(t)dt \right\}$$

where λ is the Lagrange multiplier. Take the first order condition with respect to $c(t)$,

$$B_1 e^{-(\rho-n)t} \frac{1}{c(t)} = \lambda e^{(n+g)t} e^{-R(t)}$$

Take logs on both sides,

$$\ln B_1 - (\rho - n)t - \ln c(t) = \ln \lambda + (n+g)t - R(t)$$

Take the derivative with respect to time,

$$-(\rho - n) - \frac{\dot{c(t)}}{c(t)} = n + g - r(t)$$

or

$$\frac{\dot{c(t)}}{c(t)} = r(t) - \rho - g$$

(d) Law of motion for capital per unit of effective labor,

$$\dot{k}(t) = k(t)^{\alpha} - c(t) - (n+g)k(t)$$

(e) The economy's two key dynamic equations are,

$$\frac{\dot{c(t)}}{c(t)} = \alpha k(t)^{\alpha-1} - \rho - g$$

where I have used the firm's first order condition, and

$$\dot{k}(t) = k(t)^{\alpha} - c(t) - (n+g)k(t)$$

In the long run (on the BGP) we have that $\dot{c}(t) = \dot{k}(t) = 0$. By setting $\dot{c}(t) = 0$ we can solve for the steady state level of k,

$$k^* = \left(\frac{\alpha}{\rho + g} \right)^{\frac{1}{1-\alpha}}$$

Then from the production function in intensive form, output per unit of effective labor is,

$$y^* = (k^*)^{\alpha} = \left(\frac{\alpha}{\rho + g} \right)^{\frac{\alpha}{1-\alpha}}$$

By setting $\dot{k}(t) = 0$ we can solve for the steady state level of c,

$$c^* = y^* - (n+g)k^* = \left(\frac{\alpha}{\rho+g}\right)^{\frac{\alpha}{1-\alpha}} - (n+g)\left(\frac{\alpha}{\rho+g}\right)^{\frac{1}{1-\alpha}}$$

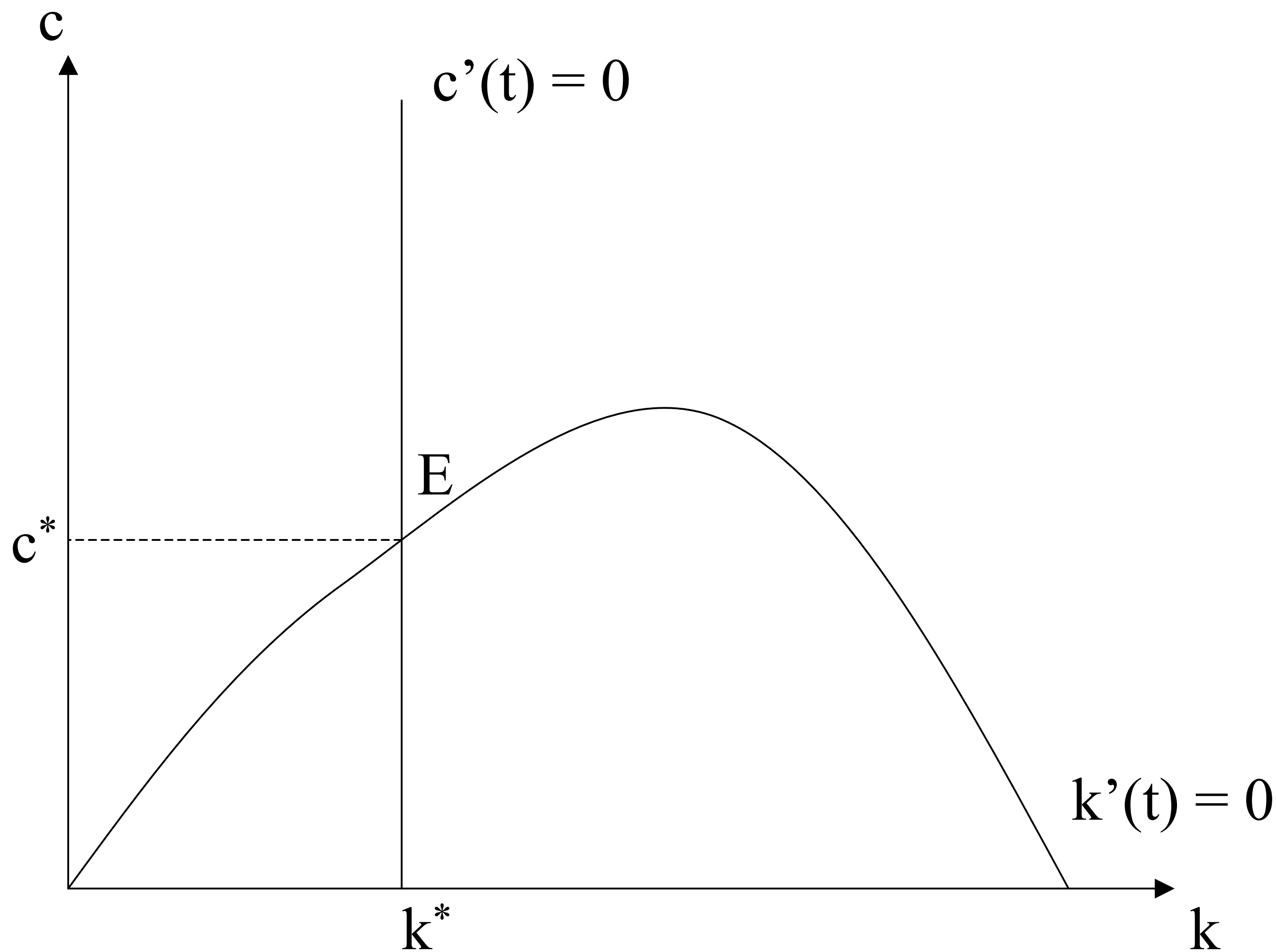
Along this balanced growth path, the growth rates of the aggregate variables K, Y are $n+g$ and the growth rates of the variables per worker $K/L, Y/L$ are g. (Note: the logic is the same as in the Solow model, i.e., use the definitions of (c, k)).

(f) Suppose the economy starts off from a level of capital per unit of effective labor below the steady state: $k(0) < k^*$. At time 0, $k(0)$ is pre-determined and cannot be adjusted. Consumption per unit of effective labor at time 0, $c(0)$, is a "control" variable, i.e., it is to be chosen. There is a whole continuum of possible choices for $c(0)$. Each one of these places the economy on a given trajectory. However, not all of these trajectories will take the economy to the steady state. In fact there is only one trajectory that takes the economy to the steady state: the saddle path. This is the only trajectory that satisfies, household optimization, law of motion for capital, the household intertemporal budget constraint and the No-Ponzi-Game condition. Thus if the economy is at $k(0)$ it chooses a level of consumption $c(0)$ that places it on the saddle path. Once the economy is on the saddle path it will go to the steady state. See Fig.4. For any point above this level of $c(0)$ the economy will diverge up and to the left, while for any point below it, it will diverge down and to the right.

(g) The competitive equilibrium is socially optimal. The conditions for the first welfare theorem hold here. That means that the competitive equilibrium is Pareto efficient (you cannot make someone better off without making someone else worse off). The reason is that there are no distortions of any kind (markets are competitive and complete, there are no externalities in production or consumption, no public goods, finite number of infinitely lived agents etc.).

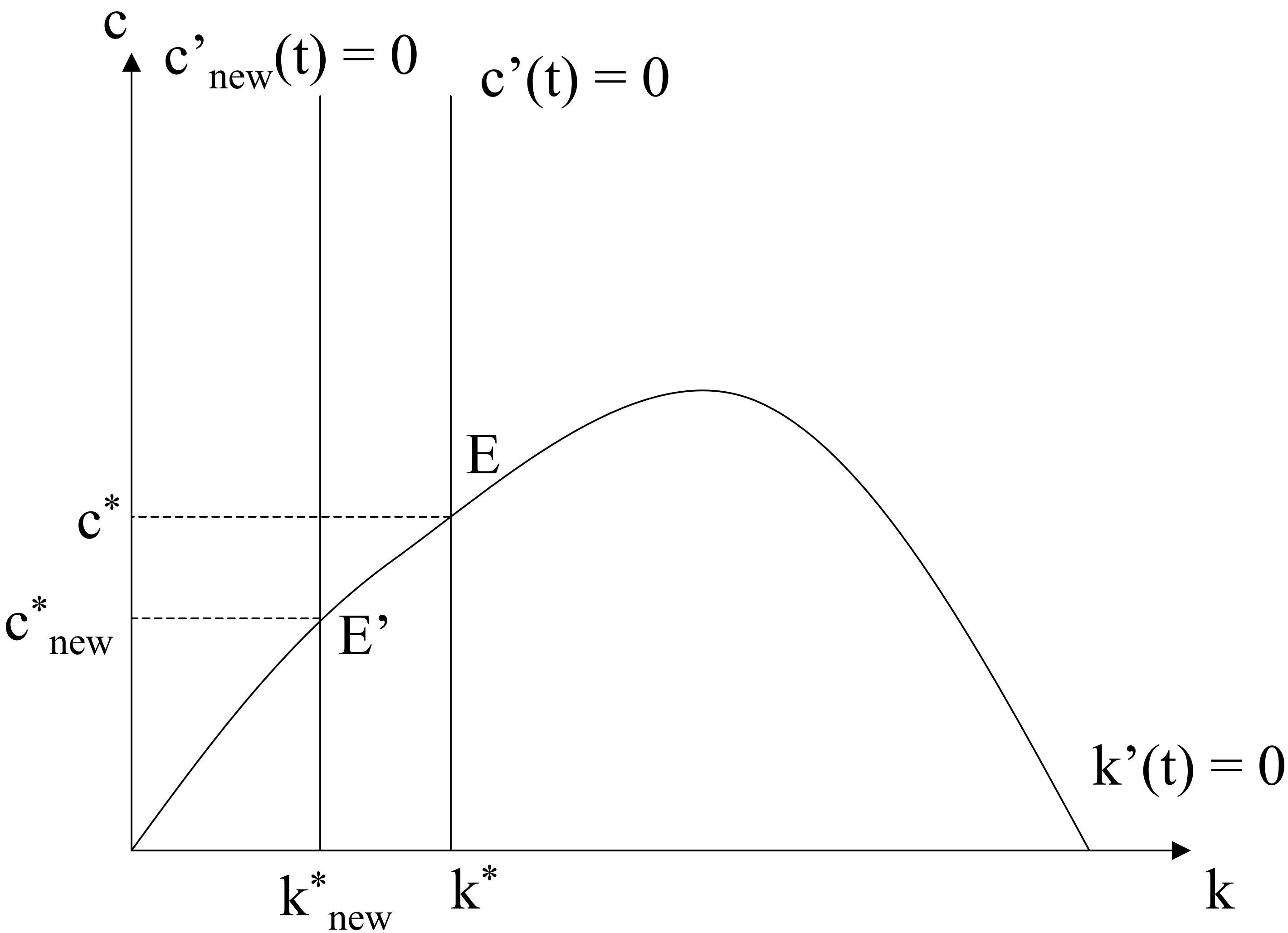
Quest.2

Fig.1: Initial Long –run Equilibrium



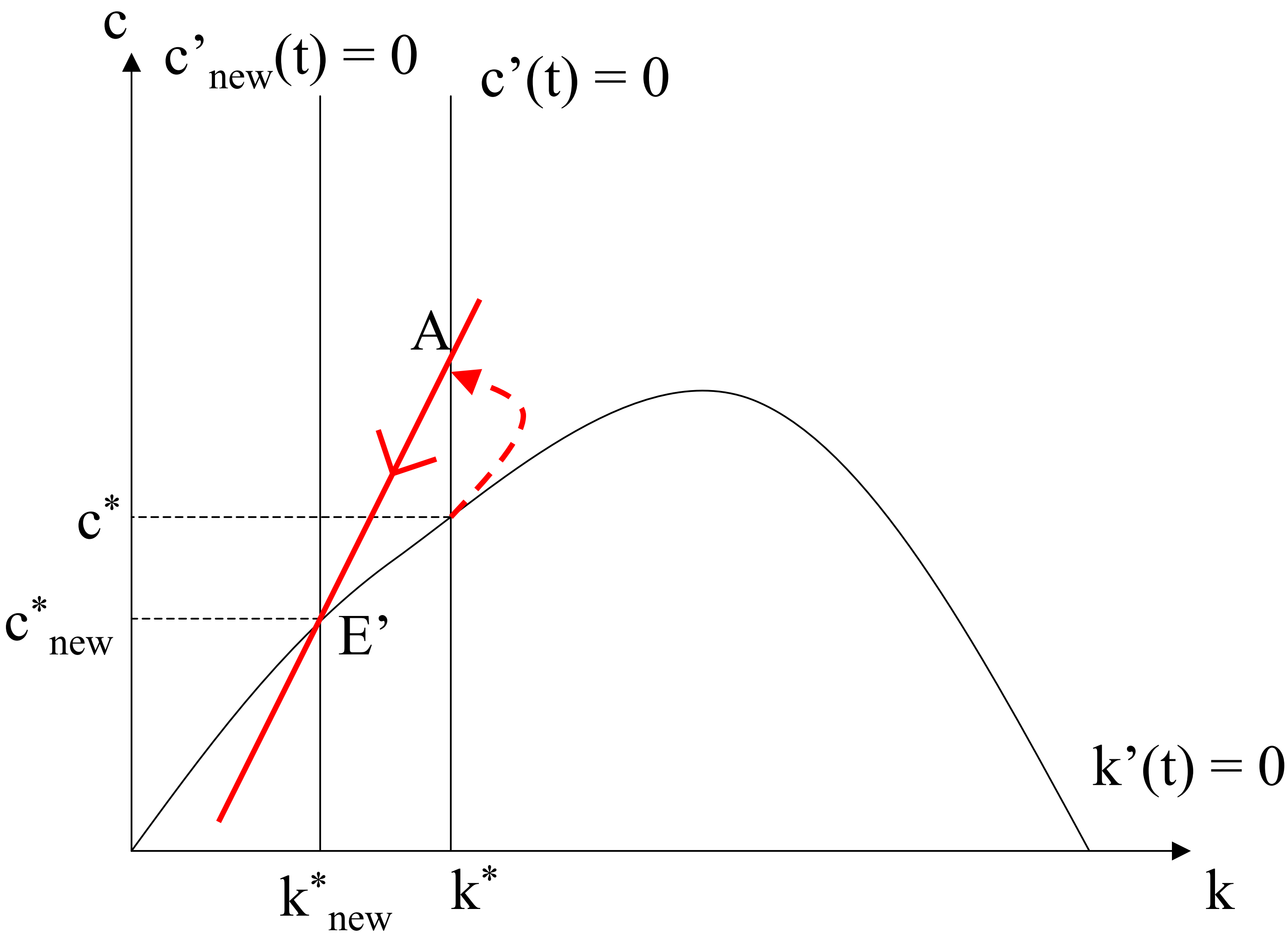
Quest.2

Fig.2: New Long –run Equilibrium

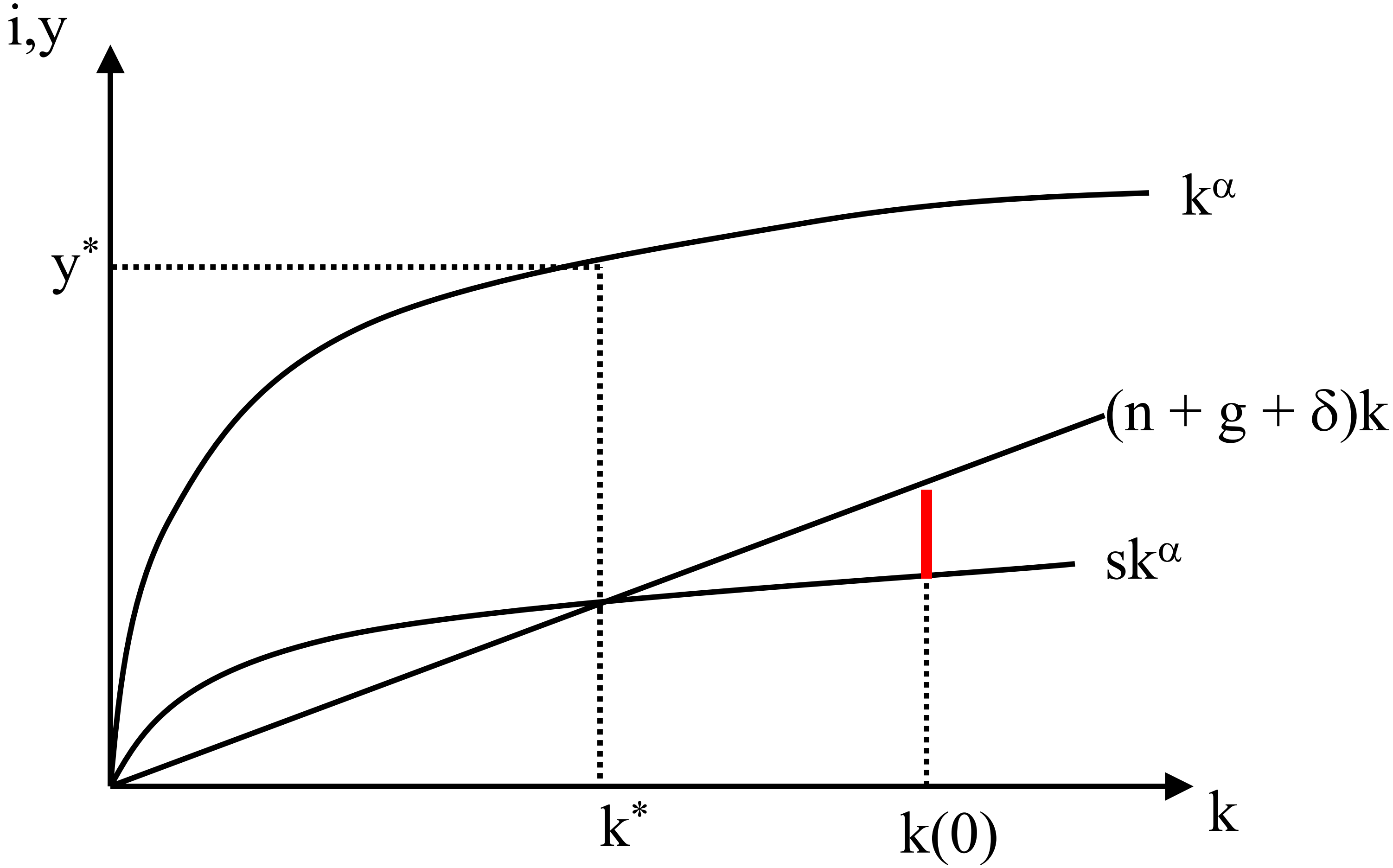


Quest.2

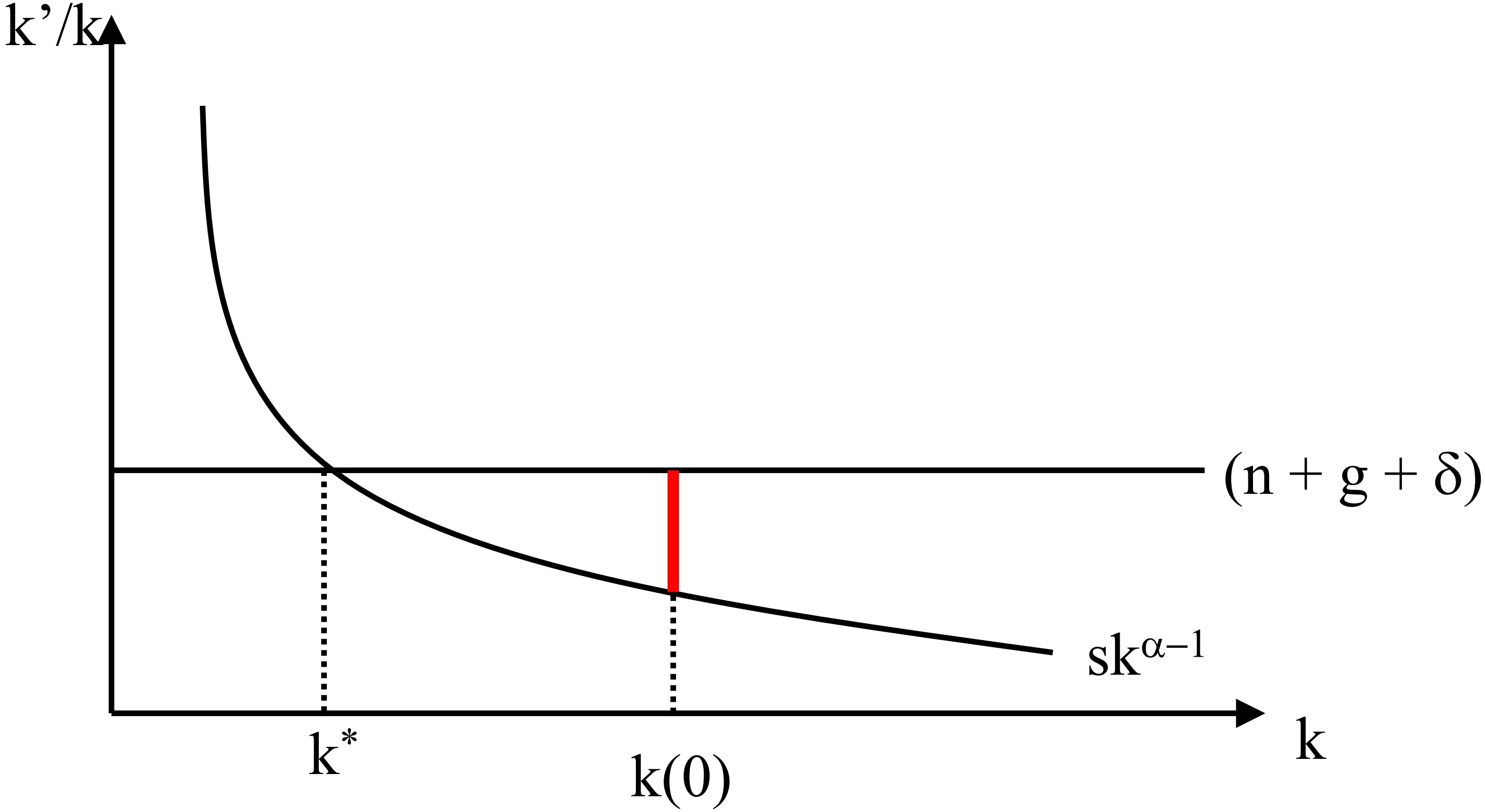
Fig.3: Transition to New Long –run Equilibrium



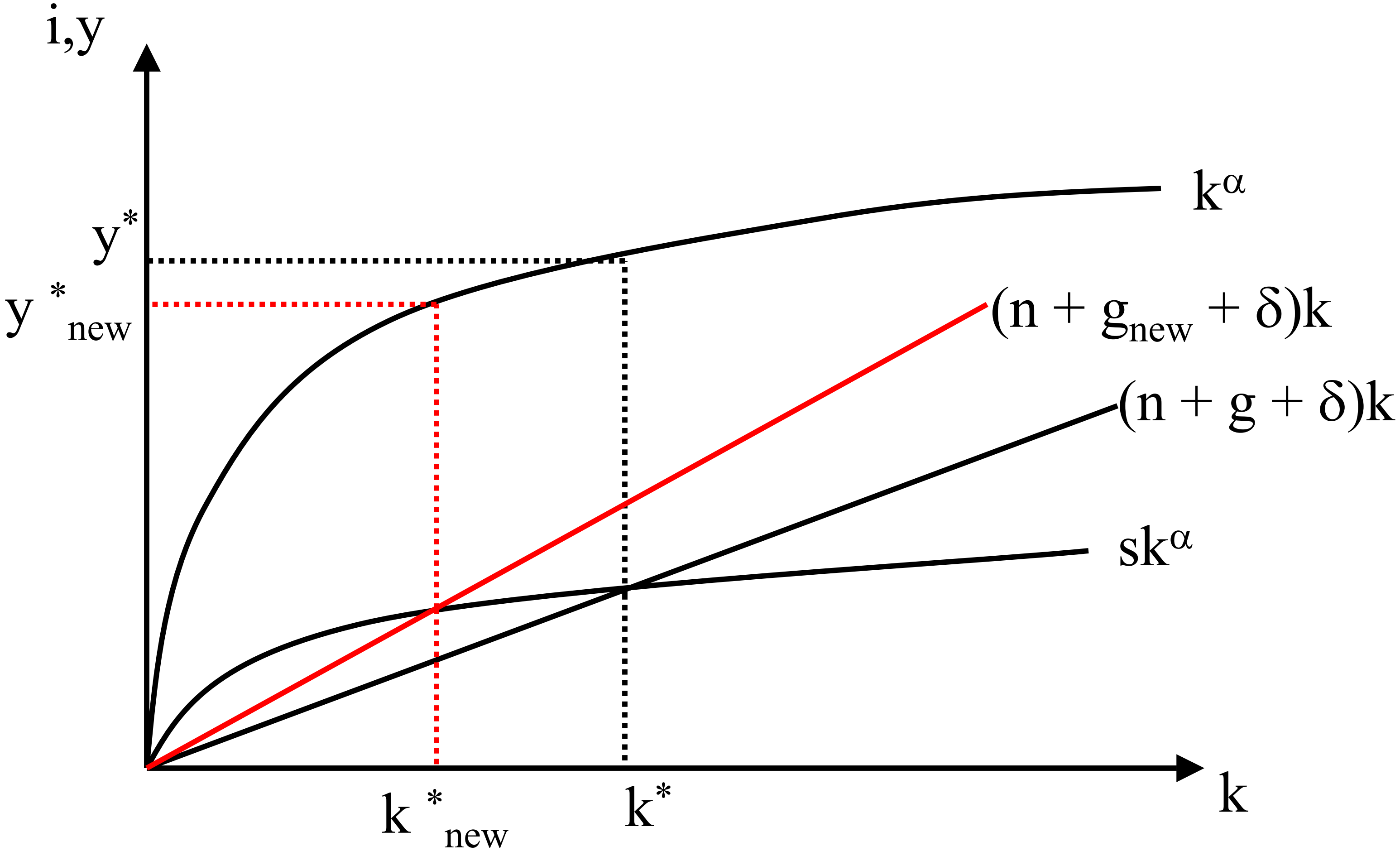
Qu.3,Fig.1: Transitional Dynamics (starting above BGP)



Que.3,Fig.2: Growth of k in transition to BGP



Qu.3,Fig.3: Increase in g in the Solow model



Quest.5

Fig.4: Adjustment to BGP Equilibrium