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Dynamic Spatial Competition Between Multi-Store Firms

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Abstract

We propose a dynamic model of an oligopoly industry characterized by spatial competition between multi-store retailers. Firms compete in prices and decide where to open or close stores depending on demand conditions and the number of competitors at different locations, and on location-specific private-information shocks. We develop an algorithm to approximate a Markov Perfect Equilibrium in our model, and propose a procedure for the estimation of the parameters of the model using panel data on number of stores, prices, and quantities at multiple geographic locations within a city. We also present numerical examples to illustrate the model and algorithm.

Keywords: Spatial competition; Store location; Industry dynamics; Sunk costs.

JEL classifications: C73, L13, L81, R10, R30.

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1 Introduction

Retail chains account for more than 60% of sales in U.S. retailing (see Hollander and Omura, 1989, and Jarmin, Klimek and Miranda, 2009). Geographic location is in many cases the most important source of product differentiation for these firms. It is also a forward-looking decision with significant non-recoverable entry costs, mainly due to capital investments which are both firm- and location-specific. Thus, the sunk cost of setting-up a new store, and the dynamic strategic behavior associated with them, is a potentially important force behind the configuration of the spatial market structure that we observe in retail markets.

Despite its relevance, there have been very few studies analyzing spatial competition as a dynamic oligopoly game. Existent models of industry dynamics often lack an explicit account of spatial competition. Although useful applications have emerged from the seminal work by Pakes and McGuire (1994) and Ericson and Pakes (1995), none have explicitly incorporated the spatial and multi-store features which are prevalent in many retailing industries.[1] The literature on spatial competition often restricts the treatment of time. Models based on the seminal work of Hotelling (1929) describe a two- or three-period framework where firms choose locations and then compete in the product market.[2] Eaton and Lipsey (1975), Schmalensee (1978), and Bonanno (1987) study the multi-store monopolist under the threat of entry. They find that an incumbent monopolist will strategically locate its stores to successfully preempt the entry of competitors. As noted by Judd (1985), the limited account of time and dynamics in this literature has very important implications on some predictions of these models. Judd notes that the aforementioned models assume that entry and location decisions are completely irreversible, with no possibility of exit or relocation. He shows that *allowing for exit* may result in non-successful spatial preemption by the incumbent.

Judd's paper emphasizes that models of spatial competition between multi-store firms need to incorporate dynamics to its full extent, allowing for endogenous entry, exit, and forward-looking strategies. That is the intention of this paper. In this context, the contribution of this paper is threefold. First, we propose a dynamic model of an oligopoly industry characterized by spatial competition between multi-store firms. In this model, firms compete in prices and decide where to open or close stores depending on the location profile of competitors, demand conditions, and location-specific private-information shocks. We define and characterize a Markov Perfect Equilibria (MPE) in this model. Our framework is a useful tool to study topics on multi-store competition that involve spatial and dynamic considerations, such as the evaluation of the welfare effects of possible mergers between multi-store firms, or studying the conditions under which spatial preemptive

[1] Ellickson and Beresteanu (2005) endogenize supermarkets' "store density," i.e., the number of stores per capita a firm owns in a market. Holmes (2011) studies the role of economies of density in explaining the spatial evolution of Wal-Mart stores since the 1950. However, spatial competition is not accounted for in these applications.

[2] Anderson, De Palma and Thisse (1992) present a compilation of static spatial competition models. There is also a large and growing literature on estimation of static structural models of spatial competition and store location. The work by West (1981a and 1981b) was seminal in this literature. Some recent important contributions to this empirical literature are Pinkse, Slade, and Brett (2002), Seim (2006), Zhu and Singh (2009), Ellickson, Houghton, and Timmins (2010), and Vitorino (2012). Also, Slade (2005) and Pinkse and Slade (2010) provide excellent surveys.

behavior is an equilibrium strategy.

A second contribution of this paper is to provide a method to compute an equilibrium of the model. The size of the state space of this dynamic game increases exponentially with the number of possible store locations. Solving exactly for an equilibrium of the model is an intractable problem even when the number of locations is not too large.[3] To deal with this computational issue we exploit an interpolation method similar to the one proposed by Rust (1997).[4]

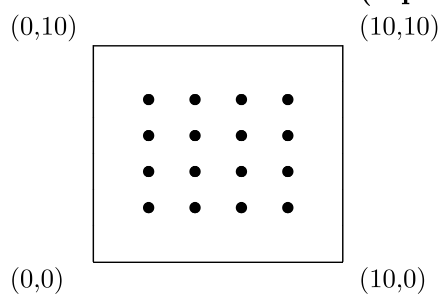
We propose a procedure for the estimation of the model using panel data on number of stores, prices, and quantities from multiple geographic locations. The method provides consistent estimates even when the dataset includes information from the equilibrium of the game in a single city, as long as the number of locations within the city is large. To illustrate the model and the algorithm, we present some numerical examples to analyze how the magnitude of the sunk cost of setting-up a new store affects firms' dynamic strategies of store location, firm value and consumer welfare.

2 Model

2.1 The Market

Consider a local market of a differentiated retail product (e.g., retail banking, supermarkets). From a geographic point of view the market is a compact set $\mathbb{C}$ in the Euclidean space $\mathbb{R}^2$. The distance between two points in the market, say a and b, is the Euclidean distance denoted by $\|a-b\|$. There is a finite set of L pre-specified locations where it is feasible for firms to operate stores. Let $\{z_1, z_2, ..., z_L\}$ be the set of geographical coordinates of these feasible locations, where $z_\ell \in \mathbb{C}$. We call each of these business locations a submarket. Figure 1 presents an example.[5]

FIGURE 1
Market and feasible business locations (represented with •)



[3] If I is the number of firms, and L is the number of locations in the models, then the number of cells in the state space is 2^{IL}. For instance, with four firms and ten locations the number of cells is greater than 10^{12}.

[4] In a companion paper (Aguirregabiria and Vicentini, 2007), we provide a manual that describes in detail our programs and procedures. This manual and the software, in GAUSS language, can be downloaded from authors' web pages.

[5] The assumption of a finite number of feasible locations is made for computational convenience. In an empirical application of the model, this assumption implies that the researcher has to discretize the set of business locations. However, there are situations where this assumption can be realistic. For instance, in Canada and US leasing contracts at shopping centers typically include a "radius restriction" clause that prohibits tenants in a shopping center from opening another store within a certain radius (see Eckert and West, 2006). Also, some countries and states have zoning laws that apply to "big box" retailers. These restrictions can be sufficiently strict such that some retailers have only a few feasible locations.

Time is discrete. At time t the market is populated by a continuum of consumers. Each consumer is characterized by a geographical location $z \in \mathbb{C}$. The geographical distribution of consumers at period t is given by the absolute measure $\phi_t(z)$ such that $\int_{\mathbb{C}} \phi_t(dz) = M_t$, where M_t is the size of the market. This measure ϕ_t evolves over time according to a discrete Markov process. Let Ω be the discrete set of possible functions ϕ_t.

There are I multi-store firms that can potentially operate in the market. We index firms by i and use $\Upsilon = \{1, 2, ..., I\}$ to represent the set of firms. At the beginning of period t a firm's network of stores is represented by the vector $n_{it} = (n_{i1t}, n_{i2t}, ..., n_{iLt})$, where $n_{i\ell t}$ is the number of stores that firm i operates in location ℓ at period t. For simplicity, we assume that a firm can have at most one store in a location, such that $n_{i\ell t} \in \{0, 1\}$. The model can be easily generalized to the case with more than one store per location and firm.[6] Overlapping of stores from different firms at the same location is allowed. The spatial market structure at period t is represented by the vector $\mathbf{n}_t = (n_{1t}, n_{2t}, ..., n_{It}) \in \{0, 1\}^{IL}$. A store in this market is identified by a pair (i, ℓ) where i represents the firm, and ℓ identifies the location.

Every period t, firms observe the spatial market structure $\mathbf{n}_t$, the state of the demand ϕ_t, and some location- and firm-specific shocks in costs which are private information of each firm. Given this information, incumbent firms compete in prices. Prices can vary over stores within the same firm. This spatial Bertrand game is static because current prices do not have any effect on future demand or profits. Furthermore, private information shocks affect fixed operating costs and entry costs but not the demand or variable costs. Therefore, these shocks do not have any influence in equilibrium prices. The resulting Bertrand prices determine equilibrium variable profits for each firm i at period t. At the end of period t, firms decide simultaneously their network of stores for next period. This choice is dynamic because of partial irreversibility in the decision to open a new store, i.e., sunk costs. Firms are allowed to open or close at most one store per period. Exogenous changes in the spatial distribution of demand (i.e., changes in ϕ_t) and firms' location-specific shocks generate entry and exit at different locations and changes over time in the spatial market structure. Firms may grow (or decline) over time expanding (contracting) their network of stores, and possibly become a dominant player (or exit from the market). The details of this model are presented in sections 2.2 to 2.4.

2.2 Consumer Behavior

The model of consumer demand that we present here builds on De Palma *et al.* (1985). We extend their model in two dimensions. First, we incorporate vertical differentiation (i.e., they assume that firms have the same qualities ω_i). And second, geographic location in our model has two dimensions while they consider a linear city.

A consumer is fully characterized by a pair (z, v), where z is her location in space and $v \in \mathbb{R}^{IL}$

[6] In our model, two stores of the same firm and at the same location are perfect substitutes. Therefore, a firm will never find optimal to have more than one store at the same location. However, it is possible to extend our logit demand model to allow for horizontal differentiation between stores with the same firm and location.

is a vector representing her idiosyncratic preferences over all possible stores. Consumer behavior is static and demand is unitary. At every period t, consumers know all active stores with their respective locations and prices. A consumer decides whether to buy or not a unit of the good and from which store to buy it. The indirect utility of consumer (z, υ) patronizing store (i, ℓ) at time t is:

$$u_{i\ell t} = \omega_{i\ell} - p_{i\ell t} - \tau \; \|z - z_\ell\| + \upsilon_{i\ell} \tag{1}$$

$\omega_{i\ell}$ is the quality of the product offered by retail chain i that, in principle, may vary exogenously across locations. All consumers agree on this measure. $p_{i\ell t}$ is the mill price charged by store (i, ℓ) at time t. The term $\tau \|z - z_\ell\|$ represents consumer's transportation costs, where τ is the unit transportation cost and $\|z - z_\ell\|$ is the Euclidean distance between the consumer's address and the store. Finally, $\upsilon_{i\ell}$ captures consumer idiosyncratic preferences for store (i, ℓ). The utility of the outside alternative (i.e., not purchasing the good) is normalized to zero.

A consumer purchases a unit of the good at store (i, ℓ) if and only if $u_{i\ell t} \geq 0$ and $u_{i\ell t} \geq u_{i'\ell' t}$ for any other store (i', ℓ'). To obtain the aggregate demand at each store we have to integrate individual demands over the distribution of (z, υ). We assume that υ is independent of z and it has a type 1 extreme value distribution with dispersion parameter μ. The parameter μ measures the importance of horizontal product differentiation, other than spatial differentiation. Integrating over υ we obtain the *local demand* for store (i, ℓ) from consumers at location z:

$$\sigma_{i\ell}(z, \mathbf{n}_t, \mathbf{p}_t) = \frac{n_{i\ell t} \; \exp\left\{\left(\omega_{i\ell} - p_{i\ell t} - \tau \; \|z - z_\ell\|\right)/\mu\right\}}{1 + \sum_{i'=1}^{I} \sum_{\ell'=1}^{L} n_{i'\ell' t} \; \exp\left\{\left(\omega_{i'\ell'} - p_{i'\ell' t} - \tau \; \|z - z_{\ell'}\|\right)/\mu\right\}} \tag{2}$$

Integrating these local demands over the spatial distribution of consumers we obtain the *aggregate demand* for store (i, ℓ) at time t:

$$s_{i\ell}(\mathbf{n}_t, \mathbf{p}_t, \phi_t) = \int_{\mathbb{C}} \sigma_{i\ell}(z, \mathbf{n}_t, \mathbf{p}_t) \; \phi_t(dz) \tag{3}$$

Consumers' substitution patterns depend directly on the distance function $\|z - z_\ell\|$, so that a store competes more fiercely against closer stores. Stores' market areas are *overlapping* because of the unobserved heterogeneity of consumers, υ. Therefore a store serves consumers from all corners of the city $\mathbb{C}$, but more so the nearby patronage. Stores will always face a positive demand and can adjust prices without facing a perfectly elastic demand. Firms face the trade-off between strategic and market share effects. As stores locate closer to each other, the more intense price competition acts as a centrifugal force of dispersion (*strategic* effect). At the same time, firms wish to locate where transportation costs are minimum, which acts as a centripetal force of agglomeration (*market share* effect). An equilibrium spatial market structure would balance these forces, along with the effect of own-firm stores cannibalization.

We note the importance of the parameters μ and τ that capture product differentiation. As $\mu \to 0$ the degree of non-spatial horizontal product differentiation becomes small and every consumer shops at the store with lowest full price from her location (i.e., quality-adjusted mill price plus transportation costs). At the limit we would observe market areas defined as Voronoi graphs (or Thiessen polygons) with well-defined market borders (see Eaton and Lipsey, 1975, or Tabuchi, 1994, among others). Transportation costs increase the importance of location, serve as a shield for market power and create incentives for firm dispersion.[7]

2.3 Price Competition

For notational simplicity we omit the time subindex in this subsection. Every period firms compete in prices taking as given their network of stores, the state of the demand, and variable costs. Firms may charge different prices at different stores. This price competition is a game of complete information. A firm variable profit function is:

$$R_i(\mathbf{n}, \mathbf{p}, \phi) = \sum\nolimits_{\ell=1}^{L} (p_{i\ell} - c_{i\ell}) \;\; s_{i\ell}(\mathbf{n}, \mathbf{p}, \phi) \tag{4}$$

$c_{i\ell}$ is the unit variable cost of firm i at location ℓ which is assumed constant over time. Each firm maximizes its variable profit by choosing its best-response vector of prices. The best response of firm i can be characterized by the first-order condition for each price $p_{i\ell}$:

$$s_{i\ell} + (p_{i\ell} - c_{i\ell}) \frac{\partial s_{i\ell}}{\partial p_{i\ell}} + \sum\nolimits_{\ell' \neq \ell} (p_{i\ell'} - c_{i\ell'}) \frac{\partial s_{i\ell'}}{\partial p_{i\ell}} = 0 \tag{5}$$

The first two terms are the price and quantity effects of $p_{i\ell}$ on the profit at its own store (i, ℓ), while the last term is the quantity effect of $p_{i\ell}$ on all other stores of firm i. In our demand system, stores of a same firm are *gross substitutes* (i.e., $\partial s_{i\ell'} / \partial p_{i\ell} > 0$ for $\ell' \neq \ell$) and therefore the third term is always positive. This implies that, *ceteris paribus,* a multi-store firm will offer higher prices than a single-store firm.

Let $\mathbf{p}$ and $\mathbf{s}(\mathbf{p})$ be vectors with dimension $IL \times 1$ of prices and demands, respectively, for every store. Following Berry (1994) and Berry *et al.* (1995), we define a square matrix $\mathbf{\Lambda}(\mathbf{p})$ of dimension $IL \times IL$ with elements:

$$\Lambda_{i\ell}^{j\ell'} = \begin{cases} -\dfrac{\partial s_{j\ell'}}{\partial p_{i\ell}} & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

We can write the entire system of best-response equations in vector form as $\mathbf{s}(\mathbf{p}) - \mathbf{\Lambda}(\mathbf{p}) \cdot (\mathbf{p} - \mathbf{c}) = 0$, or what is equivalent:

$$\mathbf{p} = \mathbf{c} + \mathbf{\Lambda}(\mathbf{p})^{-1} \cdot \mathbf{s}(\mathbf{p}) \tag{7}$$

A spatial Nash-Bertrand equilibrium is then a vector $\mathbf{p}^*$ that solves the fixed-point mapping (7). Given our assumptions on the distribution of consumer taste heterogeneity υ, the mappings $\mathbf{s}(\mathbf{p})$ and $\mathbf{\Lambda}(\mathbf{p})$ are continuously differentiable. Furthermore, it is possible to show that, for every firm i and location ℓ, an equilibrium price $p_{i\ell}^*$ in this game is always greater or equal than the constant

[7] Besides computing equilbrium prices, our Bertrand algorithm computes demand price elasticities for each location and store at these prices. These elasticities help the researcher better understand what are the actual market areas in geographic space. The detection of the relevant geographical market area has long been debated among antitrust authorities (see Willig, 1991, and Baker, 1997).

marginal cost $c_{i\ell}$ and smaller or equal than the equilibrium price at store (i, ℓ) when firm i is a monopolist with stores at every location in the city. Therefore, the vector of prices $\mathbf{p}$ belongs to a compact set: $\mathbf{p}$ belongs to the IL-dimension rectangle $[\mathbf{c}, \mathbf{p}^{Mon}]$. By Brower's fixed-point theorem, a Nash-Bertrand equilibrium exists.

The equilibrium is not necessarily unique. Multiplicity of equilibria may be a problem when we use the model for comparative statics or to evaluate the effects of public policies. A way of dealing with multiplicity of equilibria is to impose an equilibrium selection mechanism, i.e., a criterion that selects a specific type of equilibrium such that when we do comparative statics the same equilibrium type is always selected. To implement an equilibrium selection mechanism in practice, we need an algorithm that can find the specific type of equilibrium for any possible specification of the primitives of the model. We describe here an algorithm with these features that exploits the supermodularity of the this Bertrand game. The algorithm is based on Echenique (2007).

Following Vives (1999, page 32), our Bertrand game is smooth supermodular if it satisfies the following conditions: (1) the space of prices is a compact cube in the Euclidean space; (2) the profit function $R_i()$ is twice continuously differentiable in prices; (3) $\partial^2 R_i / \partial p_{i\ell} p_{jm} \geq 0$ for any $i \neq j$ and any pair of locations ℓ and m; and (4) $\partial^2 R_i / \partial p_{i\ell} p_{im} \geq 0$ for any $\ell \neq m$. As mentioned in the previous paragraph, conditions (1) and (2) are satisfied in this pricing game. A sufficient (but not necessary) condition for (3) and (4) to hold is that the local market shares $\sigma_{i\ell}(z, \mathbf{p})$ are never larger than a half. More generally, this Bertrand game is smooth supermodular if quality differences between firms are not too large and the degree of horizontal product differentiation (μ) is not too small relative to transportation costs (τ).[8]

Based on the smooth supermodularity of this Bertrand game it is simple to impose an equilibrium selection mechanism by exploiting the following Lemma due to Topkis (1979).

LEMMA: Define the best response mapping in vector form, $\mathbf{b}(\mathbf{p}) \equiv \mathbf{c} + \Lambda(\mathbf{p})^{-1} \mathbf{s}(\mathbf{p})$. *Consider the following algorithm (best response function iteration): start with an initial vector of prices* $\mathbf{p}^0$*; at*

[8] For condition (3) we have that for $i \neq j$:

$$\frac{\partial^2 R_i}{\partial p_{i\ell} \partial p_{jm}} = \left[\frac{\partial s_{i\ell}}{\partial p_{jm}} \right] + \left[(p_{i\ell} - c_{i\ell}) \frac{\partial^2 s_{i\ell}}{\partial p_{i\ell} \partial p_{jm}} \right] + \left[\sum_{\ell' \neq \ell} (p_{i\ell'} - c_{i\ell'}) \frac{\partial^2 s_{i\ell'}}{\partial p_{i\ell} \partial p_{jm}} \right]$$

The first and the third terms in brackets are always positive. For the second term, we have that: $\partial^2 s_{i\ell} / \partial p_{i\ell} \partial p_{jm} = \mu^{-1} \int \partial \sigma_{i\ell}(z) / \partial p_{jm} \left(1 - 2\sigma_{i\ell}(z)\right) \phi(dz)$. Since $\partial \sigma_{i\ell}(z) / \partial p_{jm} > 0$, a sufficient condition for this second term to be positive is that, for any location $z \in \mathbb{C}$, the local market shares $\sigma_{i\ell}(z)$ are smaller than $1/2$. This condition holds when qualities are not too large and the degree of horizontal product differentiation μ is not too small relative to transportation costs. However, it is clear that this sufficient condition is far to be necessary. Local market shares greater than $1/2$ are perfectly compatible with a positive value for $\partial^2 s_{i\ell} / \partial p_{i\ell} \partial p_{jm}$. Furthermore, it is clear that the cross-price second derivative of the profit function can be positive when $\partial^2 s_{i\ell} / \partial p_{i\ell} \partial p_{jm}$ is negative just because the other two terms can be larger in absolute value. For condition (4) we have that for $\ell \neq m$:

$$\frac{\partial^2 R_i}{\partial p_{i\ell} \partial p_{im}} = \left[\frac{\partial s_{i\ell}}{\partial p_{im}} + \frac{\partial s_{im}}{\partial p_{i\ell}} \right] + \left[(p_{i\ell} - c_{i\ell}) \frac{\partial^2 s_{i\ell}}{\partial p_{i\ell} \partial p_{im}} + (p_{im} - c_{im}) \frac{\partial^2 s_{im}}{\partial p_{i\ell} \partial p_{im}} \right] + \left[\sum_{\ell' \neq \{\ell, m\}} (p_{i\ell'} - c_{i\ell'}) \frac{\partial^2 s_{i\ell'}}{\partial p_{i\ell} \partial p_{im}} \right]$$

Again, the first and the third terms in brackets are always positive. The second term is also positive under the same conditions as mentioned above.

iteration $k \geq 1$, $\mathbf{p}^k = \mathbf{b}(\mathbf{p}^{k-1})$; stop when $\mathbf{p}^k = \mathbf{p}^{k-1}$. If the game is supermodular then: if we start with $\mathbf{p}^0 = c$, then the algorithm stops at the Nash-Bertrand equilibrium with smallest prices. Let $\mathbf{p}^{low}$ be that equilibrium. Then, if we initialize the algorithm with $\mathbf{p}^0 = \mathbf{p}^{low} + \delta$, where δ is a positive constant arbitrarily close to zero, then the algorithm stops at the Nash-Bertrand equilibrium with second-smallest prices. We can proceed in this way to obtain all the equilibria in this pricing game. Similarly, if we start with $\mathbf{p}^0 = \mathbf{p}^{Mon}$, then the algorithm stops at the Nash-Bertrand equilibrium with largest prices. And if we initialize the algorithm with $\mathbf{p}^0 = \mathbf{p}^{low} - \delta$, then the algorithm stops at the Nash-Bertrand equilibrium with second-largest prices.

Topkis (1979) proved this Lemma for supermodular games, and Topkis (1998) extended the result to a more general class of games with strategic complementarities (GSC).[9] Based on this Lemma, we can use Topkis algorithm to select always the same type of Nash-Bertrand equilibrium, e.g., the equilibrium with minimum prices. The two extremal equilibria coincide with the Pareto best and the Pareto worst equilibria from the point of view of firms (see Vives, 1999, page 152). In the numerical examples in section 4, we use Topkis algorithm to select the Nash-Bertrand equilibrium with smallest prices: the worst equilibria from the point of view of firms.[10]

Let $\mathbf{p}^*(\mathbf{n}, \phi)$ be the vector of equilibrium prices associated with a value $(\mathbf{n}, \phi)$ of the state variables. Solving this vector into the variable profit function one obtains the equilibrium variable profit function:

$$R_i^*(\mathbf{n}, \phi) \equiv R_i(\mathbf{n}, \mathbf{p}^*(\mathbf{n}, \phi), \phi) \tag{8}$$

2.4 Dynamic game

At the end of period t firms simultaneously choose their network of stores $\mathbf{n}_{t+1}$ with an understanding that they will affect their variable profits at future periods. We model the choice of store location as a game of *incomplete information*, so that each firm i has to form beliefs about other firms' choices of networks. More specifically, there are components of the entry costs and exit values of a store which are firm-specific and private information. There are two main reasons why we include incomplete information in our model. First, as shown by Doraszelski and Satterthwaite (2010), in Ericson-Pakes complete-information model of industry dynamics an equilibrium in pure strategies does not necessarily exits. Doraszelski and Satterthwaite also show that the introduction of private information variables with continuous distribution function and large support guarantees the existence of an equilibrium in this class of games of industry dynamics. Second, the recent literature on estimation of dynamic games has also considered games of incomplete information because these variables are convenient sources of unobserved heterogeneity from the point of view of the researcher (see Aguirregabiria and Mira, 2007, or Pakes, Ostrovsky, and Berry, 2007).

We assume that a firm may open or close *at most one store* per period. Given that we can

[9] See also Echenique (2007), who has developed an efficient algorithm to find all the equilibria in GSC.

[10] We can also use the Lemma to check for multiplicity of equilibria. If the smallest equilibrium coincides with the largest equilibrium, then the equilibrium is unique.

make the frequency of firms' decisions arbitrarily high, this is a plausible assumption that reduces significantly the cost of computing an equilibrium in this model. Let a_{it} be the decision of firm i at period t such that: $a_{it} = \ell_+$ represents the decision of opening a new store at location ℓ; $a_{it} = \ell_-$ means that a store at location ℓ is closed; and $a_{it} = 0$ means the firm chooses to do nothing. Therefore, the choice set is $A = \{0, \ell_+, \ell_- : \ell = 1, 2, ..., L\}$. Some of the choice alternatives in A are not be feasible for a firm given its current network n_{it}. In particular, a firm can not close a store in a submarket where it has no stores, and it cannot open a new store in a location where it already has a store. The set of feasible choices for firm i at period t is denoted $A(n_{it})$ such that $A(n_{it}) = \{0\} \cup \{\ell_+ : n_{i\ell} = 0\} \cup \{\ell_- : n_{i\ell} = 1\}$.

We represent the transition rule of market structure as $\mathbf{n}_{t+1} = \mathbf{n}_t + \mathbf{1}[a_t]$, where $\mathbf{1}[a_t]$ is a $IL \times 1$ vector such that its (i, ℓ)-element is equal to $+1$ when $a_{it} = \ell_+$, to -1 when $a_{it} = \ell_-$, and to zero otherwise. That is, the (i, ℓ)-element of $\mathbf{1}[a_t]$ is equal to $1\{a_{it} = \ell_+\} - 1\{a_{it} = \ell_-\}$, where $1\{.\}$ is the indicator function.

2.4.1 Specification of the profit function

Firm i's current profit is:

$$\Pi_{it} \quad = \quad R_i^* (\mathbf{n}_t, \phi_t) - FC_{it} - EC_{it} + EV_{it} \tag{9}$$

FC_{it} is the fixed cost of operating all the stores of firm i. EC_{it} is the entry or set-up cost of a new store. And EV_{it} is the exit value of closing a store. Fixed operating costs depend on the number of stores but also on their location.

$$FC_{it} = \sum_{\ell=1}^{L} \theta_{i\ell}^{FC} \; n_{i\ell t} \tag{10}$$

$\theta_{i\ell}^{FC}$ is the fixed cost of operating a store in submarket ℓ.[11] The specification of entry cost is:

$$EC_{it} \quad = \quad \sum_{\ell=1}^{L} 1\{a_{it} = \ell_+\} \; \left(\theta_{i\ell}^{EC} + \varepsilon_{i\ell t}^{EC}\right) \tag{11}$$

$\theta_{i\ell}^{EC}$ is the entry cost at location ℓ. The variable $\varepsilon_{i\ell t}^{EC}$ represents a firm- and location-specific component of the entry cost. This idiosyncratic shock is private information of firm i. The specification of the exit value is:

$$EV_{it} \quad = \quad \sum_{\ell=1}^{L} 1\{a_{it} = \ell_-\} \; \left(\theta_{i\ell}^{EV} + \varepsilon_{i\ell t}^{EV}\right) \tag{12}$$

$\theta_{i\ell}^{EV}$ is the scrapping or exit value of a store in location ℓ. The variable $\varepsilon_{i\ell t}^{EV}$ is a firm- and location-specific shock in the exit value of a store.

[11] The specification of fixed costs can be extended to take into account that the fixed cost of operating a network of stores may depend on the number of stores (e.g., economies of scale) and on the distance between the stores (e.g., economies of scope). Scope economies may be positively related to the proximity of own-firm stores. This form of scope economies are called *economies of density*. Holmes (2011) studies the role of economies of density in explaining the spatial evolution of Wal-Mart stores since the 1950s.

The vector of private information variables for firm i at period t is $\varepsilon_{it} = \{\varepsilon_{i\ell t}^{EC}, \varepsilon_{i\ell t}^{EV} : \ell = 1, 2, ..., L\}$. We make two assumptions on its distribution. First, we assume that ε_{it} is independent of demand conditions ϕ_t, and independently distributed across firms and over time. Independence across firms implies that a firm cannot learn about other firms' ε's by using its own private information. And independence over time means that a firm cannot use other firms' histories of previous decisions to infer their current ε's. These assumptions simplify significantly the definition and the computation of an equilibrium in this dynamic game. Second, we assume that ε_{it} has a cumulative distribution function $G_i(.)$ that is strictly increasing and continuously differentiable with respect to the Lebesgue measure in $\mathbb{R}^{2L}$. These two assumptions allow for a broad range of specifications for the ε_{it}'s, including spatially correlated shocks.

It will be convenient to distinguish two additive components in the current profit function:

$$\Pi_{it} = \pi_i\left(a_{it}, \mathbf{n}_t, \phi_t\right) + \varepsilon_{it}(a_{it}) \tag{13}$$

where $\pi_i\left(a_{it}, \mathbf{n}_t\right)$ is the current profit function excluding the private information variables, and $\varepsilon_{it}(a_{it})$ represents the private information shock associated with action a_{it}.

2.4.2 Markov Perfect Equilibrium

We consider that a firm's strategy depends only on its payoff relevant state variables $(\mathbf{n}_t, \phi_t, \varepsilon_{it})$. For the sake of notational simplicity, hereinafter we omit the state of the demand ϕ_t as an argument of the different functions. Let $\alpha \equiv \{\alpha_i(\mathbf{n}_t, \varepsilon_{it}) : i \in \Upsilon\}$ be a set of strategy functions, one for each firm, such that α_i is a function from $\{0,1\}^{IL} \times \mathbb{R}^{2L}$ into A. Given a set of strategy functions α, we can define a value function $V_i^{\alpha}(\mathbf{n}_t, \varepsilon_{it})$ that represents the value of firm i given that the other firms behave according to their strategy functions in α and firm i responds optimally. The value function V_i^{α} is the unique solution of the following Bellman equation:

$$V_i^{\alpha}(\mathbf{n}_t, \varepsilon_{it}) \quad = \max_{a_{it} \in A(n_{it})} \left\{ \; v_i^{\alpha}(a_{it}, \mathbf{n}_t) + \varepsilon_{it}(a_{it}) \; \right\} \tag{14}$$

where the functions $v_i^{\alpha}(a_{it}, \mathbf{n}_t)$ are *choice specific value functions* which are defined as:

$$\begin{aligned} v_i^{\alpha}(a_{it}, \mathbf{n}_t) \quad &\equiv \quad \pi_i\left(a_{it}, \mathbf{n}_t\right) \\ &+ \quad \beta \int V_i^{\alpha}\left(\mathbf{n}_t + \mathbf{1}[a_{it}, \alpha_{-i}(\mathbf{n}_t, \varepsilon_{-it})], \varepsilon_{i,t+1}\right) dG_i(\varepsilon_{i,t+1}) \left[\prod_{j \neq i} dG_j(\varepsilon_{jt})\right] \end{aligned} \tag{15}$$

Similarly, we can define firm i's best response function, $\alpha_i^{BR}(\mathbf{n}_t, \varepsilon_{it}, \alpha_{-i})$, as:

$$\alpha_i^{BR}(\mathbf{n}_t, \varepsilon_{it}, \alpha_{-i}) \quad = \quad \arg \max_{a_{it} \in A(n_{it})} \left\{ \; v_i^{\alpha}(a_{it}, \mathbf{n}_t) + \varepsilon_{it}(a_{it}) \; \right\} \tag{16}$$

A Markov perfect equilibrium (MPE) in this game is a set of strategy functions such that each firm's strategy maximizes the value of the firm for each possible $(\mathbf{n}_t, \varepsilon_{it})$ and taking other firms' strategies as given.

DEFINITION: A set of strategy functions $\alpha^* \equiv \{\alpha_i^*(\mathbf{n}_t, \varepsilon_{it}) : i \in \Upsilon\}$ is a MPE if and only if for any firm i and any state $(\mathbf{n}_t, \varepsilon_{it})$ we have that:

$$\alpha_i^*(\mathbf{n}_t, \varepsilon_{it}) = \alpha_i^{BR}(\mathbf{n}_t, \varepsilon_{it}; \alpha_{-i}^*) \tag{17}$$

Next, we follow Aguirregabiria and Mira (2007, pp. 7-13) to represent a MPE as a fixed point in a space of choice probabilities. The algorithm that we use to compute an equilibrium (in section 3.2) uses this representation. We start defining three objects: conditional choice probabilities; integrated value function; and best response probability function.

Conditional choice probabilities (CCPs). Given any set of strategy functions α, we can define a set of conditional choice probabilities $\mathbf{P}^\alpha \equiv \{P_i^\alpha(a_{it}|\mathbf{n}_t) : i \in \Upsilon; a_{it} \in A; \mathbf{n}_t \in \{0,1\}^{IL}\}$ such that

$$P_i^\alpha(a_{it}|\mathbf{n}_t) \equiv \Pr(\alpha_i(\mathbf{n}_t, \varepsilon_{it}) = a_{it}|\mathbf{n}_t) = \int 1\{\alpha_i(\mathbf{n}_t, \varepsilon_{it}) = a_{it}\}\ dG_i(\varepsilon_{it}) \tag{18}$$

The probabilities in $\mathbf{P}^\alpha$ represent firms' expected behavior, from the point of view of the competitors, when firms follow their respective strategies in α. Given that a_{it} and $\mathbf{n}_t$ are discrete variables with finite support, $\mathbf{P}^\alpha$ is a vector in an Euclidean space of finite dimension.[12] More precisely, $\mathbf{P}^\alpha \in [0,1]^D$ where $D = I * L * 2^{IL}$ is the number of free probabilities in the vector $\mathbf{P}^\alpha$.

Define the *integrated value function* $\bar{V}_i^\alpha(\mathbf{n}_t) \equiv \int V_i^\alpha(\mathbf{n}_t, \varepsilon_{it}) dG_i(\varepsilon_{it})$. Applying the definitions of CCPs and integrated value function to the Bellman equation in (14)-(15) we get the following *integrated Bellman equation*:

$$\bar{V}_i^\alpha(\mathbf{n}_t) \;=\; \int \max_{a_{it}} \left\{ \pi_i\left(a_{it}, \mathbf{n}_t\right) + \varepsilon_{it}(a_{it}) + \beta \sum_{a_{-it}} \bar{V}_i^\alpha\left(\mathbf{n}_t + \mathbf{1}[a_{it}, a_{-it}]\right) \left[\prod_{j \neq i} P_j^\alpha(a_{jt}|\mathbf{n}_t)\right] \right\} dG_i(\varepsilon_{it}) \tag{19}$$

The integrated value function $\bar{V}_i^\alpha$ is the unique fixed point of this Bellman equation. Notice that the fixed point mapping that defines $\bar{V}_i^\alpha$ depends on firms' strategies only through the vector of choice probabilities $\mathbf{P}^\alpha$. To emphasize this point and to define a MPE in probability space, we change the notation slightly and use the symbol $\bar{V}_i^{\mathbf{P}}$ instead of $\bar{V}_i^\alpha$ to denote the integrated value function. For the same reason, we use $v_i^{\mathbf{P}}(a_{it}, \mathbf{n}_t)$ to represent the choice-specific value functions, which can be written as:

$$v_i^{\mathbf{P}}(a_{it}, \mathbf{n}_t) \equiv \pi_i\left(a_{it}, \mathbf{n}_t\right) + \beta \sum_{a_{-it}} \bar{V}_i^{\mathbf{P}}\left(\mathbf{n}_t + \mathbf{1}[a_{it}, a_{-it}]\right) \left[\prod_{j \neq i} P_j^\alpha(a_{jt}|\mathbf{n}_t)\right] \tag{20}$$

Given these value functions, we can re-write the best response function as: $\alpha_i^{BR}(\mathbf{n}_t, \varepsilon_{it}, \mathbf{P}) = \arg\max_{a_{it} \in A(n_{it})}\{v_i^{\mathbf{P}}(a_{it}, \mathbf{n}_t) + \varepsilon_{it}(a_{it})\}$. Note that we have replaced α_{-i} by $\mathbf{P}$ as an argument in the best response function. This is because CCPs contain all the information about competitors' strategies that a firm needs to construct his best response.

[12] Note that we have assumed that ϕ_t can take only a finite number of values. Therefore, $P_i^\alpha(a_{it}|\mathbf{n}_t, \phi_t)$ also belongs to a finite-dimension Euclidean space.

The *best response probability function,* $\Psi_i(a_{it}|\mathbf{n}_t, \mathbf{P})$, is the probability that action a_{it} is firm i's best response given that the state of the market is $\mathbf{n}_t$ and the other firms behave according to their choice probabilities in $\mathbf{P}$. It is the best response function α_i^{BR} integrated over the distribution of private information variables.

$$\Psi_i(a_{it}|\mathbf{n}_t, \mathbf{P}) \equiv \int 1\left\{a_{it} = \arg\max_{a \in A(n_{it})} \left\{\ v_i^{\mathbf{P}}(a, \mathbf{n}_t) + \varepsilon_{it}(a)\ \right\}\right\} dG_i(\varepsilon_{it}) \tag{21}$$

This function maps CCPs into CCPs. The best response probability function in vector form is $\mathbf{\Psi}(\mathbf{P}) = \{\Psi_i(a_{it}|\mathbf{n}_t, \mathbf{P}) : (i, a_{it}, \mathbf{n}_t) \in \Upsilon \times A \times \{0,1\}^{IL}\}$.

Let α^* be a set of MPE strategies and let $\mathbf{P}^*$ be the vector of CCPs associated to α^*. Using the previous definitions it is simple to verify that $\mathbf{P}^*$ should be a fixed point of the mapping $\mathbf{\Psi}$. Inversely, let $\mathbf{P}^*$ be a fixed point of the mapping $\mathbf{\Psi}$, and define the set of strategy functions α^* with $\alpha_i^*(\mathbf{n}_t, \varepsilon_{it}) = \arg\max_{a_{it}}\{v_i^{\mathbf{P}^*}(a_{it}, \mathbf{n}_t) + \varepsilon_{it}(a_{it})\}$. Then, it is also simple to verify that α^* is a MPE (see Aguirregabiria and Mira, 2007, for further details). Therefore, we can represent any MPE in this model as a fixed point of the best response probability mapping. Equilibrium probabilities solve the coupled fixed-point problems defined by equations (19), (20) and (21). Given a vector of probabilities $\mathbf{P}$, we obtain value functions $\bar{V}_i^{\mathbf{P}}$ as solutions of the I Bellman equations in (19), and given these value functions, we obtain best response probabilities using (21).

Given this representation of an equilibrium, the proof of existence of a MPE is a straightforward application of Brower's theorem. The distribution $G_i(.)$ has support over the entire $\mathbb{R}^{2L}$ and it is continuous and strictly increasing with respect to every argument. This implies that the fixed-point mapping $\mathbf{\Psi}$ is continuous on the compact set $[0,1]^D$. Thus, by Brower's theorem, an equilibrium exits.

EXAMPLE: The functional forms of the integrated Bellman equation and of the best response probability mapping depend on the distribution of the private information variables. A special case in which these functions have close form expressions is when the private information variables have a type 1 extreme value distribution. Suppose that the private information shocks $\{\varepsilon_{it}(a) : a \in A\}$ are independently and identically distributed over (i,t,a) with type 1 extreme value distribution. Then, the integrated Bellman equation is:

$$\bar{V}_i^{\mathbf{P}}(\mathbf{n}_t) \quad = \quad \log\left(\sum_{a_{it} \in A(n_{it})} \exp\left\{v_i^{\mathbf{P}}(a_{it}, \mathbf{n}_t)\right\}\right) \tag{22}$$

And the best response probability function is:

$$\Psi_i(a_{it}|\mathbf{n}_t, \mathbf{P}) = \frac{\exp\left\{v_i^{\mathbf{P}}(a_{it}, \mathbf{n}_t)\right\}}{\sum_{a \in A(n_{it})} \exp\left\{v_i^{\mathbf{P}}(a, \mathbf{n}_t)\right\}} \tag{23}$$

The iid extreme value distribution is restrictive because it implies no spatial correlation between private information shocks. However, it is very convenient from a computational point of view because it avoids multiple integration over the space of ε_{it}. ■

This dynamic game can have multiple equilibria. This is an issue when we use this model for comparative statics. In principle, we could deal with multiplicity by imposing an equilibrium selection mechanism such that, for different values of the model parameters, the same equilibrium type is always selected . We illustrated in section 2.3 how imposing an equilibrium selection mechanism is relatively simple in static supermodular games. Unfortunately, it is generally difficult to establish supermodularity in dynamic games.[13]

To deal with the problem of multiple equilibria when doing comparative statics with this the dynamic game, we apply the homotopy method proposed in Aguirregabiria (2012). The method is such that it guarantees that the same 'type' of equilibrium is selected for the two values of the structural parameters for which we want to make the comparative statics exercise. We briefly describe this method in section 3.4 below.

3 Algorithms for solution and comparative statics

3.1 Computation of a Nash-Bertrand equilibrium

To compute a Nash-Bertrand equilibrium of the static pricing game, we iterate in the best response function. More specifically, we use a *Gauss-Seidel* version of the algorithm that iterates in the best response mapping, such that players take turns in best-responding instead of jointly best-responding in each iteration. Topkis (1998) has shown that the Lemma that we have presented in section 2.3 also applies to the Gauss-Seidel version of the algorithm. In fact, Topkis shows that for supermodular games the Gauss-Seidel algorithm is faster (see also Echenique, 2007).

For a given value of the state variables, we have defined the best response mapping as $\mathbf{b}(\mathbf{p}) \equiv \mathbf{c}+\mathbf{\Lambda}(\mathbf{p})^{-1}\cdot\mathbf{s}(\mathbf{p})$. Let $\mathbf{b}_{(i)}(\mathbf{p})$ be the elements of $\mathbf{b}(\mathbf{p})$ associated with the prices of firm i. Similarly, let $\mathbf{p}_{(i)}$ be the elements of the vector $\mathbf{p}$ associated with firm i. To obtain the equilibrium with smallest prices we initialize the algorithm with prices equal to marginal costs. *Step 0:* Start with the vector of prices $\mathbf{p}^0$ such that $\mathbf{p}^0_{(i)} = c_i$ for any $i \in \Upsilon$. *Step 1:* Compute aggregate demands $\mathbf{s}\left(\mathbf{p}^0\right)$ and the matrix of partial derivatives $\Lambda\left(\mathbf{p}^0\right)$ using quadrature integration (see below). *Step 2 (Gauss-Seidel iteration):* Starting with firm 1, obtain a new vector $\mathbf{p}^1_{(1)}$ as $\mathbf{p}^1_{(1)} = \mathbf{b}_{(1)}\left(\mathbf{p}^0\right)$. Then, for firm 2 $\mathbf{p}^1_{(2)} = \mathbf{b}_{(2)}\left(\mathbf{p}^1_{(1)}, \mathbf{p}^0_{(2)}, ..., \mathbf{p}^0_{(I)}\right)$, and so on for firm i $\mathbf{p}^1_{(i)} = \mathbf{b}_{(i)}\left(\mathbf{p}^1_{(1)}, ..., \mathbf{p}^1_{(i-1)}, \mathbf{p}^0_{(i)}, ..., \mathbf{p}^0_{(I)}\right)$. *Step 3:* If $\left\|\mathbf{p}^1 - \mathbf{p}^0\right\|$ is smaller than a pre-fixed small constant, then $\mathbf{p}^* = \mathbf{p}^1$. Otherwise, proceed to step 1 with $\mathbf{p}^0 = \mathbf{p}^1$. Once the price equilibrium is computed, we encode the equilibrium current variable profits of a firm given a particular state, $R^*_i(\mathbf{n}, \phi)$.

Given the logit assumption on the idiosyncratic tastes, the local demands have the closed form expression in (2). However, to obtain the vector of aggregate demands $\mathbf{s}(\mathbf{p})$ and the matrix of partial derivatives $\Lambda(\mathbf{p})$ we have to integrate local demands over consumers' addresses in the 2-dimensional city $\mathbb{C}$. We use a quadrature method with midpoint nodes (see Judd, 1998, ch. 7).

[13] Curtat (1996) provides a useful overview of the problem. More recently, Bernstein and Kök (2009) have obtained conditions for the supermodularity of a dynamic game of innovation and cost reduction in assembly networks.

We first divide $\mathbb{C}$ into a pre-specified number of mutually exclusive and adjacent rectangular cells, with each cell k having a representative node point $z_{(k)}$ in its center. For each location z in cell k we approximate the local demand $\sigma_{i\ell}(z, \mathbf{n}_t, \mathbf{p}_t)$ and the density $\phi_t(z)$ using $\sigma_{i\ell}(z_{(k)}, \mathbf{n}_t, \mathbf{p}_t)$ and $\phi_t\left(z_{(k)}\right)$, respectively. Therefore, we calculate aggregate demand for store (i, ℓ) as:

$$s_{i\ell}(\mathbf{n}_t, \mathbf{p}_t, \phi_t) = \sum\nolimits_k \sigma_{i\ell}(z_{(k)}, \mathbf{n}_t, \mathbf{p}_t)\ \phi_t\left(z_{(k)}\right)\ area(k) \tag{24}$$

where $area(k)$ is the area of the rectangular cell k.

3.2 Computation of a MPE

Consider the example where the private information variables are extreme value distributed. A MPE is a vector of probabilities $\mathbf{P}^*$ such that $\mathbf{P}^* = \mathbf{\Psi}(\mathbf{P}^*)$, where the fixed-point mapping $\mathbf{\Psi}(\mathbf{P})$ is $\{\Psi_i(a_{it}|\mathbf{n}_t, \mathbf{P}) : (i, a_{it}, \mathbf{n}_t) \in \Upsilon \times A \times \{0,1\}^{IL}\}$ with

$$\Psi_i(a_{it}|\mathbf{n}_t, \mathbf{P}) = \frac{\exp\left\{\pi_i\left(a_{it}, \mathbf{n}_t\right) + \beta \sum\limits_{a_{-it}} \bar{V}_i^{\mathbf{P}}\left(\mathbf{n}_t + \mathbf{1}[a_{it}, a_{-it}]\right)\left[\prod\limits_{j \neq i} P_j(a_{jt}|\mathbf{n}_t)\right]\right\}}{\sum\limits_a \exp\left\{\pi_i\left(a, \mathbf{n}_t\right) + \beta \sum\limits_{a_{-it}} \bar{V}_i^{\mathbf{P}}\left(\mathbf{n}_t + \mathbf{1}[a, a_{-it}]\right)\left[\prod\limits_{j \neq i} P_j(a_{jt}|\mathbf{n}_t)\right]\right\}} \tag{25}$$

and the value function $\bar{V}_i^{\mathbf{P}}$ solves the Bellman equation

$$\bar{V}_i^{\mathbf{P}}(\mathbf{n}_t) \quad = \quad \log\left(\sum_{a_{it}} \exp\left\{\pi_i\left(a_{it}, \mathbf{n}_t\right) + \beta \sum_{a_{-it}} \bar{V}_i^{\mathbf{P}}\left(\mathbf{n}_t + \mathbf{1}[a_{it}, a_{-it}]\right)\left[\prod_{j \neq i} P_j(a_{jt}|\mathbf{n}_t)\right]\right\}\right) \tag{26}$$

To obtain a MPE we iterate in the best response function $\mathbf{\Psi}$ using Gauss-Seidel iterations. The algorithm proceeds as follows. *Step 0:* Initialize the algorithm with a vector of probabilities $\mathbf{P}^0$. *Step 1:* Starting with firm 1, and given $\left(\mathbf{P}_2^0, \mathbf{P}_3^0, ..., \mathbf{P}_I^0\right)$ fixed, we obtain the value function $\bar{V}_1^{\mathbf{P}^0}$ by applying value function iterations in the Bellman equation (26). Given $\bar{V}_1^{\mathbf{P}^0}$, we use the best response probability mapping in (25) to obtain a new vector of CCPs for firm 1: $\mathbf{P}_1^1 = \{\Psi_1(a_{it}|\mathbf{n}_t, \mathbf{P}^0)\}$. Then, we proceed with firm 2. Given $\left(\mathbf{P}_1^1, \mathbf{P}_3^0, ..., \mathbf{P}_I^0\right)$ fixed, we obtain the value function $\bar{V}_2^{\left(\mathbf{P}_1^1, \mathbf{P}_3^0, ..., \mathbf{P}_I^0\right)}$ by using value function iterations in the Bellman equation (26). Then, we update firm 2's CCPs as $\mathbf{P}_2^1 = \{\Psi_2(a_{it}|\mathbf{n}_t, \mathbf{P}_1^1, \mathbf{P}_3^0, ..., \mathbf{P}_I^0)\}$. We proceed in this way to update the CCPs of the I firms. *Step 3:* If $\left\|\mathbf{P}^1 - \mathbf{P}^0\right\|$ is smaller than a pre-fixed small constant, then $\mathbf{P}^* = \mathbf{P}^1$. Otherwise, proceed to step 1with $\mathbf{P}^0 = \mathbf{P}^1$.

The most serious burden for the computation of an equilibrium in our model comes from the space-memory requirements.[14] Value functions and choice probabilities should be stored in high-speed memory because they are required for value function iteration and for the calculation of

[14]Under the assumption that firms cannot open or close more than one store per period, the computation of the expected value of next period value function is not a serious computational issue if the number of firms is small. To calculate the expected value $\sum_{a_{-it}} \bar{V}_i^{\mathbf{P}}(\mathbf{n}_t + \mathbf{1}[a_{it}, a_{-it}]) \prod_{j \neq i} P_j(a_{jt}|\mathbf{n}_t)$, we have to perform only $(1+L)^I$ sums and products, instead of the much larger number of 2^{IL} operations which are required in the general case. For instance, in a duopoly model with 40 locations we have that $2^{IL} \simeq 10^{12}$ and $(1+L)^I = 1681$.

best response probabilities. Given that the vector of state variables $\mathbf{n}_t$ is discrete, value functions and choice probability functions can be described as vectors in Euclidean spaces: i.e., $\bar{V}_i^{\mathbf{P}} \in \mathbb{R}^{2^{IL}}$ and $\mathbf{P} \in [0,1]^{IL(2^{IL})}$. In applications with many locations (or firms), the dimension of these Euclidean spaces can be very large. For instance, in a duopoly model with 40 locations we have that $2^{IL} \simeq 10^{12}$. This magnitude of memory space is rarely available.

There are two general approaches to deal with this computational issue. One is to approximate the value function using a parametric family of surfaces, such as polynomials or nonlinear basis functions derived from neural networks (see Bertsekas and Tsitsiklis, 1996). The other approach is to store $\bar{V}_i^{\mathbf{P}}$ and $\mathbf{P}$ only over a subset of the state space and use interpolation to obtain values of these functions at other points. In this paper we consider the interpolation approach.[15]

Let $S = \{\mathbf{n}_1, \mathbf{n}_2, ..., \mathbf{n}_{|S|}\}$ be a subset of the actual state space $\{0,1\}^{IL}$. The number of elements in the subset S is given by the amount of high-speed memory in our computer. The selection of the grid points in S can be done in different ways. For instance, we can take $|S|$ random draws from a uniform distribution over the set $\{0,1\}^{IL}$. Let $\bar{\mathbf{V}}_i^{\mathbf{P|S}}$ be a vector of values restricted to the subset S: i.e., $\bar{\mathbf{V}}_i^{\mathbf{P|S}} = \{\bar{V}_i^{\mathbf{P|S}}(\mathbf{n}_t) : \mathbf{n}_t \in S\}$. The vector $\bar{\mathbf{V}}_i^{\mathbf{P|S}}$ is the unique fixed point of the following Bellman equation: for any $\mathbf{n}_t \in S$,

$$\bar{V}_i^{\mathbf{P|S}}(\mathbf{n}_t) \quad = \quad \log\left(\sum_{a_{it}} \exp\left\{\pi_i\left(a_{it}, \mathbf{n}_t\right) + \beta \sum_{a_{-it}} \Gamma_i^{\bar{\mathbf{V}}^{\mathbf{P|S}}}\left(\mathbf{n}_t + \mathbf{1}[a_{it}, a_{-it}]\right)\left[\prod_{j \neq i} P_j(a_{jt}|\mathbf{n}_t)\right]\right\}\right) \tag{27}$$

where $\Gamma_i^{\bar{\mathbf{V}}^{\mathbf{P|S}}}(\mathbf{n})$ is the interpolation function. Different interpolating functions may be considered. For instance,

$$\Gamma_i^{\bar{\mathbf{V}}^{\mathbf{P|S}}}(\mathbf{n}) = \begin{cases} \bar{V}_i^{\mathbf{P|S}}(\mathbf{n}) & if \quad \mathbf{n} \in S \\ \\ \gamma_i^{(0)} + \sum_{\ell=1}^{L}\left(\sum_{j=1}^{I} \gamma_{ij\ell}^{(1)}\, n_{j\ell} + \sum_{j \neq i} \gamma_{ij\ell}^{(2)}\, n_{i\ell}\, n_{j\ell}\right) & if \quad \mathbf{n} \notin S \end{cases} \tag{28}$$

where $\gamma' s$ are parameters which are obtained by an OLS regression of $\bar{V}_i^{\mathbf{P|S}}(\mathbf{n})$ on $\{n_{j\ell},\, n_{i\ell} * n_{j\ell}\}$ for values of $\mathbf{n}$ in the set S. Note that the total number of γ parameters is $I(1 + L(2I-1))$, and the total memory requirements to store the value function $\bar{V}_i^{\mathbf{P|S}}(.)$ is $|S| + I(1 + L(2I-1))$. Given the vectors $\bar{\mathbf{V}}_i^{\mathbf{P|S}}$ for every firm i, an approximation to the MPE is a vector of probabilities $\mathbf{P}^* = \{P_i(a_{it}|\mathbf{n}_t) : (i, a_{it}, \mathbf{n}_t) \in \Upsilon \times A \times S\}$ such that $\mathbf{P}^* = \mathbf{\Psi}^{(S)}(\mathbf{P}^*)$, where the fixed-point mapping $\mathbf{\Psi}^{(S)}(\mathbf{P})$ is $\{\Psi_i(a_{it}|\mathbf{n}_t, \mathbf{P}) : (i, a_{it}, \mathbf{n}_t) \in \Upsilon \times A \times S\}$. That is, the vector of choice probabilities and the equilibrium mapping are restricted to the subspace S of the state space.

We have assumed that private information variables are independently extreme value distributed. This assumption is made for convenience because it avoids numerical integration to calculate

[15] This interpolation approach goes back at least to Larson and Casti (1982). More recently, Rust (1997) has proposed a method of interpolation that exploits randomization in the selection of the grid points. See Rust (1996) for an excellent survey on numerical methods for dynamic programming that includes a discussion of interpolation techniques.

the multiple integral $\int V_i^{\mathbf{P}}(\mathbf{n}_t, \varepsilon_{it}) dG_i(\varepsilon_{it})$. However, the assumption of no spatial correlation between shocks at different locations is not innocuous. For instance, this correlation can generate spatial agglomeration of stores. Relaxing the extreme value assumption requires one to use simulation techniques to approximate multiple dimensional integrals. See Bajari, Benkard and Levin (2007) and Ryan (2012) for the application of simulation techniques in the context of dynamic oligopoly games.

3.3 Comparative statics

In this subsection we describe how to apply the approach in Aguirregabiria (2012) to our model. Let $\boldsymbol{\theta}$ be the vector of structural parameters of the model, and let include this vector explicitly as an argument in the equilibrium mapping, $\Psi(\mathbf{P}, \boldsymbol{\theta})$. An equilibrium of the model associated with $\boldsymbol{\theta}$ is a solution to the fixed-point problem $\mathbf{P} = \Psi(\mathbf{P}, \boldsymbol{\theta})$. Let $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_*$ be two values of $\boldsymbol{\theta}$. We want to study how the equilibrium of the model, describe by $\mathbf{P}$, changes when we change structural parameters from $\boldsymbol{\theta}_0$ to $\boldsymbol{\theta}_*$ but *keeping fixed the type of equilibrium.* The later condition is key. In the comparative statics exercise that we are interested in here, we control for the type of equilibrium. Given that we do not model any equilibrium selection mechanism, our comparative statics do not consider the possibility of switching from one type to other type of equilibrium.

Suppose that the model has a countable set of equilibrium 'types' that we index by $k \in \{1, 2, ...\}$. Let $\Pi_k(\boldsymbol{\theta})$ be the vector of choice probabilities that represents equilibrium type k when the vector of parameters is $\boldsymbol{\theta}$, such that it satisfies the equilibrium restrictions $\Pi_k(\boldsymbol{\theta}) = \Psi(\Pi_k(\boldsymbol{\theta}), \boldsymbol{\theta})$. Note that some equilibrium types may exist only for a subset of points in the parameter space Θ. Under our conditions on the mapping $\Psi(\mathbf{P}, \boldsymbol{\theta})$, the equilibrium probability functions $\Pi_k(\boldsymbol{\theta})$ are continuous in $\boldsymbol{\theta}$. Therefore, the implementation of the comparative statics exercise consists in computing the equilibrium probablity-vectors $\mathbf{P}_0 \equiv \Pi_k(\boldsymbol{\theta}_0)$ and $\mathbf{P}_* \equiv \Pi_k(\boldsymbol{\theta}_*)$ for some *common* equilibrium type k selected by the researcher.

Suppose that we have computed $\mathbf{P}_0$ using the algorithm described above and some arbitrary initial values of firms' choice probabilities. The challenge is how to implement a procedure that selects the same type of equilibrium for $\boldsymbol{\theta}_*$. Under the condition that the Jacobian matrix $\mathbf{I} - \partial\Psi(\mathbf{P}_0, \boldsymbol{\theta}_0)/\partial\mathbf{P}'$ is non-singular, the following is a first order Taylor approximation to $\Pi_k(\boldsymbol{\theta}_*)$:

$$\Pi_k(\boldsymbol{\theta}_*) \simeq \mathbf{P}_0 + \left(\mathbf{I} - \frac{\partial\Psi(\mathbf{P}_0, \boldsymbol{\theta}_0)}{\partial\mathbf{P}'}\right)^{-1} \frac{\partial\Psi(\mathbf{P}_0, \boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}'} \left(\boldsymbol{\theta}_* - \boldsymbol{\theta}_0\right) \tag{29}$$

When $\| \boldsymbol{\theta}_* - \boldsymbol{\theta}_0 \|^2$ is small, the right-hand-side of this expression provides a good approximation to $\Pi_k(\boldsymbol{\theta}_*)$. To improve the accuracy of this approximation, we may combine this approach with iterations in the equilibrium mapping. Suppose that the equilibrium type k is Lyapunov stable, i.e., the Jacobian matrix $\partial\Psi(\Pi_k(\boldsymbol{\theta}), \boldsymbol{\theta})/\partial\mathbf{P}'$ has all its eigenvalues in the unit circle. This implies that there is a neighborhood of $\mathbf{P}_*$, say $\mathcal{N}$, such that if we iterate in the equilibrium mapping $\Psi(., \boldsymbol{\theta}_*)$ starting with a $\mathbf{P} \in \mathcal{N}$, then we converge to $\mathbf{P}_*$. The neighborhood $\mathcal{N}$ is called the *dominion of attraction* of the stable equilibrium $\mathbf{P}_*$. Suppose that the Taylor approximation is precise enough such that it belongs to the dominion of attraction of $\mathbf{P}_*$. Then, by iterating in the equilibrium mapping $\Psi(., \boldsymbol{\theta}_*)$ starting with the Taylor approximation we will obtain the equilibrium $\Pi_k(\boldsymbol{\theta}_*)$.

4 Structural estimation

The parameters of the model can be consistently estimated even if we have data only from a single city. That is, we can apply law of large numbers and central limit theorems as the number of locations L within the city goes to infinity, and the number of firms and time periods is finite. Suppose that the researcher has access to a panel dataset that includes the following information: (1) firms' store networks in a city over T periods of time, where T is small (i.e., it could be even $T = 2$), $\{n_{i\ell t}\}$ for every (i, ℓ) and $t = \{1, ..., T\}$; (2) population, ϕ_ℓ, consumer demographics and exogenous socioeconomic variables at the locations level, x_ℓ; and (3) prices and quantities $\{p_{i\ell t}, q_{i\ell t}\}$ for every active store (i, ℓ) and period t.

To complete the specification of the econometric model, we need to make some assumptions on the primitives $\omega_{i\ell}$, $c_{i\ell}$, $\theta_{i\ell}^{FC}$, $\theta_{i\ell}^{EC}$, and $\theta_{i\ell}^{EV}$. We assume that these primitives vary freely across firms, but they vary over locations only according to the observable socioeconomic variables in x_ℓ.

$$\omega_{i\ell} = \theta_i^\omega x_\ell;\ c_{i\ell} = \theta_i^c x_\ell;\ \theta_{i\ell}^{FC} = \theta_i^{FC} x_\ell;\ \theta_{i\ell}^{EC} = \theta_i^{EC} x_\ell \tag{30}$$

where θ_i^ω, θ_i^c, θ_i^{FC}, θ_i^{EC}, and θ_i^{EV} are vectors of parameters. This assumption rules out the possibility of unobserved heterogeneity across locations ℓ, and therefore it may be a strong assumption in applications without a rich set of observables in x_ℓ.[16]

Let $\mathbf{x}$ be the vector$\{x_\ell : \ell = 1, 2, ..., L\}$, that is a description of the "landscape" of observable socioeconomic characteristics in the city. All the analysis is conditional on $\mathbf{x}$. Given $\mathbf{x}$, we can interpret the store networks in $\mathbf{n}$ as a single realization of a spatial stochastic process. In terms of the econometric analysis, this has similarities with time series econometrics in the sense that a time series is a single realization from a stochastic process. Though we observe a single realization of this stochastic process, we can still estimate consistently the parameters of that process as the number of locations L goes to infinity.

For estimation purposes, it is convenient to distinguish three subvectors in $\boldsymbol{\theta}$. That is, $\boldsymbol{\theta} = \{\boldsymbol{\theta}_D$, $\boldsymbol{\theta}_{MC}$, $\boldsymbol{\theta}_{FC}\}$ where $\boldsymbol{\theta}_D$ represents demand parameters $\{\mu,\ \tau,\ \theta_i^\omega : i = 1, 2, ..., I\}$, $\boldsymbol{\theta}_{MC}$ contains parameters in marginal costs $\{\theta_i^c : i\}$, and $\boldsymbol{\theta}_{FC}$ contains parameters in fixed costs, entry costs, and exit values, $\{\theta_i^{FC},\ \theta_i^{EC},\ \theta_i^{EV} : i\}$.

The estimation of parameters in demand and marginal costs can follow the well-known approach in Berry, Levinsohn, and Pakes (1996) and Nevo (2001) for the estimation of demand of differentiated products, i.e., the so called BLP method. In fact, given the assumption of no unobserved location heterogeneity in $\omega_{i\ell}$, the estimation of $\boldsymbol{\theta}_D$ can be much simpler than in the BLP approach because there is no endogeneity of prices and no need to invert the relationship between market

[16] This is a common assumption in some empirical applications, such as Seim (2006).

shares and average utilities. Therefore, a possible estimator of $\boldsymbol{\theta}_D$ is a Nonlinear Least Squares estimator:

$$\widehat{\boldsymbol{\theta}}_D = \arg\min_{\boldsymbol{\theta}_D} \sum_{i=1}^{I} \sum_{\ell=1}^{L} \left[S_{i\ell} - \sum\nolimits_k \sigma_{i\ell}(z_{(k)}, \mathbf{n}, \mathbf{p}; \boldsymbol{\theta}_D)\ \phi\left(z_{(k)}\right) \right]^2 \tag{31}$$

where $S_{i\ell}$ is the observed market share of store (i, ℓ) (equals zero if that store does not exits), and $\sigma_{i\ell}(z_{(k)}, \mathbf{n}, \mathbf{p}; \boldsymbol{\theta}_D)$ is the multinomial logit probability that provides the proportion of consumers in location $z_{(k)}$ patronizing store (i, ℓ). Given this estimation of demand, marginal costs parameters in $\boldsymbol{\theta}_{MC}$ can be also estimated by least squares using the marginal conditions of optimality for prices in equation (5). And given consistent estimators $\widehat{\boldsymbol{\theta}}_D$ and $\widehat{\boldsymbol{\theta}}_{MC}$, and the algorithm for computing the Bertrand equilibrium, we can construct estimates of equilibrium variable profits, $\widehat{R}_i(\mathbf{n}_t)$, for any observed or hypothetical market structure $\mathbf{n}_t$.

For the estimation of $\boldsymbol{\theta}_{FC}$, we exploit the restrictions impose by the equilibrium of the dynamic game. For arbitrary values of CCPs and $\boldsymbol{\theta}$, we can define the pseudo likelihood function:

$$\begin{aligned} Q(\mathbf{P}, \boldsymbol{\theta}) \quad = & \quad \sum_{i,t,\ell} 1\{n_{i\ell t} = n_{i\ell t-1}\}\ \ln \Psi_i(0|\mathbf{n}_t, \mathbf{P}, \boldsymbol{\theta}) \\ + & \quad \sum_{i,t,\ell} 1\{n_{i\ell t-1} = 0,\, n_{i\ell t} = 1\}\ \ln \Psi_i(\ell_+|\mathbf{n}_t, \mathbf{P}, \boldsymbol{\theta}) \\ + & \quad \sum_{i,t,\ell} 1\{n_{i\ell t-1} = 1,\, n_{i\ell t} = 0\}\ \ln \Psi_i(\ell_-|\mathbf{n}_t, \mathbf{P}, \boldsymbol{\theta}) \end{aligned} \tag{32}$$

A consistent estimator of $\boldsymbol{\theta}_{FC}$ can be obtained by maximizing the pseudo likelihood function $Q(\mathbf{P}, \boldsymbol{\theta}_{FC}, \widehat{\boldsymbol{\theta}}_D, \widehat{\boldsymbol{\theta}}_{MC})$ with respect to $(\mathbf{P}, \boldsymbol{\theta}_{FC})$ and to the equilibrium constraints $\mathbf{P} = \Psi(\mathbf{P}, \boldsymbol{\theta}_{FC}, \widehat{\boldsymbol{\theta}}_D, \widehat{\boldsymbol{\theta}}_{MC})$. The solution to this optimization problem can be implemented using different methods such as the nested fixed point algorithm (Rust, 1987), nested pseudo likelihood (Aguirregabiria and Mira, 2002 and 2007), or MPEC (Su and Judd, 2012).

Alternatively, if the equilibrium choice probabilities that generate the data, say $\mathbf{P}_0$, can be estimated nonparametrically and consistently in a first step, then $\boldsymbol{\theta}_{FC}$ can be estimated consistently using a two-step method: $\widehat{\boldsymbol{\theta}}_{FC} = \arg\max_{\boldsymbol{\theta}_{FC}}\ Q(\widehat{\mathbf{P}}_0, \boldsymbol{\theta}_{FC}, \widehat{\boldsymbol{\theta}}_D, \widehat{\boldsymbol{\theta}}_{MC})$, where $\widehat{\mathbf{P}}_0$ is the the nonparametric estimator of $\mathbf{P}_0$. In terms of computation time, this two-step method is much cheaper than the previous methods, though it is also less efficient and has larger finite sample bias. Perhaps, for the type of data that we analyze in this paper, an additional issue of the two-step method is that it is not obvious how to obtain a nonparametric estimator of $\mathbf{P}_0$ that is consistent as L goes and I and T are fixed. However, kernel methods and other type of methods exist for the nonparametric estimation of spatial discrete stochastic processes (see Anselin, 2010, for a recent survey).

5 Entry costs and the dynamics of store location

We apply our model to analyze how changes in the cost of setting-up a store (entry costs) affect firms' strategies, firm value and consumer welfare.

5.1 Benchmark model

The following parameters are constant over our experiments.

(a) Market. The market is a square city of 10×10 kilometers: $\mathbb{C} = [0, 10]^2$. Consumers (households) are uniformly distributed on $\mathbb{C}$ and the population size is equal to $100,000$ households. Population size and the geographical distribution of consumers are constant over time. There are 16 business locations ($L = 16$) which form a uniform grid in the square city. The coordinates of business locations are the 16 points that result of the intersection of coordinates $(2, 4, 6, 8)$ in the horizontal and vertical axis, i.e., $z_1 = (2, 2)$, $z_2 = (2, 4)$, ..., $z_{15} = (8, 6)$, and $z_{16} = (8, 8)$. The unit transportation cost τ, which includes the opportunity cost of travel time, is \$5 per kilometer.[17]

(b) Product and Firms. Firms are supermarket chains. The product under consideration is the weekly shopping basket of a family in a supermarket. Therefore, ω_i represents the (average) willingness to pay for the weekly shopping basket in supermarket chain i. There are two firms in this city. These firms are identical in terms of the quality of their products. We fix $\omega_1 = \omega_2 = \$135$. One period in the model is a calendar year and the discount factor is set at $\beta = 0.9$. Note that households' willingness to pay ($\omega = \$135$) and the transportation cost ($\tau = \$5/km$) correspond to weekly shopping, while the frequency of firms' decisions is annual. Therefore, the correct definition of market size should be the number of households times the number of weeks per year: $M = 100,000 \times 52 = 5,200,000$ household-weeks. The parameter μ, that measures the degree of (non-spatial) horizontal product differentiation, is equal to \$2.

(c) Firms' costs. Firms are also identical in their cost structures. A firm's marginal cost to provide the weekly shopping basket of a household is \$100. Fixed operating costs and exit values are set to zero. The common knowledge component of entry costs, θ_{EC}, is constant across locations. We compute an equilibrium of the dynamic game under 21 different values of θ_{EC} between \$0 and \$20 million. To give an idea of the relative magnitude of this range of entry costs, note that the total variable profit of a monopolist with one store at every locations is close to \$2.5 million. Therefore, the range of values of θ_{EC} goes from 0% to 800% of the annual variable profits of that monopolist. Though we think that an entry cost above 400% is unrealistically high, we have considered very high values of the entry cost in our analysis in order to understand the effects of these costs in the limit when they generate almost zero entry. The private information parts of entry costs and exit values are independently and independently and identically distributed with extreme value type 1 distribution.

Table 1 presents a summary of the Nash-Bertrand equilibrium outcome under different market structures. Profits and consumer surplus are concave functions of the number of stores, either for a monopolist or a duopolist. There are decreasing returns, in profits and in consumer surplus, of an

[17] Suppose that most of the transportation cost comes from the opportunity cost of travel time. If the average hourly wage is $\$30/hour$ and the average transportation speed in this city is $6km/hour$, then $\tau = (\$30/hour)/(6km/hour) = \$5$ per kilometer.

additional store. Competition reduces profits per firm and increases consumer surplus. The closer the stores of the two firms, the smaller price-cost margins and profits, and the larger consumer surplus.

Table 1
Nash-Bertrand Equilibrium[1]

Market Structure		**Price-Cost Margin**	**Variable Profits**[2]	**Consumer Surplus**[3]	**Total Surplus**[3]
Monopoly with:	*1 store*	15.2%	25.15	14.16	39.31
	2 stores	17.9%	45.89	19.65	65.54
	3 stores	18.5%	67.71	27.55	95.26
	4 stores	19.4%	87.06	31.65	118.7
Duopoly	*1 store*	3.7%	12.17	45.64	57.81
Same locations:	*2 stores*	3.8%	16.57	73.70	90.27
	3 stores	4.0%	19.93	99.51	119.4
	4 stores	4.0%	20.80	117.2	138.0
Duopoly	*Zero distance*	3.7%	12.17	45.64	57.81
Different locations	*Small distance*	16.2%	45.27	24.23	69.50
one store for each firm	*Large distance*	15.5%	49.77	27.15	76.93

Note 1: Parameter values: City [0,10]×[0,10]; four locations (2,2), (2,8), (8,2) and (8,8); $\mu = 2$; $\tau = 5$; $\omega = 135$; $c = 100$.
Note 2: Annual variable profits, in million dollars, of all firms, i.e., variable profits per person-week × 100,000 × 52.
Note 3: Annual surplus in million dollars.

(d) Interpolation algorithm. The number of points in the state space of this dynamic game is $2^{IL} = 2^{32} \simeq 4.3 * 10^8$. We do not have enough memory space to compute exactly an equilibrium of this dynamic game. Instead, we use the interpolation method described in section 3.2. We now explain our selection of the set of points S and of the interpolation function Γ. The grid S contains $20,000$ points which are random draws from a uniform distribution over the whole state space. More precisely, let $\mathbf{n}^{(S)} \equiv \{n_{i\ell}^{(S)} : i = 1,2;\ \ell = 1,2,...,16\}$ be a point in the grid S. To obtain a point in this grid we generate the values $n_{i\ell}^{(S)} \in \{0,1\}$ as independent random draws from a Bernoulli distribution with probability 0.5. Our specification of the interpolation function exploits several features of our example. First, firms are identical and we impose symmetry in the equilibrium. Therefore, the interpolating function should be symmetric across firms. Second, given that consumers are uniformly distributed over the city, the key feature that represents the "quality" of a business location is its distance to the center of the city. We can distinguish three regions, A, B and C, such that two locations within the same region have the same distance to the center of the city. Figure 2 shows these regions. Third, the average distance between the stores of a firm summarizes the degree of cannibalization between these stores. And fourth, the average distance

between the stores of the two firms summarizes the degree of substitution between the two firms. Based on these ideas, we use the following interpolation function:

$$\Gamma_i^{\bar{\mathbf{V}}^{\mathbf{P}|\mathbf{S}}}(\mathbf{n}) = \begin{cases} \bar{V}_i^{\mathbf{P}|\mathbf{S}}(\mathbf{n}) & if \quad \mathbf{n} \in S \\ \begin{array}{l} \text{Second order polynomial in the following variables} \\ \{n_{iA},\ n_{iB},\ n_{iC},\ n_{-iA},\ n_{-iB},\ n_{-iC}, \\ \quad d_{iA},\ d_{iB},\ d_{iC},\ d_{-iA},\ d_{-iB},\ d_{-iC}, \\ \quad d_A,\ d_B,\ d_C\} \end{array} & if \quad \mathbf{n} \notin S \end{cases} \tag{33}$$

where n_{iR} is the number of stores that firm i has in region R; d_{iR} is the average distance between firm $i's$ stores in region R; and d_R is the average distance between the stores of firm 1 and firm 2 in region R. The number of parameters in this interpolation function is 136.

We have made some sensitivity analysis to validate our approximation to the exact solution. To validate the number of grid points in S, we have solved the model using 5000, 10000, 15000 and 20000 grid points. While the solution with 5000 points present some differences with respect to the solution with 20000 points, the other three solutions are almost identical. To validate the interpolation function (33) we have applied this function to similar but much smaller problems for which we can compare the approximation to the exact solution of the game. We have considered the same dynamic game but with 4 locations instead of 16. For this simpler game the state space has $2^{10} = 1024$ cells, and the city has only two regions (A and B) according to the distance to the city center. The interpolation function is a second order polynomial in the variables $\{n_{iA}, n_{iB}, n_{-iA}, n_{-iB}, d_i, d_{-i}, d\}$ and it has 28 parameters. The experiment shows that this interpolation function provides an excellent approximation to the true value function and to the equilibrium choice probabilities in this example. More generally, we conjecture that for specifications where firms are symmetric, consumers are uniformly distributed and business locations have a symmetric spatial structure, this type of interpolation function may provide a good approximation to the exact solution of the game.

FIGURE 2
Subregions A, B and C used for interpolation

(0,10) (10,10)

C	**B**	**B**	**C**
B	**A**	**A**	**B**
B	**A**	**A**	**B**
C	**B**	**B**	**C**

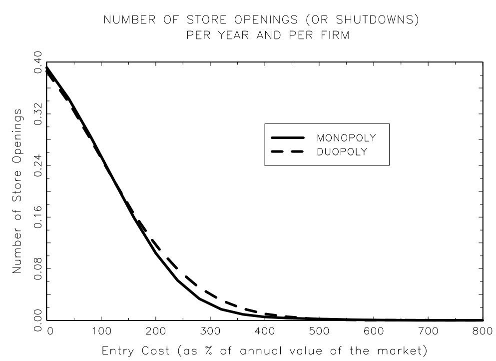
(0,0) (10,0)

5.2 Results

Figures 3 to 7 summarize the results of our numerical experiments. Each of these figures presents an outcome variable of the game (e.g., average number of stores) in the vertical axis as a function of the entry cost parameter θ_{EC} in the horizontal axis. To analyze the effects of competition we represent these functions both for a duopoly model and for a monopolist without threat of potential entrants. In the horizontal axis, the entry cost is measured as a percentage of the annual variable profit of a monopolist with stores at every business location. Some of the outcome variables (e.g., number of stores, consumer welfare, value of a firm, average distance between stores) are calculated using the steady-state distribution of state variables which is implied by the Markov perfect equilibrium.

(a) Figure 3: Store turnover: openings and shutdowns.[18] Figure 3 presents the average number of store openings per year and firm. As one would expect, an increase in the entry cost always reduces store openings and shutdowns. For most values of the entry cost, store turnover is similar under monopoly and duopoly. Nevertheless, for entry costs between 200% and 400% we observe that competition can generate some extra store turnover relative to monopoly.

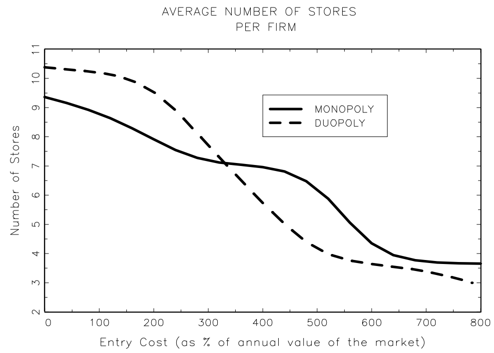
Figure 3



(b) Figure 4: Number of stores. Entry costs have a negative effect both on the creation of new stores and on store shutdowns. For our benchmark model we obtain that an increase in the entry cost has always a negative effect on the number of stores per firm. Interestingly, this negative effect is stronger under duopoly than under monopoly. In fact, this implies that the number of stores per firm is not always larger under monopoly that under duopoly. For entry costs larger than 350%, a monopolist has more stores than a duopolist.

[18] In steady-state the number of store openings should be equal to the number of store shutdowns.

Figure 4



(c) Figures 5, 6 and 7: Consumer welfare and value of a firm. Given the negative effect of entry costs on the number of stores, consumer welfare also declines with entry costs. Furthermore, despite entry costs have a stronger effect on the number of stores under duopoly, consumer welfare is always around 50% larger under duopoly than under monopoly. This is because the monopolist offsets a reduction in consumer transportation costs with higher prices. The value of a firm, in figure 6, and total welfare, in figure 7, decline monotonically with entry costs. Welfare is always greater under duopoly.

Figure 5

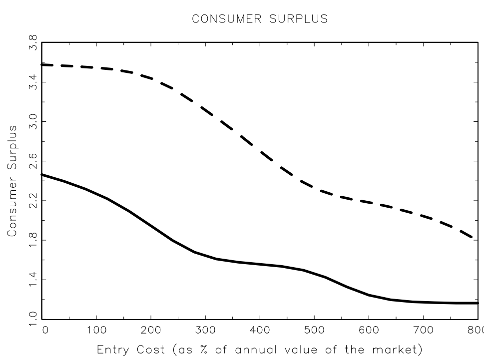


Figure 6

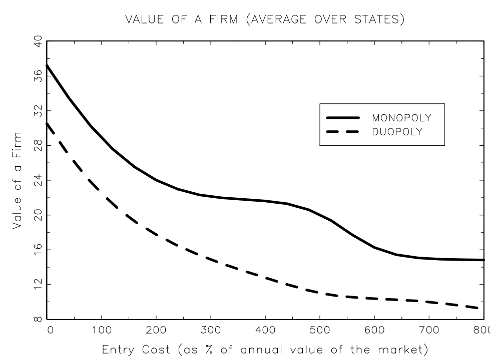
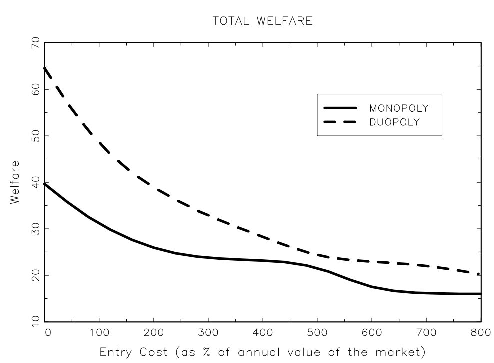


Figure 7



6 Conclusion

This paper proposes a dynamic model of an oligopoly industry characterized by spatial competition between multi-store firms. Firms compete in prices and decide where to open or close stores depending on the spatial market structure. We define and characterize a Markov Perfect Equilibria (MPE) in this model. Our framework is a useful tool to study multi-store competition issues that involve spatial and dynamic considerations. An algorithm to compute an equilibrium of the model is proposed. The algorithm exploits interpolation techniques. We also propose a procedure for the consistent estimation of the parameters of the model using panel data on store location, prices and quantities from multiple locations in a single city. We illustrate the model and the algorithm with several numerical experiments that analyze the competitive and welfare effects of the sunk cost of setting-up a store.

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