Web Appendix:
The Dynamics of the Transfer and Renewal of Patents

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Abstract

This appendix presents the proofs of the results of the model presented in "The Dynamics of the Transfer and Renewal of Patents."

Lemma 1 $E_{g^e}[V_{a+1}(x', y'|z)]$ is (i) continuous and weakly increasing in z, and (ii) weakly decreasing in a.

Proof. The proof of the parts (i) and (ii) are obtained by induction on a.

Part (i). Since $E_{g^e}[V_{L+1}(x', y'|z)] = 0$ for all z, the initial condition of the inductive argument is satisfied. Now, suppose that the result holds for $a + 1$, it suffices to show that the result also holds for a. The hypothesis of the inductive argument is that $E_{g^e}[V_{a+2}(x', y'|z)]$ is continuous and weakly increasing in z. Recall that $V_{a+1}(x, y) = \max\{V^S_{a+1}(x, y), V^K_{a+1}(x, y), 0\}$ and that $V^S(x, y) = y - c_a - \tau + \beta E_{g^e}[V_{a+2}(x', y'|y], V^K(x, y) = x - c_a + \beta E_{g^e}[V_{a+2}(x', y'|x]$. So, $V_{a+1}(x, y)$ is also continuous and weakly increasing in $z \in \{y, x, 0\}$. Finally, we want to show that if $V_{a+1}(x, y)$ is also continuous and weakly increasing in $z \in \{y, x, 0\}$, then $E_{g^e}[V_{a+1}(x', y'|z)]$ is also continuos and weakly increasing in z. Since $F_{g^e}(.)$ is independent of z,we can show that if $z_1 \geq z_2$, then $E_{g^e}[V_{a+1}(x', y'|z_1)] \geq E_{g^e}[V_{a+1}(x', y'|z_2)]$.

$$
\begin{aligned}
E_{g^e}[V_{a+1}(x', y'|z_1)] &= \int_{u^e} V_{a+1}(\delta z, \delta z u^e_a|z_1) F_{g^e}(\mathrm{d}u^e) \\
&\geq \int_{u^e} V_{a+1}(\delta z, \delta z u^e_a|z_2) F_{g^e}(\mathrm{d}u^e) \\
&= E_{g^e}[V_{a+1}(x', y'|z_2)]
\end{aligned}
$$

To prove continuity, take any $z \in R^+$. $E_{g^e}[V_{a+1}(x', y'|z)]$ will be continuous at z if for every sequence $\{z_n\}$ such that $\lim(z_n) = z$, we can show that $\lim_{z_n \to z}(E_{g^e}[V_{a+1}(x', y'|z_n)]) = E_{g^e}[V_{a+1}(x', y'|z)]$. Since $F_{g^e}(.)$ is independent of z,

$$
\begin{aligned}
\lim_{z_n \to z} E_{g^e}[V_{a+1}(x', y'|z_n)] &= \lim_{z_n \to z} \int_{u^e} V_{a+1}(\delta z_n, \delta z_n u^e_a|z_n) F_{g^e}(\mathrm{d}u^e) \\
&= \int_{u^e} \lim_{z_n \to z} (V_{a+1}(\delta z_n, \delta z_n u^e_a|z_n)) F_{g^e}(\mathrm{d}u^e) \\
&= \int_{u^e} V_{a+1}(\delta z, \delta z u^e_a|z)) F_{g^e}(\mathrm{d}u^e)
\end{aligned}
$$

where second step follows, in particular, because $F_{g^e}(\mathrm{d}u^e)$ is independent of z, and the last step follows because $V_{a+1}(\delta z, \delta z u^e_a|z)$ is continuos in z (i.e., $\lim_{z_n \to z}(V_{a+1}(\delta z_n, \delta z_n u^e_a|z_n)) = V_{a+1}(\delta z, \delta z u^e_a|z))$).

Part (ii). We want to show that $E_{g^e}[V_{a+1}(x', y'|z)]$ is weakly decreasing in a. The initial condition of the inductive argument requires to show that $E_{g^e}[V_{L+1}(x', y'|z)] \leq E_{g^e}[V_L(x', y'|z)]$. Since patents are active for L periods, then $E_{g^e}[V_{L+1}(x', y'|z)] = 0$. Moreover, by definition of $V_L(x', y'|z) \geq 0$, so it must be the case that $E_{g^e}[V_L(x', y'|z)]$. The induction hypothesis is that $E_{g^e}[V_{a+2}(x', y'|z)] \geq E_{g^e}[V_{a+1}(x', y'|z)]$. It suffices to show that $E_{g^e}[V_{a+1}(x', y'|z)] \geq E_{g^e}[V_a(x', y'|z)]$. Recall that $V_{a+1}(x', y'|z) = \max\{y' - c_{a+1} - \tau + \beta E_{g^e}[V_{a+2}(x'', y''|y')], x' -$

$c_{a+1} + \beta E_{g^e}[V_{a+2}(x'', y''|x')], 0\}$

$\leq \max\{y' - c_a - \tau + \beta E_{g^e}[V_{a+1}(x'', y''|y)], x' - c_a + \beta E_{g^e}[V_{a+1}(x'', y''|x')]\} = V_a(x', y'|z),$

where the inequality holds because $c_a \leq c_{a+1}$ and because of the induction hypothesis. Finally, we can show that

$$\begin{aligned} E_{g^e}[V_{a+1}(x', y'|z)] &= \int_{u^e} V_{a+1}(\delta z, \delta z u_a^e|z) F_{g^e}(\mathrm{d}u^e) \\ &\leq \int_{u^e} V_a(\delta z, \delta z u_a^e|z) F_{g^e}(\mathrm{d}u^e) \\ &= E_{g^e}[V_a(x', y'|z)] \end{aligned}$$

where the inequality follows because $V_{a+1}(x', y'|z) \leq V_a(x', y'|z)$ and $F_{g^e}(.)$ is independent of a. ■

Lemma 2 For a fixed τ, g^e, and a , we can show that: (a) If $V_a^S(x,y) - V_a^K(x,y) \geq 0$, then $V_a^S(x,y) - V_a^K(x,y)$ is weakly increasing in x, (b) the $V_a^K(x,y) - V_a^E(x,y)$ is weakly increasing in x.

Proof. For convenience, let me define $\widetilde{V}_a(x)$ as $V_a^K(x,y)$, that is

$$\widetilde{V}_a(x) = x - c_a + \beta E_{g^e}[V_{a+1}(\delta x, \delta x g_a^e)]$$

So, since $V_a^S(x,y) = V_a^K(x,y) - \tau$, then $V_a^S(x,y) = \widetilde{V}_a(y) - \tau$.

Let us start proving part (a) of the Lemma. That is the case when the owner of the patent is indifferent between selling the patent or keep it (i.e., any per period revenue such $x > \widehat{x}_a$).

The proof is by induction on a.

1) Suppose $a = L$ (i.e., the last period).

$$V_L^S(x,y) - V_L^K(x,y) = xg^e - \tau - x = x(g^e - 1) - \tau$$

Recall that $V_L^S(x,y) - V_L^K(x,y) \geq 0$, so it must the case that $g^e > 0$. Then, $V_L^S(x,y) - V_L^K(x,y)$ is decreasing in x.

2) Suppose that the result holds for $a+1$. So, the induction hypothesis is that for a fixed x, the $[V_{a+1}^S(x,y) - V_{a+1}^K(x,y)]$ is weakly increasing in x. Then, it suffices to show that for a fixed x, the $[V_a^S(x,y) - V_a^K(x,y)]$ is weakly increasing in x. Without loss of generality we can consider that $y > x$.

We can rewrite $[V_a^S(x,y) - V_a^K(x,y)]$ as

$$\begin{aligned} V_a^S(x,y) - V_a^K(x,y) &= y - c_a - \tau + \beta E_{g^e}[V_{a+1}(\delta y, \delta y g_a^e)] \\ &\quad -x + c_a + \tau - \beta E_{g^e}[V_{a+1}(\delta x, \delta x g_a^e)] \\ &= (y - x - \tau) + \beta E_{g^e}[V_{a+1}(\delta y, \delta y g_a^e) - V_{a+1}(\delta x, \delta x g_a^e)] \end{aligned}$$

Since $y > x$ and $y = x g_{a-1}^e$ then It is obvious to show that $(y - x - \tau)$ is weakly increasing in x. Let us look at the second term $E_{g^e}[V_{a+1}(\delta y, \delta y g_a^e) - V_{a+1}(\delta x, \delta x g_a^e)]$. Since F_{g^e} is independent of x, it just remain to be shown that $[V_{a+1}(\delta y, \delta y g_a^e) - V_{a+1}(\delta x, \delta x g_a^e)]$ is weakly increasing in x.

We can rewrite the equivalent of the induction hypothesis as

$$[V_{a+1}^S(x,y) - V_{a+1}^K(x,y)] = \widetilde{V}_{a+1}(y) - \tau - \widetilde{V}_{a+1}(x)$$

which is weakly increasing in x.

There are three cases to study.

Case (1): fix x and consider a realization of g_a^e such $V_{a+1}(\delta y, \delta y g_a^e) = V_{a+1}^S(\delta y, \delta y g_a^e)$, and $V_{a+1}(\delta x, \delta x g_a^e) = V_{a+1}^K(\delta x, \delta x g_a^e)$. That is in K region with $(\delta x, \delta x g_a^e)$ and in S region with $(\delta y, \delta y g_a^e)$. Let $\lambda = g_{a-1}^e g_a^e$. Then,

$$\begin{aligned} V_{a+1}(\delta y, \delta y g_a^e) - V_{a+1}(\delta x, \delta x g_a^e) &= V_{a+1}^S(\delta y, \delta y g_a^e) - V_{a+1}^K(\delta x, \delta x g_a^e) \\ &= V_{a+1}^S(\delta x g_{a-1}^e, \delta x g_{a-1}^e g_a^e) - V_{a+1}^K(\delta x, \delta x g_a^e) \\ &= V_{a+1}^S(\lambda \delta x, \lambda \delta x g_a^e) - V_{a+1}^K(\delta x, \delta x g_a^e) \\ &= \widetilde{V}_{a+1}(\delta x g_{a-1}^e g_a^e) - \tau - \widetilde{V}_{a+1}(\delta x) \\ &= \widetilde{V}_{a+1}(\lambda \delta x) - \tau - \widetilde{V}_{a+1}(\delta x) \end{aligned}$$

where the last expression is an increasing transformation of the equivalent of the induction hypothesis (i.e., $\widetilde{V}_{a+1}(y) - \tau - \widetilde{V}_{a+1}(x)$). Therefore, $V_{a+1}(\delta y, \delta y g_a^e) - V_{a+1}(\delta x, \delta x g_a^e)$ is weakly increasing in x.

Case (2): fix x and consider a realization of g_a^e such that, in K region with $(\delta y, \delta y g_a^e)$, in K region with $(\delta x, \delta x g_a^e)$. Let $\lambda = g_{a-1}^e$. Then,

$$\begin{aligned} V_{a+1}(\delta y, \delta y g_a^e) - V_{a+1}(\delta x, \delta x g_a^e) &= V_{a+1}^K(\delta y, \delta y g_a^e) - V_{a+1}^K(\delta x, \delta x g_a^e) \\ &= V_{a+1}^K(\delta x g_{a-1}^e, \delta x g_{a-1}^e g_a^e) - V_{a+1}^K(\delta x, \delta x g_a^e) \\ &= \widetilde{V}_{a+1}(\delta x g_{a-1}^e) - \widetilde{V}_{a+1}(\delta x) \\ &= \widetilde{V}_{a+1}(\lambda \delta x) - \widetilde{V}_{a+1}(\delta x) \end{aligned}$$

where the last expression is an increasing transformation of the equivalent of the induction hypothesis (i.e., $\widetilde{V}_{a+1}(y) - \tau - \widetilde{V}_{a+1}(x)$). Therefore, $V_{a+1}(\delta y, \delta y g_a^e) - V_{a+1}(\delta x, \delta x g_a^e)$ is weakly increasing in x.

Case (3): fix x and consider a realization of g_a^e such that, in S region with $(\delta y, \delta y g_a^e)$ and in S region with $(\delta x, \delta x g_a^e)$. Let $\lambda = g_{a-1}^e$. Then,

$$\begin{aligned} V_{a+1}(\delta y, \delta y g_a^e) - V_{a+1}(\delta x, \delta x g_a^e) &= V_{a+1}^S(\delta y, \delta y g_a^e) - V_{a+1}^S(\delta x, \delta x g_a^e) \\ &= V_{a+1}^S(\delta x g_{a-1}^e, \delta x g_{a-1}^e g_a^e) - V_{a+1}^S(\delta x, \delta x g_a^e) \\ &= \widetilde{V}_{a+1}(\delta x g_{a-1}^e g_a^e) - \widetilde{V}_{a+1}(\delta x g_a^e) \\ &= \widetilde{V}_{a+1}(\lambda \delta x g_a^e) - \widetilde{V}_{a+1}(\delta x g_a^e) \end{aligned}$$

where the last expression is an increasing transformation of the equivalent of the induction hypothesis (i.e., $\widetilde{V}_{a+1}(y) - \tau - \widetilde{V}_{a+1}(x)$). Therefore, $V_{a+1}(\delta y, \delta y g_a^e) - V_{a+1}(\delta x, \delta x g_a^e)$ is weakly increasing in x.

Let us now prove part (b) of the Lemma. We want to show that the $V_a^K(x,y) - V_a^E(x,y)$ is weakly increasing in x. We know that $V_a^E(x,y) = 0$. So, $V_a^K(x,y) - V_a^E(x,y) = V_a^K(x,y)$. For a fixed g^e, by Lemma 1 we know that $V_a^K(x,y)$ is increasing in x. Therefore, $V_a^K(x,y) - V_a^E(x,y)$ is also increasing in x. ■

Proposition 1 There are cutoff rules $\widehat{g}_a^e(x,\tau)$ and $\widehat{x}_a$ such that (a) for sufficiently high x (i.e. $x \geq \widehat{x}_a$), there is a $\widehat{g}_a^e(x,\tau)$ such that the owner is indifferent between selling and keeping (i.e., $V_a^S(x, x\widehat{g}_a^e(x,\tau)) = V_a^K(x, x\widehat{g}_a^e(x,\tau))$); (b) and for sufficiently low x (i.e., $x \leq \widehat{x}_a$) there is exist a $\widehat{g}_a^e(x,\tau)$ such that the owner is indifferent between selling and letting the patent to expire (i.e., $V_a^S(x, x\widehat{g}_a^e(x,\tau)) = 0$).

Proof. Let us start showing that there exist a cutoff rule $\widehat{g}_a^e(x,\tau)$ defined as the improvement factor that makes a potential seller indifferent between selling or not.

We first focus when the seller is indifferent between selling and letting the patent expire. Lemma 1 implies that, for a fixed a, $\widetilde{V}_a(x)$ is an increasing function of x. We can show for any $\tau > 0$, when $g^e = 0$, $\widetilde{V}_a(xg^e) - \tau < 0$. Moreover, we can also show that for any $\tau > 0$ there exist a sufficiently high g^e, for example $\overline{g^e}$, that $\widetilde{V}_a(x\overline{g^e}) - \tau > 0$. Therefore, by Bolzano's theorem, there exist a $\widehat{g}_a^e(x,\tau)$ such that

$$\begin{aligned} \widetilde{V}_a(x\widehat{g}_a^e(x,\tau)) - \tau &= 0 \\ V_a^S(x, x\widehat{g}_a^e(x,\tau)) &= 0 \end{aligned}$$

Second, we focus on the case in which the potential seller is indifferent between selling and keeping the patent. That is $V_a^S(x,y) = V_a^K(x,y)$, which implies that

$$\widetilde{V}_a(x\widehat{g}_a^e(x,\tau)) - \tau = \widetilde{V}_a(x)$$

Now, let us consider a $\tau > 0$ and a $g^e = 1$. We can show that $V_a^S(x,y) - V_a^K(x,y) < 0$ because $V_a^S(x,y) = V_a^S(x,x) - \tau = V_a^K(x,y) - \tau$. On the other hand, for fixed τ and fixed x, Lemma 1 shows that $V_a^S(x,y)$ is increasing in y. So, for sufficiently high g^e we can show that $V_a^S(x,y) - V_a^K(x,y) > 0$. Finally, by Lemma 2 we know that for a fixed τ, the difference $(V_a^S(x,y) - V_a^K(x,y))$ is increasing in x. By Bolzano's theorem, there exist a $\widehat{g}_a^e(x,\tau)$ such that $V_a^S(x, x\widehat{g}_a^e(x,\tau)) = V_a^K(x, x\widehat{g}_a^e(x,\tau))$, which is that

$$\widetilde{V}_a(x\widehat{g}_a^e(x,\tau)) - \tau = \widetilde{V}_a(x)$$

■

Proposition 2 (Selection Effect) The cutoff rule $\widehat{g}_a^e(x,\tau)$ is weakly decreasing as a function of x.

Proof. For convenience, let me define $\widetilde{V}_a(x)$ as $V_a^K(x,y)$, that is

$$\widetilde{V}_a(x) = x - c_a + \beta E_{g^e}[V_{a+1}(\delta x, \delta x g_a^e)]$$

So, since $V_a^S(x,y) = V_a^K(x,y) - \tau$, then $V_a^S(x,y) = \widetilde{V}_a(y) - \tau$.

Let us first consider when the owner of the patent is indifferent between selling the patent or let it allow to expire. That is any per period revenue such $x \leq \widehat{x}_a$. The cutoff rule $\widehat{g}_a^e(x,\tau)$ is defined as follows.

$$\begin{aligned} V_a^S(x, x\widehat{g}_a^e(x,\tau)) &= 0 \\ \widetilde{V}_a(x\widehat{g}_a^e(x,\tau)) - \tau &= 0 \end{aligned}$$

From Lemma 1 we know that for a fixed g^e the function $V_a^S(x, xg^e) = \widetilde{V}_a(xg^e) - \tau$ is weakly increasing in x. So, for the above equality to hold, it must be the case that if x increases then $\widehat{g}_a^e(x,\tau)$ decreases.

Let us now consider when the owner of the patent is indifferent between selling the patent or keep it. That is any per period revenue such $x > \widehat{x}_a$.

The proof is by induction on a.

1) Suppose $a = L$ (i.e., the last period). A seller is indifferent between selling and keeping a

patent if

$$\begin{aligned} V_L^S(x,y) &= V_L^K(x,y) \\ x\widehat{g}_a^e(x,\tau) - \tau &= x \end{aligned}$$

And we can show that $\widehat{g}_a^e(x,\tau)$ is decreasing in x

$$\widehat{g}_a^e(x,\tau) = 1 + \frac{\tau}{x}$$

2) Suppose that $\widehat{g}_a^e(x,\tau)$ is weakly decreasing in x for $a+1$. We want to show that the result holds for a. Notice that the induction hypothesis (i.e., $\widehat{g}_a^e(x,\tau)$ is weakly decreasing in x for $a+1$) is equivalent to showing that for a fixed x, the $[V_{a+1}^S(x,y) - V_{a+1}^K(x,y)]$ is weakly increasing in x. Therefore, in order to prove that $\widehat{g}_a^e(x,\tau)$ in weakly decreasing in x for age a, it suffices to show that for a fixed x, the difference $[V_a^S(x,y) - V_a^K(x,y)]$ is weakly increasing in x.

Note that $\widehat{g}_a^e(x,\tau)$ for any a is only defined for x such that $\widehat{g}_a^e(x,\tau) \geq 1 \ \forall a \in [1,L]$. So, without loss of generality we can consider $y > x$. Finally, by Lemma 2, see especially part (2) of the proof, we show that for a fixed τ and g^e, $[V_a^S(x,y) - V_a^K(x,y)]$ is weakly increasing in x. As argued above, this results is equivalent to showing that the function $\widehat{g}_a^e(x,\tau)$ is weakly decreasing in x. ■

Proposition 3 (Horizon Effect) For any $x > 0$ and $\tau > 0$, the cutoff rule $\widehat{g}_a^e(x,\tau)$ is weakly increasing as a function of a.

Proof. For convenience, let me define $\widetilde{V}_a(x)$ as $V_a^K(x,y)$ as

$$\widetilde{V}_a(x) = x - c_a + \beta E_{g^e}[V_{a+1}(\delta x, \delta x g_a^e)]$$

So, since $V_a^S(x,y) = V_a^K(x,y) - \tau$, then $V_a^S(x,y) = \widetilde{V}_a(y) - \tau$.

Showing that for a fixed x and τ the cutoff rule $\widehat{g}_a^e(x,\tau)$ is weakly increasing as a function of a is equivalent to proving that for a fixed x and g^e, $[V_a^S(x,y) - V_a^K(x,y)]$ is indeed decreasing in a. That is,

$$[V_a^S(x,y) - V_a^K(x,y)] > [V_{a+1}^S(x,y) - V_{a+1}^K(x,y)]$$

And rearranging the induction hypothesis we obtain,

$$\begin{aligned}
\widetilde{V}_a(y) - \tau - \widetilde{V}_a(x) &> \widetilde{V}_{a+1}(y) - \tau - \widetilde{V}_{a+1}(x) \\
\widetilde{V}_a(y) - \widetilde{V}_a(x) &> \widetilde{V}_{a+1}(y) - \widetilde{V}_{a+1}(x) \\
y - c_a + \beta E_{g^e}[V_{a+1}(\delta y, \delta y g_a^e)] - x + c_a - \beta E_{g^e}[V_{a+1}(\delta x, \delta x g_a^e)] &> y - c_{a+1} + \beta E_{g^e}[V_{a+2}(\delta y, \delta y g_{a+1}^e)] \\
&\quad -x + c_{a+1} - \beta E_{g^e}[V_{a+2}(\delta x, \delta x g_{a+1}^e)] \\
y - x + \beta E_{g^e}[V_{a+1}(\delta y, \delta y g_a^e)] - \beta E_{g^e}[V_{a+1}(\delta x, \delta x g_a^e)] &> y - x + \beta E_{g^e}[V_{a+2}(\delta y, \delta y g_{a+1}^e)] \\
&\quad -\beta E_{g^e}[V_{a+2}(\delta x, \delta x g_{a+1}^e)] \\
E_{g^e}[V_{a+1}(\delta y, \delta y g_a^e)] - E_{g^e}[V_{a+1}(\delta x, \delta x g_a^e) &> E_{g^e}[V_{a+2}(\delta y, \delta y g_{a+1}^e)] \\
&\quad -E_{g^e}[V_{a+2}(\delta x, \delta x g_{a+1}^e)]
\end{aligned}$$

Without loss of generality, consider per period revenues x and y, $y > x$, such that $[V_a^S(x,y) - V_a^K(x,y)] \geq 0$.

The proof is by induction.

1) Suppose $a + 1 = L$, so $a = L - 1$. We want to show that

$$[V_L^S(x,y) - V_L^K(x,y)] > [V_{L+1}^S(x,y) - V_{L+1}^K(x,y)]$$

Let us start defining

$$V_L^S(x,y) - V_L^K(x,y) = \widetilde{V}_a(y) - \tau - \widetilde{V}_a(x) = y - \tau - x$$

$$\begin{aligned}
V_{L-1}^S(x,y) - V_{L-1}^K(x,y) &= y + \beta E_{g^e}[V_L(\delta y, \delta y g_{L-1}^e)] - \tau - x + \beta E_{g^e}[V_L(\delta x, \delta x g_{L-1}^e)] \\
&= y - \tau - x + \beta E_{g^e}[V_L(\delta y, \delta y g_{L-1}^e) - V_L(\delta x, \delta x g_{L-1}^e)]
\end{aligned}$$

In Lemma 1 we showed that for a fixed a, $V_a(x,y)$ was weakly increasing in x. So, it must be the case that $V_L(\delta y, \delta y g_{L-1}^e) \geq V_L(\delta x, \delta x g_{L-1}^e)$. The expectation of a random variable that is larger or equal than zero is also larger or equal than zero. So, $E_{g^e}[V_L(\delta y, \delta y g_{L-1}^e) - V_L(\delta x, \delta x g_{L-1}^e)] \geq 0$. Then

$$\begin{aligned}
V_{L-1}^S(x,y) - V_{L-1}^K(x,y) &= y - \tau - x + \beta E_{g^e}[V_L(\delta y, \delta y g_{L-1}^e) - V_L(\delta x, \delta x g_{L-1}^e)] \\
&\geq y - \tau - x \\
&= V_L^S(x,y) - V_L^K(x,y)
\end{aligned}$$

Therefore, $[V_L^S(x,y) - V_L^K(x,y)] > [V_{L+1}^S(x,y) - V_{L+1}^K(x,y)]$.

2) Now, suppose that the relationship holds for $a+1$. So, the induction hypothesis is

$$\begin{aligned} [V_{a+1}^S(x,y) - V_{a+1}^K(x,y)] &> [V_{a+2}^S(x,y) - V_{a+2}^K(x,y)] \\ \widetilde{V}_{a+1}(y) - \tau - \widetilde{V}_{a+1}(x) &> \widetilde{V}_{a+2}(y) - \tau - \widetilde{V}_{a+2}(x) \\ \widetilde{V}_{a+1}(y) - \widetilde{V}_{a+1}(x) &> \widetilde{V}_{a+2}(y) - \widetilde{V}_{a+2}(x) \end{aligned}$$

We will show that it holds for a, that is

$$[V_a^S(x,y) - V_a^K(x,y)] > [V_{a+1}^S(x,y) - V_{a+1}^K(x,y)]$$

We know that

$$\begin{aligned} V_a^S(x,y) - V_a^K(x,y) &= \widetilde{V}_a(y) - \tau - \widetilde{V}_a(x) \\ &= y - \tau + \beta E_{g^e}[V_{a+1}(\delta y, \delta y g_a^e)] - x - \beta E_{g^e}[V_{a+1}(\delta x, \delta x g_a^e)] \\ &= y - \tau - x + \beta(E_{g^e}[V_{a+1}(\delta y, \delta y g_a^e)] - E_{g^e}[V_{a+1}(\delta x, \delta x g_a^e)]) \\ &> y - \tau - x + \beta(E_{g^e}[V_{a+2}(\delta y, \delta y g_{a+1}^e)] - E_{g^e}[V_{a+2}(\delta x, \delta x g_{a+1}^e)]) \\ &= y - \tau - c_a + \beta E_{g^e}[V_{a+2}(\delta y, \delta y g_{a+1}^e)] - x + c_a - \beta E_{g^e}[V_{a+2}(\delta x, \delta x g_{a+1}^e)] \\ &= V_{a+1}^S(x,y) - V_{a+1}^K(x,y) \end{aligned}$$

■

Proposition 4 (Pakes and Schankerman) If the schedule of renewal fees $\{c_a\}_{a=1}^{a=L}$ is weakly increasing in the age of the patent, then the cutoff rule $\widehat{x}_a$ is weakly increasing as a function of a.

Proof. Recall that the cutoff rule $\widehat{x}_a$ is defined as the per period return that makes the owner of the patent indifferent between keeping and allow the patent to expire. That is

$$V_a^K(\widehat{x}_a, \widehat{x}_a g^e) = 0$$

Similarly,

$$V_{a+1}^K(\widehat{x}_{a+1}, \widehat{x}_{a+1} g^e) = 0$$

So,

$$V_{a+1}^K(\widehat{x}_{a+1}, \widehat{x}_{a+1} g^e) = V_a^K(\widehat{x}_a, \widehat{x}_a g^e)$$

By Lemma 1 we know that $V_a^K(.,.)$ is a weakly increasing function of a and x. So, it must be the case that $\widehat{x}_{a+1} \geq \widehat{x}_a$ for the equality to hold. ■