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Optimal Auctions with Information Acquisition

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Abstract

This paper studies optimal auction design in a private value setting with endogenous information acquisition. First, we develop a general framework for modeling information acquisition when a seller wants to sell an object to one of several potential buyers who can each gather information about their valuations prior to participation. We then show that under certain conditions, standard auctions with a reserve price remain optimal, but the optimal reserve price lies between the mean valuation and the standard reserve price in Myerson (1981). We provide sufficient conditions under which the value of information to the seller is positive, and also characterize the necessary and sufficient conditions under which equilibrium information acquisition in private value auctions is socially excessive. The key to the analysis is the insight that buyer incentives to acquire information become stronger as the reserve price moves toward the mean valuation.

KEYWORDS: optimal auctions, information acquisition, rotation order, informational efficiency

JEL CLASSIFICATION: C70, D44, D82, D86

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1 Introduction

The efficient use of information dispersed in society is, as Hayek (1945) emphasized, a fundamental issue in economics, and one that has been the focus of important research in mechanism design. A typical assumption in the literature is that the information held by market participants is exogenous; yet in many real world situations, agents' information about the goods and services being traded is acquired rather than endowed.[1] When information acquisition is endogenous, the selling mechanism proposed by the seller affects not only buyers' incentives to reveal the information they gathered ex post, but also their incentives to acquire information ex ante. Not surprisingly, if the information structure is endogenous, the ex post optimal selling mechanism, such as the one characterized in Myerson (1981), may not be optimal ex ante.

The purpose of this paper is to study how a seller should design the selling mechanism when information acquisition is endogenous and costly for buyers. We first develop a convenient and general framework to model information acquisition in an independent private value setting, where a seller wants to sell an object to one of several potential buyers and where the buyers can each covertly acquire information about their valuations. In the model, a buyer acquires information by increasing the precision of the signal he receives, and after receiving this each buyer forms a posterior estimate of his valuation which will depend on both the realization and the informativeness of his signal in the following way: buyer valuation estimates in a private value setting move *apart* as more information is acquired, i.e., the distribution of posterior estimates conditional on a more informative signal is more spread out.[2] The resulting family of distributions of the posterior estimates with different signals are rotation-ordered[3] – the information order we use to rank the informativeness of signals.

We apply this framework to analyze monopoly pricing and optimal auction design with endogenous information acquisition. Since an increase in information leads to an increase in the dispersion of buyers' valuation estimates, increased information acquisition has two competing effects on the seller's revenue. On the one hand, it increases the potential social surplus – the difference between the highest valuation estimate among buyers and the seller's reservation value. On the other hand, it also increases buyers' private information, raising their information rents.[4] Given that the

[1] For example, consumers collect information about the characteristics of products and match this information with their private preferences to determine their valuations before their purchase decision. In a take-over bidding, buyers gather costly information about potential syergies between their own assets and assets of the target firm to determine how much they should bid.

[2] For example, suppose a consumer tries a newly opened restaurant and finds the food spicy. He will like the restaurant more if he loves spicy food, and he will like the restaurant less if he dislikes spicy food.

[3] If two signals are rotation-ordered, then the two distributions of posterior estimates generated by these signals cross each other only once. The rotation order was recently introduced by Johnson and Myatt (2006) in order to model how advertising, marketing and product design affect the dispersion of consumers' valuations and lead to a rotation (rather than a shift) of the market demand curve. It meets the Blackwell's (1951) criterion of informativeness in the sense that a buyer can achieve a higher expected payoff under incentive compatible mechanisms when basing his decision on the realization of a more informative signal.

[4] This should be contrasted to results in the common value setting where more information links buyer valuations *together* and thus always benefits the seller (Milgrom and Weber (1982), and Ottaviani and Prat (2001)).

seller's revenue is the difference between social surplus and information rents, her task is therefore to choose a selling mechanism that balances these two forces.

We start by considering optimal auctions with a single buyer, whose true valuation is normally distributed and ex-ante unobservable to both parties. The buyer receives a noisy signal – the sum of his true valuation and a normally distributed noise – and can increase the informativeness of his signal by reducing the variance of the noise, though at an increasing cost.[5] In this case, we show that the optimal selling mechanism is to post a (reserve) price, so we can also reinterpret the seller's optimization problem as a monopoly pricing problem with endogenous information acquisition by consumers.

Since the buyer always prefers a low reserve price, it may seem at first glance that a lower reserve price always gives the buyer a higher incentive to gather information. Yet the buyer's incentives to acquire information depend on his *relative* gain from information acquisition rather than on his *absolute* payoff; indeed, incentives to collect information increase as the reserve price moves toward the mean valuation, either from above or from below. To understand this observation, consider the case where information acquisition is binary. If the reserve price is very high or very low, new information is unlikely to change the buyer's purchase decision. In contrast, if the reserve price is close to the mean valuation, new information is valuable because it helps the buyer make the right decision – namely, to buy or not to buy.

The observation that incentives to acquire information become stronger as the reserve price moves closer to the mean valuation, together with the fact that more information leads to an increase in the dispersion of the buyer's valuation estimates, yields a surprising result: the optimal reserve price is *always* adjusted downward compared to the standard reserve price in this setting.[6] To see this, note that when the standard reserve price is higher than the mean valuation, more information increases the probability of trade and benefits the seller. Therefore, the seller will adjust the reserve price downward to induce the buyer to acquire more information. On the other hand, when the standard reserve price is lower than the mean valuation, more information reduces the probability of trade and hurts the seller. Hence, the seller again adjusts the reserve price downward, but this time to induce the buyer to acquire *less* information.

The same observation can be extended to a general setting with many buyers and rotation-ordered information structures. But the analysis of the general model is more subtle and complicated, primarily because the optimal selling mechanism no longer admits the simple form of a posted price. Rather, a feasible mechanism has to provide buyers with the right incentives to collect information in the information acquisition stage (moral hazard) and be incentive compatible in the information revelation stage (adverse selection). We use the standard *first-order approach* to tackle the moral hazard problem, replacing the information acquisition constraints by the first-order conditions of the buyers' maximization problems (Mirrlees (1999), and Rogerson (1985)).[7]

[5] The resulting information structure will be rotation-ordered, with the rotation point at the mean valuation.

[6] The *standard reserve price* is defined as the optimal reserve price in Myerson (1981) where information is exogenous. In the case of one bidder, the standard reserve price is simply the optimal monopoly price with exogenous information.

[7] A condition, analogous to CDFC in Rogerson (1985), is shown to be sufficient for the first-order approach to be

Our problem here is complicated because more information may hurt or benefit the seller, while in the standard moral hazard problem, the principal always benefits from higher effort (without considering the incentive cost of inducing higher effort).

In order to identify the seller's information preference, we focus for tractability on the symmetric equilibrium in which all buyers acquire the same level of information. Given that buyers are ex ante symmetric, this restriction is quite natural because symmetric mechanisms are easier to implement and are less likely to cause legal disputes.[8] Using Rogerson's technique, we show that the value of information to the seller is positive when the number of bidders is not too small. Then we show that standard auctions[9] with a reserve price are optimal, but the reserve price has to be adjusted toward the mean valuation to induce buyers to acquire more information. Further, the buyers' incentives to collect information are socially excessive in standard auctions with a reserve price lower than the mean valuation. As a robustness check, we also investigate the optimal (asymmetric) selling mechanism in a setting with discrete information acquisition. We dispense with both the first-order approach and the symmetric restriction on equilibrium information choices, and show that the optimal selling mechanism in this setting involves price discrimination against "strong" bidders, though to a weaker extent as compared with the case with exogenous information.

In sum, our analysis makes it clear that endogenous information pushes down the price level in a monopoly pricing setting. We also show that the optimal selling mechanisms identified in Myerson (1981) are robust to endogenous information acquisition if buyers are induced to acquire the same level of information in equilibrium. This implies that the seller should still use standard auctions to allocate the object as long as she appropriately adjusts the reserve price to incorporate buyer incentives to acquire information. The general framework we develop to model information acquisition in a private value setting can also apply to mechanism design problems when agents can make investment prior to the auction. For instance, Lichtenberg (1988) finds strong evidence of private R&D investment prior to government procurement auctions. In this vein, our framework can be used to investigate how the government should design procurement auctions in order to promote private R&D investment.

The remainder of the paper is organized as follows. Section 2 discusses the related literature, Section 3 introduces the model, and Section 4 studies optimal auctions with a single bidder and the Gaussian specification. Section 5 then presents the analysis of optimal auctions with many bidders, and Section 6 concludes.[10]

valid when the support of buyers' posterior estimates is invariant to buyers' information choices. See Appendix B for a set of sufficient conditions under which the first-order approach is valid.

[8]Nevertheless, this is an important restriction. In principle, the seller may become better off by implementing an asymmetric equilibrium rather than a symmetric one.

[9]In this paper, we use standard auctions to refer to the four commonly used auction formats: first-price sealed-bid auctions, Vickery auctions, English auctions, and Dutch auctions.

[10]All omitted proofs are relegated to Appendix A. Appendix B provides sufficient conditions for the first-order approach to be valid.

2 Related Literature

This paper is related to the growing literature on information and mechanism design, extending the principal-agent model with information acquisition to a multi-agent setting.[11] Cremer and Khalil (1992) and Cremer, Khalil, and Rochet (1998a) (1998b) introduce endogenous information acquisition into the Baron and Myerson (1982) regulation model, and illustrate how the optimal contract has to be modified in order to give the agent incentives to acquire information. Szalay (2005) extends their framework to a setting with continuous information acquisition, and demonstrates that their findings are robust. Our model shares a similar information structure and a focus on the interim participation constraint, though we incorporate strategic interactions among bidders.

Our analysis is also related to studies on information acquisition in given auctions. Matthews (1984) studies information acquisition in a common value auction and investigates whether the equilibrium price fully reveals bidders' information. Stegeman (1996) shows that first and second price auctions with independent private values result in the same incentives for information acquisition, while Persico (2000) shows that the incentive to acquire information is stronger in the first-price auction than in the second-price auction if bidders' valuations are affiliated.[12] In contrast, the current paper studies the optimal mechanism that maximizes the seller's revenue, rather than studying information acquisition under given auction formats.[13]

A third strand of related literature studies optimal auctions when the seller controls either the access to information sources or the timing of information acquisition. The information order used in the present paper, the rotation order, was first introduced by Johnson and Myatt (2006). They use it to show that a firm's profits are a U-shaped function of the dispersion of consumers' valuations, so a monopolist will pursue extreme positions, providing either a minimal or maximal amount of information. Eso and Szentes (2007) study optimal auctions in a setting where the seller controls the access to information sources. They show that the seller will fully reveal her information and can extract all of the benefit from the released information.[14] In these models, the seller makes the information decision, rather than the buyers.

Several papers study the optimal selling mechanism in a setting where buyers make the information decision, but the seller controls the timing of information acquisition. These models (hereafter referred to as "entry models") impose an ex-ante participation constraint, so the buyers' information decision is essentially an entry decision. The optimal selling mechanism typically consists of a participation fee followed by a second price auction with no reserve price, with the participation fee being equal to the bidders' expected rent from attending the auction (see for example, Levin and Smith (1994) and Ye (2004)).[15]

[11]For a broad survey of the literature on information and mechanism design, see Bergemann and Välimäki (2006a).

[12]See Ye (2006) and Compte and Jehiel (2006) for an analysis of information acquisition in dynamic auctions.

[13]Bergemann and Välimäki (2002) also study information acquisition and mechanism design, but their focus is efficient mechanisms.

[14]Bergemann and Pesendorfer (2007) characterize the optimal information structure in the optimal auctions, while Ganuza and Penalva (2006) study the seller's optimal disclosure policy when the information is costly.

[15]Similarly, with an ex-ante participation constraint, Cremer, Spiegel, and Zheng (2003) construct a sequential

In contrast to these papers, where the seller can control either access to information sources or the timing of information acquisition (*centralized* information acquisition), information acquisition in the current paper is *decentralized*: buyers make the information decision, and can acquire information prior to participation. Thus, we impose an interim rather than an ex-ante participation constraint.[16] The relationship between our model and the existing literature can be partially summarized in the following table.

	given auction formats	mechanism design approach
centralized information acquisition	optimal disclosure in auctions	entry models
decentralized information acquisition	information acquisition in auctions	our model

3 The Model

A seller wants to sell a single object to n ex-ante symmetric buyers (or bidders), indexed by $i \in \{1, 2, ..., n\}$.[17] Both the seller and buyers are risk neutral. The buyers' true valuations $\{\omega_i : i = 1, ..., n\}$, unknown ex-ante, are independently drawn from a common distribution F with support $[\underline{\omega}, \overline{\omega}]$. F has a strict positive and differentiable density f and mean μ. A buyer with valuation ω_i gets utility u_i if he wins the object and pays t_i:

$$u_i = \omega_i - t_i.$$

The seller's valuation for the object is normalized to be zero.

3.1 The Information Structure

Buyer i can acquire a costly signal s_i about ω_i, with $s_i \in [\underline{s}, \overline{s}] \subseteq \mathbb{R}$. Signals received by different buyers are independent. Buyer i acquires information by choosing a joint distribution of (s_i, ω_i) from a family of joint distributions $G_{\alpha_i} : \mathbb{R} \times [\underline{\omega}, \overline{\omega}] \to [0, 1]$, indexed by $\alpha_i \in [\underline{\alpha}, \overline{\alpha}]$. Each fixed α_i corresponds to a statistical experiment, and the signal with higher α_i is more informative in a sense to be defined later. We refer to the joint distribution G_{α_i}, or simply α_i, as the information structure. The cost of performing an experiment α_i is $C(\alpha_i)$, which is assumed to be convex in α_i. A buyer can conduct the experiment $\underline{\alpha}$ at no cost, so $\underline{\alpha}$ is interpreted as the endowed signal.

Let $G_{\alpha_i}(\cdot|\omega_i)$ denote the prior distribution of signal s_i conditional on ω_i, and $G_{\alpha_i}(\cdot|s_i)$ denote the posterior distribution of ω_i conditional on s_i. With a little abuse of notation, $G_{\alpha_i}(\omega_i)$ and $G_{\alpha_i}(s_i)$ are used to denote the marginal distributions of ω_i and s_i, respectively. They are defined in the usual way – that is, $G_{\alpha_i}(\omega_i) = \mathbb{E}_{s_i}[G_{\alpha_i}(\omega_i|s_i)]$ and $G_{\alpha_i}(s_i) = \mathbb{E}_{\omega_i}[G_{\alpha_i}(s_i|\omega_i)]$. Consistency

selling mechanism in which the seller charges a positive entry fee and extracts the full surplus from buyers.

[16] Cremer, Spiegel, and Zheng (2007) also analyze optimal auctions where buyers can acquire information prior to participation, but the seller, rather than the buyer, pays the information cost.

[17] The analysis can be extended to a multi-unit setting where each buyer has a unit demand.

requires that $G_{\alpha_i}(\omega_i) = F(\omega_i)$ for all α_i and i. We use $g_{\alpha_i}(s_i, \omega_i)$, $g_{\alpha_i}(\cdot|\omega_i)$, $g_{\alpha_i}(\cdot|s_i)$, $g_{\alpha_i}(\omega_i)$ and $g_{\alpha_i}(s_i)$ to denote the corresponding densities.

A buyer who observes a signal s_i from experiment α_i will update his prior belief about ω_i according to Bayes' rule:

$$g_{\alpha_i}(\omega_i|s_i) = \frac{g_{\alpha_i}(s_i|\omega_i) f(\omega_i)}{\int_{\underline{\omega}}^{\overline{\omega}} g_{\alpha_i}(s_i|\omega_i) f(\omega_i) d\omega_i}$$

Let $v_i(s_i, \alpha_i)$ denote buyer i's revised estimate of ω_i after performing experiment α_i and observing s_i :

$$v_i(s_i, \alpha_i) \equiv \mathbb{E}[\omega_i|s_i, \alpha_i] = \int_{\underline{\omega}}^{\overline{\omega}} \omega_i g_{\alpha_i}(\omega_i|s_i) d\omega_i$$

To simplify notation, we use v_i to denote $v_i(s_i, \alpha_i)$, and use v to denote the n-vector $(v_1, ..., v_n)$. Occasionally, we also write v as (v_i, v_{-i}), where $v_{-i} = (v_1, ...v_{i-1}, v_{i+1}, ..., v_n)$. Throughout the paper we assume that the densities $\{g_{\alpha_i}(\cdot|\omega_i)\}$ have monotone likelihood ratio property (MLRP), so that $v_i(s_i, \alpha_i)$ is increasing in s_i, i.e., a higher s_i leads to a higher posterior estimate, given the information choice α_i (Milgrom (1981)). Let H_{α_i} denote the distribution of v_i with corresponding density h_{α_i}. Then the family of distributions $\{H_{\alpha_i}\}$ have the same mean because

$$\mathbb{E}_{s_i}[v_i(s_i, \alpha_i)] = \mathbb{E}[\omega_i] = \mu.$$

For bidder i, different information choices $\{\alpha_i\}$ lead to different distributions $\{H_{\alpha_i}\}$. So choosing α_i is equivalent to choosing an H_{α_i} from the family of distributions $\{H_{\alpha_i}\}$. In what follows, we will extensively work with the posterior estimate v_i and its distribution H_{α_i}.

3.2 Timing

The timing of the game is as follows: the seller first proposes a selling mechanism; after observing the mechanism, each buyer decides how much information to acquire; after the signals are realized, each buyer decides whether to participate; each participating buyer submits a report about his private information; and finally, an outcome, consisting of an allocation of the object and payments, is realized. Figure 1 summarizes the timing of the game:

$$- - -| - - - - \quad - - -| - - - \quad - - - - - - - -| - - - - - - \quad - - - - -| - - - - \quad - - -| - - - \longrightarrow$$

seller announces mechanism	buyer i chooses α_i	buyer i observes s_i and decides whether to participate	buyers report private information	outcome realized

Figure 1. The timing of the game

The payoff structure, the timing of the game, the information structure $\{G_{\alpha_i}\}$ and distribution F are assumed to be common knowledge. The solution concept is Bayesian Nash equilibrium.

3.3 Mechanisms

In our setting, the buyer's private information is two-dimensional, consisting of the information choice α_i and the realized signal s_i. This suggests that the design problem here is multi-dimensional and could potentially be very complicated. However, similar to Biais, Martimort, and Rochet (2002) and Szalay (2005), one single variable, the posterior estimate $v_i(\alpha_i, s_i)$, completely captures the dependence of buyer i's valuation on the two-dimensional information. Furthermore, the seller cannot screen the two pieces of information separately. For example, suppose there are two buyers, i and j, with the same posterior estimate $(v_i = v_j)$, but $\alpha_i \neq \alpha_j$. If the seller wants to favor the buyer with α_i, then buyer j can always report to have α_i. Therefore, the posterior estimate v_i is the only variable that the seller can use to screen different buyers.

We can thus invoke the Revelation Principle to focus on the direct revelation mechanisms $\{q_i(v), t_i(v)\}_{i=1}^n$:

$$\begin{aligned} q_i &: [\underline{\omega}, \overline{\omega}]^n \to [0,1], \\ t_i &: [\underline{\omega}, \overline{\omega}]^n \to \mathbb{R}, \end{aligned}$$

where $q_i(v)$ denotes the probability of winning the object for bidder i when the vector of report is v, and $t_i(v)$ denotes bidder i's corresponding payment. Let $Q_i(v_i)$ and $T_i(v_i)$ be the expected probability of winning and the expected payment conditional on v_i, respectively. The interim utility of bidder i who has a posterior estimate v_i and reports v_i' is

$$U_i(v_i, v_i') = v_i Q_i(v_i') - T_i(v_i').$$

Define $u_i(v_i) = U_i(v_i, v_i)$: the payoff of bidder i who has a posterior estimate v_i and reports truthfully.

A feasible mechanism has to satisfy the individual rationality constraint (IR):

$$u_i(v_i) = U_i(v_i, v_i) \geq 0, \quad \forall v_i \in [\underline{\omega}, \overline{\omega}], \tag{IR}$$

and the incentive compatibility constraint (IC):

$$U_i(v_i, v_i) \geq U_i(v_i, v_i'), \quad \forall v_i, v_i' \in [\underline{\omega}, \overline{\omega}]. \tag{IC}$$

With endogenous information acquisition, a feasible mechanism also has to satisfy the information acquisition constraint (IA): no bidder has an incentive to deviate from the equilibrium choice α_i^* :

$$\alpha_i^* \in \arg\max_{\alpha_i} \left\{ \mathbb{E}_{v,\alpha_{-i}^*}\left[u_i(v_i(s_i, \alpha_i))\right] - C(\alpha_i) \right\}. \tag{IA}$$

Note that $\mathbb{E}_{v,\alpha_{-i}^*}\left[u_i(v_i(s_i, \alpha_i))\right]$ is bidder i's expected payoff by choosing α_i conditional on other bidders choosing α_j^*, $j \neq i$.

The seller chooses mechanism $\{q_i(v), t_i(v)\}_{i=1}^n$ and a vector of information choices $(\alpha_1^*, ..., \alpha_n^*)$ to maximize her expected sum of payment from all bidders,

$$\pi_s = \mathbb{E}_{v,(\alpha_1^*,...,\alpha_n^*)} \sum_{i=1}^n T_i(v_i),$$

subject to (IA), (IC), and (IR).

3.4 Information Order

In order to analyze a model with general information structures, we need an information order to rank the informativeness of different signals. Since the relevant variable for screening is the posterior estimate v_i and there is one-to-one mapping between the information choice α_i and the distribution H_{α_i} of v_i, we would like to have an information order that directly ranks H_{α_i}. The rotation order, recently introduced by Johnson and Myatt (2006), meets this requirement.

Definition 1 (Rotation Order)
The family of distributions $\{H_{\alpha_i}\}$ is rotation-ordered if, for every α_i, there exists a rotation point $v^+_{\alpha_i}$, such that

$$v_i \gtrless v^+_{\alpha_i} \Leftrightarrow \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} \lesseqgtr 0. \tag{1}$$

Two distributions ordered in terms of rotation cross only once: the distribution with lower α_i crosses the distribution with higher α_i from below. As shown below, the rotation order implies second-order stochastic dominance. However, the reverse is not true: two distributions ordered in terms of second-order stochastic dominance can cross each other more than once.

Lemma 1 (Rotation Order Implies Second Order Stochastic Dominance)
If a family of distributions $\{H_{\alpha_i}\}$ is rotation-ordered and they all have the same mean, then they are also ordered in terms of second-order stochastic dominance.

Proof. See Theorem 2.A.17 in Shaked and Shanthikumar (1994). ■

Following Blackwell (1951) (1953), we say that one signal is more informative than the other if a decision-maker can achieve a higher expected utility when basing a decision on the realization of the more informative signal. We extend Blackwell's information criterion to our multi-agent setting by applying his criterion to each bidder while fixing other bidders' information choices.

Proposition 1 (Rotation Order and Blackwell's Criterion)
Suppose that $\{H_{\alpha_i}\}$ is rotation-ordered and $\alpha'_i > \alpha''_i$. Then under any mechanism $\{q_i(v), t_i(v)\}$ that is incentive compatible, bidder i achieves a higher expected payoff with information choice α'_i than information choice α''_i.

The above result is intuitive. Because the bidder i's interim payoff $u(v_i)$ is convex in v_i under any incentive compatible mechanism (Rochet (1987)), and because $H_{\alpha'_i}$ second-order stochastic dominates $H_{\alpha''_i}$ (by Lemma 1), bidder i's expected payoff is higher under the more risky prospect $H_{\alpha'_i}$. Therefore, if $\{H_\alpha\}$ is rotation-ordered and $\alpha'_i > \alpha''_i$, then a signal with α'_i is indeed more informative than a signal with α''_i because α'_i corresponds to a higher expected payoff for bidder i.

4 Optimal Auctions with One Bidder and Gaussian Specification

We start with a simple model with one buyer. If the buyer's information is exogenous, Riley and Zeckhauser (1983) show that the optimal selling mechanism is to post a non-negotiable price. With endogenous information, their logic still applies and a posted price is optimal.[18] Therefore, we can also reinterpret the seller's optimization problem as a monopoly pricing problem with endogenous information.

We first examine the buyer's information decision and show that the marginal value of information to the buyer increases as the reserve price (or monopoly price) moves toward the mean valuation. Then we analyze the seller's information preferences and demonstrate that the seller would prefer a more informed buyer if and only if the monopoly price is above the mean valuation. The above two observations lead to the main result of this section: the optimal monopoly price with endogenous information is always lower than the standard monopoly price with exogenous information.

For the purposes of illustration, we focus on a special but important rotation-ordered information structure: the *Gaussian* specification, though it is straightforward to extend the analysis to general rotation-ordered information structures.

4.1 Gaussian Specification

The buyer's true valuations ω_i are drawn from a normal distribution with mean μ and precision β : $\omega_i \sim N(\mu, 1/\beta)$. Lowering β has the consequence that the prior distribution becomes more spread out, yielding more potential gains from information acquisition.

The buyer can observe a costly signal s_i:

$$s_i = \omega_i + \varepsilon_i,$$

where the additive error ε_i is independent of ω_i, and $\varepsilon_i \sim N(0, 1/\alpha_i)$. The higher is the α_i, the more precise is the signal. Thus we interpret α_i as the informativeness (precision) of the buyer's signal. α_i is assumed to have two parts:

$$\alpha_i = \underline{\alpha} + \gamma_i.$$

The first part, $\underline{\alpha}$, is the endowed signal precision; the second term γ_i is the additional precision obtained by investing in information acquisition. For illustration purposes, the cost of information is assumed to be linear in the incremental precision. That is,

$$C(\alpha_i) = c\gamma_i = c(\alpha_i - \underline{\alpha}),$$

where c is the constant marginal cost of one additional unit of precision.

After observing a signal s_i with precision α_i, the buyer updates his belief of ω_i. By the standard normal updating technique, the posterior valuation distribution conditional on the signal s_i will be normal:

$$\omega_i | s_i \sim N\left(\frac{\beta\mu + \alpha_i s_i}{\alpha_i + \beta}, \frac{1}{\alpha_i + \beta}\right).$$

[18] The one-bidder model is a special case of the general model we study later. As shown in the next section, after incorporating the information acquisition constraint, the seller's objective function will be the Lagrangian specified in (10). If there is only one bidder, it reduces to a simple form similar to the one analyzed in Riley and Zeckhauser (1983). Therefore, their proof of the optimality of the posted price mechanism still applies here.

It immediately follows that

$$v_i(s_i, \alpha_i) \equiv \mathbb{E}(\omega_i | s_i, \alpha_i) = \frac{\beta\mu + \alpha_i s_i}{\alpha_i + \beta}.$$

Thus the distribution of the posterior estimate v_i, $H_{\alpha_i}(v_i)$, is normal:

$$v_i \sim N\left(\mu, \sigma^2(\alpha_i)\right), \text{ where } \sigma(\alpha_i) = \sqrt{\frac{\alpha_i}{(\alpha_i + \beta)\beta}}.$$

Note that the variance of v_i is increasing in the information choice α_i. So the distribution H_{α_i} will be more spread out for a more precise signal. The following two graphs capture the relationship between two distributions of the posterior estimate with different information choices.

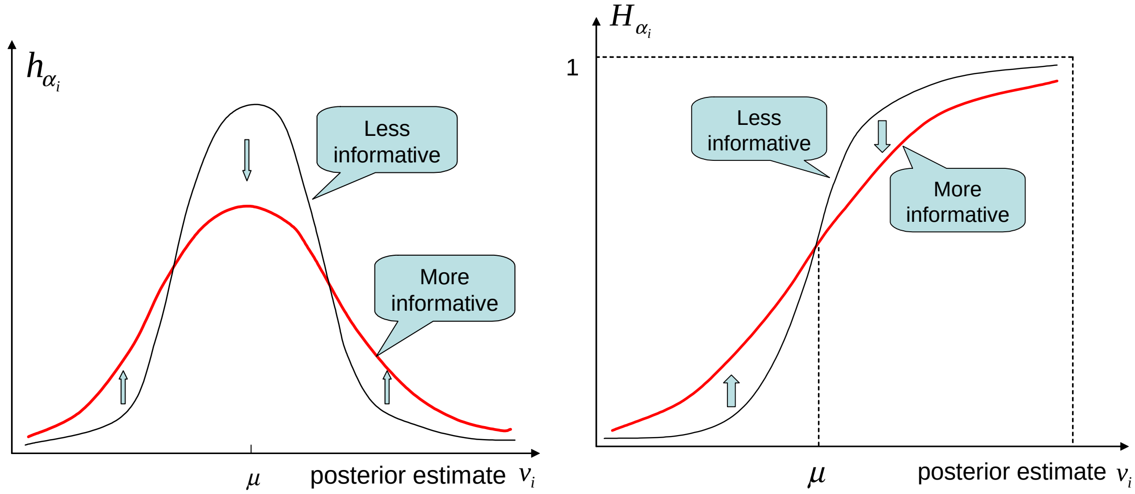


Figure 2. PDF and CDF of the posterior estimate with different signals

The left graph in Figure 2 shows that the density of the posterior estimate with a more informative signal is more dispersed than the one with a less informative signal. The right graph shows that the distribution with a less informative signal crosses the distribution with a more informative one from below at the mean valuation. With some algebra, we can show

$$v_i \gtreqless \mu \Leftrightarrow \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} \lesseqgtr 0. \tag{2}$$

Therefore, $\{H_{\alpha_i}\}$ are rotation-ordered and the rotation point $v_{\alpha_i}^+ = \mu$ for all α_i.

4.2 Marginal Value of Information to the Buyer

Given the reserve price r, the buyer chooses α_i to maximize his expected payoff:

$$\max_{\alpha_i} \int_r^\infty (v_i - r)\, h_{\alpha_i}(v_i)\, dv_i - c(\alpha_i - \underline{\alpha})$$

The marginal value of information (MVI) to the buyer is given by

$$MVI \equiv \frac{\partial\left[\int_r^\infty (v_i - r)\, h_{\alpha_i}(v_i)\, dv_i\right]}{\partial \alpha_i} = -\int_r^\infty \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} dv_i.$$

The following proposition shows how the marginal value of information to the buyer varies with respect to the reserve price.

Proposition 2 (Marginal Value of Information to the Buyer)
The marginal value of information to the buyer increases as the reserve price r moves toward the mean valuation μ, and achieves maximum at $r = \mu$.

Proof. With some algebra, we can show that

$$\frac{\partial [MVI]}{\partial r} = \frac{\partial H_{\alpha_i}(r)}{\partial \alpha_i} = -\frac{(r-\mu)}{2\sqrt{2\pi}} \exp\left(-\frac{(r-\mu)^2}{2\sigma^2}\right)\sqrt{\frac{\beta^3}{\alpha_i^3(\alpha_i+\beta)}}. \tag{3}$$

Therefore, as $r \to \mu$, MVI increases. ■

This finding is crucial in understanding other results obtained in the paper. To understand it better, consider a discrete version of the marginal value of information with two information choices α_i and α_i' ($\alpha_i' > \alpha_i$). The buyer's gain from having signal α_i' rather than α_i is

$$\Delta VI = \int_r^{\infty} (H_{\alpha_i}(v_i) - H_{\alpha_i'}(v_i)) dv_i. \tag{4}$$

Since the two distributions have the same mean, we have

$$\mu = \int_{-\infty}^{\infty} (1 - H_{\alpha_i'}(v_i)) dv_i = \int_{-\infty}^{\infty} (1 - H_{\alpha_i}(v_i)) dv_i.$$

Therefore, we can also write the gain from more information as

$$\Delta VI = \int_{-\infty}^{r} (H_{\alpha_i'}(v_i) - H_{\alpha_i}(v_i)) dv_i. \tag{5}$$

The following two graphs illustrate the buyer's gain from more information.[19]

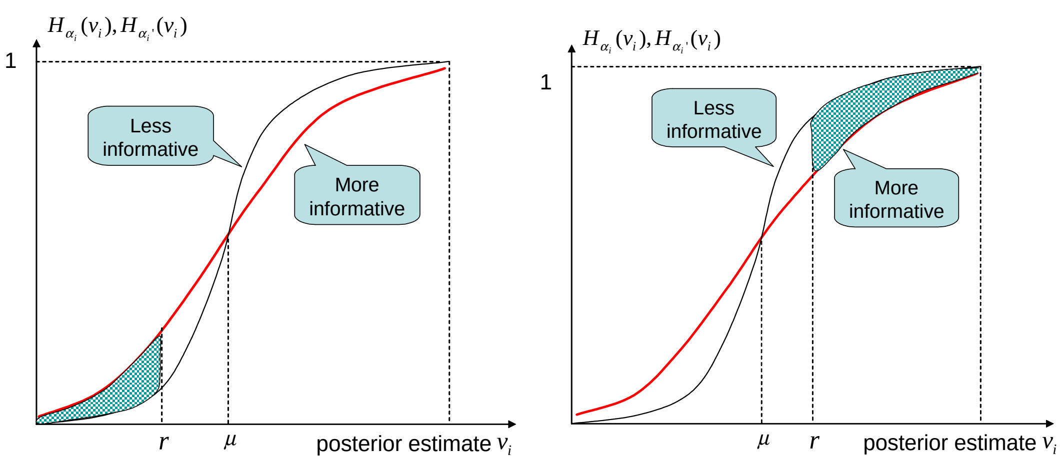


Figure 3: Buyer's gain from more information
Left ($r \geq \mu$): buyer's gain from more information (shaded area) decreases as r increases
Right ($r \leq \mu$): buyer's gain from more information (shaded area) increases as r increases

[19] I would like to thank Ben Polak for suggesting these two graphs.

Given the reserve price r, the payoff of the buyer with signal α_i is the area above the distribution H_{α_i} but below one and to the right of reserve price r. When $r \geq \mu$, the buyer's relative gain with signal α_i' rather than α_i is the shaded area in the left graph (see also expression (4)). On the other hand, when $r \leq \mu$, according to expression (5), the buyer's gain from more information is the shaded area in the right graph. In both cases, the shaded area expands as r moves toward μ and achieves maximum at the mean valuation.

Figure 3 also illustrates that the buyer's gain from a more informative signal is always positive. Under mild conditions, the buyer's expected payoff is an increasing concave function of α_i. Hence, the solution to the buyer's maximization problem will be unique, and the buyer's information choice will be decreasing in the information cost c (see Proposition 4 below).

4.3 Seller's Pricing Decision

For the seller, she chooses r and equilibrium α^* to maximize her revenue. That is

$$\begin{aligned} & \max_{r,\alpha^*} \; r\left(1 - H_{\alpha^*}(r)\right) \\ s.t. \quad : \quad & \alpha^* \in \arg\max_{\alpha_i} \int_r^\infty (v_i - r)\, h_{\alpha_i}(v_i)\, dv_i - c\left(\alpha_i - \underline{\alpha}\right). \end{aligned}$$

The buyer's (agent) information choice is unobservable to the seller (principal), and the seller sets r to align the buyer's incentives to her own. Thus, we can interpret it as a principal-agent problem. The standard way to solve this problem, the so-called *first-order approach,* is to assume that the second-order condition of the agent's maximization problem is satisfied, and use the first-order condition to replace the incentive constraint. We will assume the second-order condition is satisfied for now, and discuss it in detail at the end of this subsection.

Then, we can replace the buyer's optimization problem with the first-order condition, and rewrite the seller's optimization problem as[20]

$$\begin{aligned} & \max_{r,\alpha^*} \; r\left(1 - H_{\alpha^*}(r)\right) \\ s.t. \quad : \quad & -\int_r^\infty \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} dv_i - c = 0. \end{aligned} \qquad [\lambda]$$

Let λ be the Lagrangian multiplier for the constraint. We write the Lagrangian in a way such that a positive value of λ means that the seller benefits from a reduction in the information cost; in other words, the seller prefers a more informed buyer.

Lemma 2
For a fixed reserve price r, the seller's revenue increases in α^* if and only if $r > \mu$, and the seller's revenue decreases in α^* if and only if $r < \mu$.

Proof. Immediate from the definition of the seller's revenue and property (2) of the Gaussian specification. ■

[20] To simplify notation, in what follows, we will use $\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}$ to denote $\frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i}|_{\alpha_i = \alpha^*}$.

The intuition for this result is straightforward by looking at Figure 3. Suppose the buyer's information choice increases from α_i to α_i'. If $r > \mu$ (left figure), then more information increases the probability of trade from $(1 - H_{\alpha_i}(r))$ to $(1 - H_{\alpha_i'}(r))$. More information will therefore benefit the seller. In contrast, if $r < \mu$ (right figure), more information decreases the probability of trade from $(1 - H_{\alpha_i}(r))$ to $(1 - H_{\alpha_i'}(r))$, so more information will hurt the seller.

If we reinterpret our model as a monopoly pricing problem with a continuum of consumers, then this result links to the main findings in Johnson and Myatt (2006). To see this, we classify all markets into either *niche* markets or *mass* markets following Bergemann and Välimäki (2006b), and Johnson and Myatt (2006):

Definition 2 (Niche Market and Mass Market)
A market is said to be a niche (mass) market if the monopoly price is higher (lower) than the mean valuation μ.

Therefore, the lemma states that the monopolist would prefer better informed consumers if she is in a niche market. In contrast, the monopolist in a mass market will prefer not well-informed consumers. This result immediately leads to the key insight in Johnson and Myatt (2006): if information is free, then a seller in the niche (mass) market will provide the maximal (minimal) amount information to consumers to maintain its niche (mass) position.

Before stating our results about the optimal reserve price, we need to define a benchmark: the *standard reserve price* when information is endowed rather than acquired.

Definition 3 (Standard Reserve Price)
The standard reserve price r_α *is the optimal reserve price when the buyer's signal* α *is exogenous. That is*

$$r_\alpha \in \arg\max_r r\left(1 - H_\alpha(r)\right) \Rightarrow r_\alpha - \frac{1 - H_\alpha(r_\alpha)}{h_\alpha(r_\alpha)} = 0.$$

In particular, we will denote $r_{\underline{\alpha}}$ as the standard reserve price when no additional information (other than the endowed signal $\underline{\alpha}$) is acquired. Since normal distributions have an increasing hazard rate, r_α is uniquely defined for each H_α. The seller's optimal pricing rule can thus be stated as follows:

Proposition 3 (Optimal Reserve Price)
For a fixed β, *there exists a* $\widehat{\mu}$ *such that*

$$\begin{cases} \mu < r^* < r_{\alpha^*} & \text{if} \quad \mu < \widehat{\mu} \\ r^* = r_{\alpha^*} = \mu & \text{if} \quad \mu = \widehat{\mu} \\ r^* < r_{\alpha^*} < \mu & \text{if} \quad \mu > \widehat{\mu} \end{cases}.$$

Therefore, the optimal reserve price r^* *with endogenous information is always (weakly) lower than the standard reserve price* r_{α^*}.

In order to understand the seller's optimal pricing strategy, we can decompose the effect of a price increase on the seller's profits in three parts:

$$\frac{d\pi_s}{dr}|_{r=r^*} = \underbrace{1 - H_{\alpha^*}(r^*)}_{A} + \underbrace{[-r^* h_{\alpha^*}(r^*)]}_{B} + \underbrace{\left[-r^* \frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{\partial \alpha^*}{\partial r}\right]}_{C}.$$

First, the seller's profits increase given that a trade is made (term A). Second, for a fixed information choice, a price increase will reduce the probability of trade (term B). Third, with endogenous information acquisition, a price increase will affect the buyer's incentive to acquire information, thereby the probability of trade (term C). The first two terms are standard, while the last one is specific to the setting with endogenous information acquisition.

If $r^* > \mu$, $\partial H_{\alpha^*}(r^*)/\partial \alpha^* < 0$ by (2), and an increase in r discourages information acquisition: $\partial \alpha^*/\partial r < 0$, so term C is negative. If $r^* < \mu$, $\partial H_{\alpha^*}(r^*)/\partial \alpha^* > 0$ by (2), and the incentives to gather information are higher for a higher r: $\partial \alpha^*/\partial r > 0$. Again, term C is negative. Therefore, for $r^* \neq \mu$, the marginal gain to the seller from raising price r here is always smaller than the gain in a setting with exogenous information. As a result, the seller will charge a lower price: $r^* < r_{\alpha^*}$. For $r^* = \mu$, a marginal increase in price does not affect buyer's incentive to acquire information, so $r^* = r_{\alpha^*}$.

We conclude this subsection by presenting sufficient conditions for the second-order condition of the buyer's maximization problem to be satisfied. Under these conditions, the first-order approach is valid and the buyer's expected payoff is globally concave in the information choice α_i.

Proposition 4 (Validity of the First Order Approach)
If $r \in [\mu - 2\sigma(\underline{\alpha}), \mu + 2\sigma(\underline{\alpha})]$ and $\underline{\alpha} \geq \beta$, the second-order condition of the buyer's maximization problem is satisfied.

These conditions are stronger than necessary and are not very restrictive. Note that more than 95% of the normal density is within two standard deviations of the mean. Thus, the first condition is to ensure that the probability of trade under r will be higher than 2.5% but lower than 97.5%. In other words, the reserve price r is neither extremely high nor extremely low ensuring that the probability of trade is neither close to 1 nor close to 0. The second condition $\underline{\alpha} \geq \beta$ is to ensure $\alpha_i > \beta$ for all α_i.[21] It requires that signals be informative relative to the prior.

5 Optimal Auctions with Many Bidders

In the single-bidder model, the strategic interaction among bidders is absent, so the simple posted price mechanism is optimal. This section studies optimal auctions with many bidders and rotation-ordered information structures, and shows that most of the insights from the previous section carry through. Specifically, we show that: 1) A bidder's incentives to acquire information increase

[21] Under this condition, the equilibrium information level is away from zero. Therefore, we can avoid the non-concavity of the value of information identified in Radner and Stiglitz (1984).

as the reserve price moves toward the rotation point; 2) If we restrict attention to the symmetric mechanism that induces all bidders to acquire the same level of information, standard auctions with an adjusted reserve price are optimal; 3) If information decision is discrete, then the optimal selling mechanism reduces the level of price discrimination against (stochastically) strong bidders compared to the case with exogenous information; 4) We derive a necessary and sufficient condition under which the bidders' incentives to acquire information are socially excessive in standard auctions.

One insight that cannot be carried over from the one-bidder case, however, concerns the seller's information preferences. If there are sufficiently many bidders, the seller will encourage information acquisition — even when the standard reserve price is lower than the mean valuation (rotation point). We show that, in many cases, the seller will prefer that bidders acquire more information, as long as the number of bidders is large.

5.1 Marginal Value of Information to the Buyer

As stated in Section 3, the seller's optimization problem is to choose a menu $\{q_i(v_i, v_{-i}), t_i(v_i, v_{-i})\}$ and a vector of information choices $(\alpha_1^*, \alpha_2^*, ..., \alpha_n^*)$ to maximize her revenue subject to (IC) (IR) and (IA) constraints.

It is well-known (Myerson (1981), and Rochet (1987)) that the incentive compatibility constraint (IC) is equivalent to the following two conditions:

$$u_i(v_i) = u_i(\underline{\omega}) + \int_{\underline{\omega}}^{v_i} Q_i(x)\, dx, \tag{6}$$

and

$$Q_i(v_i) \text{ is increasing in } v_i. \tag{7}$$

With equation (6), we can write the individual rationality constraint (IR) simply as $u_i(\underline{\omega}) \geq 0$.

The information acquisition constraint (IA) requires that α_i^* be bidder i's best response given that other bidders chooses $\alpha_{-i}^* = (\alpha_1^*, ..., \alpha_{i-1}^*, \alpha_{i+1}^*, ..., \alpha_n^*)$. That is, for all i,

$$\alpha_i^* \in \arg\max_{\alpha_i} \mathbb{E}_{v_{-i},\alpha_{-i}^*} \left\{ \int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}} [1 - H_{\alpha_i}(v_i)]\, q_i(v_i, v_{-i})\, dv_i - C(\alpha_i) \right\}.$$

The subscript α_{-i}^* of the expectation operator is to remind the readers that the expectation depends on the information choice α_{-i}^* of bidder i's opponents. The subscript α_i in the lower and upper limits $(\underline{\omega}_{\alpha_i}, \overline{\omega}_{\alpha_i})$ is to emphasize the fact that the support of the posterior estimates may depend on the information choice α_i. The first-order condition is

$$-\mathbb{E}_{v_{-i},\alpha_{-i}^*} \int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}} \frac{\partial H_{\alpha_i^*}(v_i)}{\partial \alpha_i^*} q_i(v_i, v_{-i})\, dv_i - C'(\alpha_i^*) = 0. \tag{8}$$

Let r_i denote the personalized reserve price for bidder i in the optimal auction. That is, if bidder i is allocated the object with positive probability, then his posterior estimate is at least r_i:

$$q_i(v_i, v_{-i}) > 0 \Rightarrow v_i \geq r_i.$$

The marginal value of information to bidder i with reserve price r_i is

$$MVI = -\mathbb{E}_{v_{-i},\alpha_{-i}^*}\left[\int_{r_i}^{\overline{\omega}_{\alpha_i}} \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} q_i(v_i, v_{-i})\, dv_i\right].$$

Theorem 1 (Marginal Value of Information to a Buyer)
The marginal value of information to a buyer increases as r_i moves toward the rotation point $v_{\alpha_i}^+$ if and only if signals are rotation-ordered.

Proof. Sufficiency. Notice that

$$\frac{\partial [MVI]}{\partial r_i} = \mathbb{E}_{v_{-i},\alpha_{-i}^*}\left[\frac{\partial H_{\alpha_i}(r_i)}{\partial \alpha_i} q_i(r_i, v_{-i})\right]. \tag{9}$$

Rotation order implies that $\partial H_{\alpha_i}(r_i)/\partial\alpha_i > 0$ if $r_i < v_{\alpha_i}^+$ and $\partial H_{\alpha_i}(r_i)/\partial\alpha_i < 0$ if $r_i > v_{\alpha_i}^+$. Therefore, MVI is increasing in r_i if $r_i < v_{\alpha_i}^+$ and decreasing in r_i if $r_i > v_{\alpha_i}^+$.

Necessity. Suppose signals are not rotation-ordered. Then the two distributions associated with two different α_i's must cross at least twice. Without loss of generality, suppose the other crossing points is lower than $v_{\alpha_i}^+$. Then we can find a $r_i < v_{\alpha_i}^+$ such that

$$\frac{\partial H_{\alpha_i}(r_i)}{\partial \alpha_i} < 0.$$

Thus, MVI decreases as r_i moves toward $v_{\alpha_i}^+$ by (9), a contradiction. ■

Theorem 1 generalizes Proposition 2 to a setting with many bidders and rotation-ordered information structures. Therefore, if a seller wants to induce buyer i to acquire more information, she should set the reserve price r_i closer to the rotation point $v_{\alpha_i}^+$.

5.2 Characterization of Symmetric Optimal Auctions

In order to characterize optimal auctions, we need to simplify the (IA) constraints. If the first-order approach is valid, we can can replace bidder i's optimization problem by (8). It is valid if the second-order condition of bidders' optimization problem is satisfied, which we will assume for now and relegate detailed discussions to Appendix B. In principle, the equilibrium information choices could be different: $\alpha_i^* \neq \alpha_j^*$, for $i \neq j$, so there will be a system of n first-order conditions: one for each bidder.

We use Lagrangian approach to incorporate the n first-order conditions. As in the standard moral hazard model, the main difficulty lies in the determination of the sign of the Lagrangian multiplier of these first-order conditions (Rogerson (1985)). The seller's maximization problem here, however, is substantially more complicated in three ways. First, we have n agents and n first-order conditions. Second, unlike in the standard moral hazard model where higher effort always benefits the principal if it is costless to induce effort, more information here may hurt the seller, as we can see from Lemma 2. Finally, the seller has to give bidders not only incentives to acquire information, but also incentives to tell the truth, i.e., our model is a mixed model with moral hazard and adverse selection. As such, some restrictions on the model are necessary in order to characterize the optimal selling mechanism.

In this subsection, we restrict our attention to symmetric mechanisms with $\alpha_1^* = \cdots = \alpha_n^* = \alpha^*$. This restriction helps reduce the system of first-order conditions to a single equation (8).[22] Replacing the incentive constraint by equation (6) and (7), and replacing the (IA) constraint by (8), we can transform the seller's optimization problem from the allocation-transfer space into the allocation-utility space:

$$\max_{q_i, u(\underline{\omega}), \alpha^*} \left\{ \mathbb{E}_{v,\alpha^*} \sum_{i=1}^{n} \left[\left(v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} \right) q_i(v_i, v_{-i}) \right] - n u_i(\underline{\omega}) \right\}$$

subject to

$$0 \leq q_i(v_i, v_{-i}) \leq 1; \sum_{i=1}^{n} q_i(v_i, v_{-i}) \leq 1, \tag{Regularity}$$

$$Q_i(v_i) \text{ is increasing in } v_i, \tag{Monotonicity}$$

$$u_i(\underline{\omega}) \geq 0, \tag{IR}$$

$$\mathbb{E}_{v,\alpha^*} \left[-\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} \frac{1}{h_{\alpha^*}(v_i)} q_i(v_i, v_{-i}) \right] - C'(\alpha^*) = 0. \tag{IA}$$

It is easy to see that the (IR) constraint must be binding. For now we ignore the regularity constraint and the monotonicity constraint, and verify them later. Then the only remaining constraint is the (IA) constraint. Let λ denote the Lagrangian multiplier for the (IA) constraint, and write the Lagrangian for the seller's maximization problem as

$$\mathcal{L} = \mathbb{E}_{v,\alpha^*} \sum_{i=1}^{n} \left[\left(v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n} \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} \frac{1}{h_{\alpha^*}(v_i)} \right) q_i(v_i, v_{-i}) \right] - \lambda C'(\alpha^*). \tag{10}$$

Then a positive λ implies that the seller's revenue increases as the marginal cost of information decreases. The virtual surplus function $J^*(v_i)$ can be defined as

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n} \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} \frac{1}{h_{\alpha^*}(v_i)}. \tag{11}$$

In order to characterize the optimal solution to the seller's optimization problem, we make the following assumptions:

Assumption 1 (Rotation Order)
The family of distributions of the posterior estimates, $\{H_{\alpha_i}\}$, is rotation-ordered and the rotation point is μ for all α_i.

[22] A sufficient condition for the existence of a symmetric equilibrium is that there exists a α^* satisfying both the first-order condition and the second-order condition of the buyer's maximization problem. If we assume $\lim_{\alpha \to \underline{\alpha}} C'(\alpha) = 0$, and $\lim_{\alpha \to \overline{\alpha}} C'(\alpha) = \kappa$ (where κ is a large positive number), then there must exist a α^* satisfies the first-order condition (8). If the cost function is sufficiently convex, that is, $C''(\alpha)$ is sufficient large, then the second-order condition is satisfied (See Appendix B for more detail). A quadradic cost function $C(\alpha) = \kappa_0 (\alpha - \underline{\alpha})^2$ with large κ_0 meets all the requirements.

Assumption 2 (Monotonicity)

$$-\frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i}\frac{1}{h_{\alpha_i}(v_i)} \text{ is increasing in } v_i \text{ for all } \alpha_i \text{ and } v_i \in \left[\underline{\omega}_{\alpha_i}, \overline{\omega}_{\alpha_i}\right].$$

Assumption 3 (Regularity)

$$v_i - \frac{1-H_{\alpha_i}(v_i)}{h_{\alpha_i}(v_i)} \text{ is increasing in } v_i \text{ for all } \alpha_i \text{ and } v_i \in \left[\underline{\omega}_{\alpha_i}, \overline{\omega}_{\alpha_i}\right].$$

Assumption 1 assumes that the signals are rotation-ordered and the rotation point $v_{\alpha_i}^{+}$ is μ for all α_i. The assumption $v_{\alpha_i}^{+} = \mu$ is not critical, but it greatly eases our presentation. We will discuss it later. Assumption 2 is stronger than the rotation order assumption, and it says that the expected gain from more information is higher for the buyers with higher v_i.[23] Finally, Assumption 3 is a regularity assumption.

Both the rotation order assumption and the regularity assumption are mild assumptions. The monotonicity assumption is relatively more restrictive, but two commonly used information technologies in the literature, the Gaussian specification and the truth-or-noise technology (Lewis and Sappington (1994)), satisfy all three assumptions.

Definition 4 (Truth-or-noise Technology)
The buyers' true valuations $\{\omega_i\}$ are independently drawn from a distribution F, and F has an increasing hazard rate. Buyer i can acquire a costly signal s_i about ω_i. With probability $\alpha_i \in [\underline{\alpha}, 1]$, the signal s_i perfectly matches the true valuation ω_i, and with probability $1-\alpha_i$, s_i is a noise independently drawn from F.

Under the truth-or-noise specification, the signal s_i sometimes perfectly reveals buyer i's valuation ω_i, but is noise otherwise.

Lemma 3 (All Assumptions Hold for the Two Leading Examples)
Both the Gaussian specification and the truth-or-noise technology generate a family of distributions $\{H_{\alpha_i}\}$ that satisfies Assumptions 1, 2, and 3.

Note that Assumption 1 does not imply that the underlying distribution F is symmetric. For example, for the truth-or-noise technology, the underlying distribution F could be convex or concave, but the rotation point is still μ.

In order to characterize the symmetric optimal auction, we first need to identify the seller's information preferences, that is, the sign of the Lagrangian multiplier λ for the (IA) constraint. It turns out that this is the most difficult part of the analysis. We use the technique in Rogerson

[23] Indeed, the monotonicity assumption, together with the mean-preserving property of our information structures, implies rotation order. To see this, first note that $\frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i}$ cannot always be positive or negative, otherwise it will imply first order stochastic dominance which violates the fact that the family of distributions $\{H_{\alpha_i}\}$ have the same mean. Therefore, if monotonicity assumption holds, $\frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i}$ must change sign from positive to negative only once. That is, $\{H_{\alpha_i}\}$ is rotation ordered.

(1985) to sign λ: we relax the (IA) constraint to an inequality constraint, characterize the optimal solution of the relaxed problem, and then verify that (IA) constraint is binding in the optimal solution under the set of conditions in Lemma 4 below.

Lemma 4
Suppose the first-order approach is valid. The seller benefits from a reduction of marginal cost ($\lambda > 0$), when either one of the following two sets of conditions is satisfied:
(1) Assumptions 1-3 hold and $r_{\underline{\alpha}} > \mu$.
(2) The Gaussian specification or the truth-or-noise technology, and large n.

The first condition implies $r^* > \mu$, which is sufficient for $\lambda > 0$. Recall that, in the case of one bidder, the seller prefers more information if $r^* > \mu$. An increase in the number of bidders only strengthens the seller's preference for more information. The second condition should be contrasted with Lemma 2 in the case of one bidder. The strategic interaction between buyers, which is absent in the one-bidder model, plays a crucial part here. To see this, note that the seller's revenue is determined by the valuation of the marginal bidder (for example, the second highest bidder) and the reserve price. With many bidders, the valuation of the marginal bidder will be higher than the mean valuation. This valuation is likely to be higher when more information is acquired. In the case with one bidder, however, a seller will prefer a more informed buyer only when the optimal reserve price is higher than the mean valuation (niche market).

Now, we can present a simple rule for adjusting the reserve price in optimal auctions with information acquisition:[24]

Theorem 2 (Simple Rule for Adjusting the Reserve Price)
Suppose the first-order approach is valid, Assumptions 1 and 3 hold. If $\lambda > 0$, then the optimal reserve price r^* is closer to the mean valuation μ than the standard reserve price r_{α^*}. Specifically,

$$\begin{cases} \mu \leq r^* < r_{\alpha^*} & \text{if} \quad r_{\alpha^*} > \mu \\ r^* = \mu & \text{if} \quad r_{\alpha^*} = \mu \\ r_{\alpha^*} < r^* \leq \mu & \text{if} \quad r_{\alpha^*} < \mu \end{cases} .$$

If $\lambda < 0$, then $r^* < r_{\alpha^*} < \mu$.

Theorem 2 is conceptually a direct consequence of Theorem 1, and generalizes Proposition 3. It characterizes the relationship between the optimal reserve price in our setting and the standard reserve price in Myerson's optimal auctions. If the seller wants to encourage information acquisition, she has to set the optimal reserve price between the mean valuation and the standard reserve price because the bidders' incentives to acquire information are stronger when the reserve price is closer to the mean valuation.

[24]The next two theorems will characterize the optimal selling mechanism contingent on the sign of the endogenous Lagrangian multiplier λ. With Lemma 4, we can always restate the theorems by replacing the condition $\lambda > 0$ by the exogenous condition (1) or (2). However, since both condition (1) and (2) are not necessary for $\lambda > 0$, we state our theorems in terms of the sign of λ.

This result is important in practice when the seller is concerned about bidders' incentives to acquire information. The reserve price is always the most important decision she has to make other than choosing the auction format. Theorem 2 identifies a simple rule to adjust the reserve price when endogenous information acquisition is important.

This result also has important implication for empirical studies. The empirical auction literature has attempted to evaluate the optimality of a seller's reserve price policy. Most of these studies assume exogenous information and do not consider the bidders' incentives to acquire information. They use observed bids and the equilibrium bidding behavior to recover the distribution of bidders' valuations, and then compare the actual reserve price with the standard reserve price calculated from the estimated distribution. Our results indicate that, in situations where information acquisition is important, the standard reserve price may not be an appropriate benchmark for comparison.

The next result shows that under the stronger Assumption 2, standard auctions with an appropriately chosen reserve price are optimal.

Theorem 3 (Optimal Auctions)
Suppose the first-order approach is valid, Assumptions 1, 2 and 3 hold, and $\lambda > 0$. Then standard auctions with the reserve price r^* adjusted according to Theorem 2 are optimal.

An immediate consequence of Theorem 3 is the revenue equivalence among all standard auctions, because the allocation rule is the same across all standard auctions. Furthermore, since the bidders' expected gain from information acquisition is the same for all standard auctions, the equilibrium amount of information acquired is the same across standard auctions as well.

The restriction of symmetric equilibrium is important for the above result. If bidders are allowed to acquire different levels of information in equilibrium, the revenue equivalence theorem fails. If we assume that information acquisition is discrete rather than continuous, however, we can characterize the optimal selling mechanism without the symmetric restriction, as shown in the following subsection.

5.3 Asymmetric Mechanisms with Discrete Information Acquisition

In this subsection, we assume information acquisition is discrete, so we can drop both the symmetric restriction and the first-order approach. Specifically, bidders are assumed to be ex ante symmetric and endowed with signal with precision α_0. Each bidder can opt to receive a signal α_1 that is more informative than signal α_0 in terms of rotation order, but he has to incur a cost c. The distribution of bidder i's posterior estimates v_i is denoted by $H_0(\cdot)$ if he does not acquire information, and $H_1(\cdot)$ if he acquires information. Let h_0 and h_1 denote the corresponding densities.

By revelation principle, we can restrict to the direct revelation mechanism $\{q_i(v), t_i(v)\}$. Without loss of generality, suppose the seller wants to induce first m bidders $(0 \leq m \leq n)$ to acquire additional information. Then the mechanism must satisfy the standard (IC), (IR) constraints,

$$\begin{aligned} u_i(v_i) &= u_i(\underline{\omega}) + \int_{\underline{\omega}}^{v_i} Q_i(v_i)\, dv_i \text{ and } Q_i(v_i) \text{ is increasing in } v_i, \text{ for all } i \text{ and } v_i \\ u_i(v_i) &\geq 0, \text{ for all } i \text{ and } v_i \end{aligned}$$

and information acquisition (IA) constraints:

$$\begin{aligned} \int_{\underline{\omega}}^{\overline{\omega}} u_i(v_i)\, dH_1(v_i) - \int_{\underline{\omega}}^{\overline{\omega}} u_i(v_i)\, dH_0(v_i) &\geq c, \text{ for } i \leq m \\ \int_{\underline{\omega}}^{\overline{\omega}} u_i(v_i)\, dH_1(v_i) - \int_{\underline{\omega}}^{\overline{\omega}} u_i(v_i)\, dH_0(v_i) &\leq c, \text{ for } i > m \end{aligned}$$

That is, the mechanism has to ensure that the first m bidders have incentives to acquire information and the remaining $(n-m)$ bidders have incentives *not* to acquire information.

Let λ_i denote the Lagrangian multiplier of the information acquisition constraint for bidder i, $i = 1, ..., n$. Then the Lagrangian can be written and simplified as

$$\begin{aligned} \mathcal{L} &= \sum_{i=1}^{m} \left\{ \int_{\underline{\omega}}^{\overline{\omega}} \left[v_i - \frac{1 - H_1(v_i)}{h_1(v_i)} + \lambda_i \frac{H_0(v_i) - H_1(v_i)}{h_1(v_i)} \right] Q_i(v_i)\, dH_1(v_i) - \lambda_i c \right\} \\ &\quad + \sum_{i=m+1}^{n} \left\{ \int_{\underline{\omega}}^{\overline{\omega}} \left[v_i - \frac{1 - H_0(v_i)}{h_0(v_i)} - \lambda_i \frac{H_0(v_i) - H_1(v_i)}{h_0(v_i)} \right] Q_i(v_i)\, dH_0(v_i) + \lambda_i c \right\} \end{aligned} \tag{12}$$

Consider two bidders, i and j: bidder i acquires information and bidder j does not. Then the distributions of the posterior estimates of bidder i and j are H_1 and H_0, respectively. Let r_0 and r_1 solve

$$r_0 - \frac{1 - H_0(r_0)}{h_0(r_0)} = 0 \text{ and } r_1 - \frac{1 - H_1(r_1)}{h_1(r_1)} = 0.$$

Since information structures are rotation-ordered,

$$\begin{aligned} H_0(x) > H_1(x) &\quad \text{if} \quad x > v^+ \\ H_0(x) < H_1(x) &\quad \text{if} \quad x < v^+ \end{aligned},$$

where v^+ is the rotation point. Therefore, conditional on the posterior estimate $x > v^+$, informed bidder i is stochastically "stronger" than bidder j, but conditional on $x < v^+$, uninformed bidder j is stochastically "stronger" than bidder i. Myerson (1981) demonstrates that the optimal auction should discriminate against "strong" bidders. Interestingly, endogenous information acquisition reduces the level of ex-post discrimination against "strong" bidders, as shown in the following theorem.

Theorem 4 (Asymmetric Mechanism)
With endogenous information, the optimal reserve price for informed bidder i, r_i^*, lies between r_1 and v^+, and the optimal reserve price for uninformed bidder j, r_j^*, is set such that r_0 is between r_j^* and v^+. Moreover, the mechanism reduces the level of price discrimination against "strong" bidders if signals are rotation-ordered.

Proof. The proof of the first part is identical to the proof of Theorem 2. For the second part, notice that the difference between the virtual surplus functions for bidder i and bidder j is:

$$J_i(x) - J_j(x) = \left(x - \frac{1 - H_1(x)}{h_1(x)}\right) - \left(x - \frac{1 - H_0(x)}{h_0(x)}\right) + \lambda_i \frac{H_0(x) - H_1(x)}{h_1(v_i)} + \lambda_j \frac{H_0(x) - H_1(x)}{h_0(x)}.$$

Since $\lambda_i, \lambda_j \geq 0$,

$$\begin{array}{ll} J_i(x) - J_j(x) \geq \left(x - \frac{1-H_1(x)}{h_1(x)}\right) - \left(x - \frac{1-H_0(x)}{h_0(x)}\right) & \text{if} \quad x > v^+ \\ J_i(x) - J_j(x) \leq \left(x - \frac{1-H_1(x)}{h_1(x)}\right) - \left(x - \frac{1-H_0(x)}{h_0(x)}\right) & \text{if} \quad x < v^+ \end{array}.$$

Thus, compared to the case with exogenous information, informed bidder i is treated more favorably if the winning bid is higher than v^+, and uninformed bidder j is treated more favorably otherwise. In both cases, endogenous information acquisition reduces price discrimination against "strong" bidders. ■

The first part of the theorem shows that the optimal rule for adjusting reserve price identified in Theorem 2 is still valid in this discrete setting. The seller would set the optimal reserve price closer to the rotation point in order give bidder i incentives to acquire information. The second part suggests that the seller would soften price discrimination in order to provide bidders with appropriate incentives either to or not to acquire information.

5.4 Informational Efficiency

Theorem 3 states that standard auctions with an adjusted reserve price are optimal when we restrict the equilibrium to be symmetric. This subsection will examine the informational efficiency of standard auctions and obtain a slightly more general results that can apply to optimal symmetric auctions.

We use a symmetric benchmark in which the social optimal information choice α^{FB} is the same for all bidders. For all i :

$$\alpha^{FB} \in \arg\max_{\alpha_i} \int_0^{\varpi_{\alpha_i}} \left(1 - H_{\alpha_i}^n(v_i)\right) dv_i - nC(\alpha_i).$$

At information level α_i, the marginal value of information to the social planner is

$$MVI^{FB}(\alpha_i) = -n \int_0^{\varpi_{\alpha_i}} \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} H_{\alpha_i}^{n-1}(v_i)\, dv_i. \tag{13}$$

Recall that, at information level α_i, the marginal value of information to the bidder i is

$$MVI(\alpha_i) = -\int_r^{\varpi_{\alpha_i}} \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} H_{\alpha_i}^{n-1}(v_i)\, dv_i. \tag{14}$$

Since the social planner has to pay n times the individual information cost, we normalize the social value of information by multiplying $1/n$. The difference between the social and individual gain from acquiring information is

$$\Delta(\alpha_i, n) = \frac{1}{n} MVI^{FB}(\alpha_i) - MVI(\alpha_i) = -\int_0^r \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} H_{\alpha_i}^{n-1}(v_i)\, dv_i. \tag{15}$$

By definition of rotation order, if $r < \mu$, $\Delta(\alpha_i, n) < 0.$ That is, information acquisition in auctions with $r < \mu$ is socially excessive. Thus, we have proved the following result.

Proposition 5 (Informational Efficiency)
Suppose Assumption 1 holds. There exists $\delta > 0$ such that bidders have socially excessive incentives to acquire information in standard auctions if and only if $r < \mu + \delta$.

When $r = 0$, the bidders' incentive to acquire information coincides with the social optimum, which can be easily seen from equation (15).[25] As r increases, the buyers' incentive to acquire information increases, reaches maximum at $r = \mu$, and declines afterwards. Consequently, there exists a $\delta > 0$, such that the individual incentive to acquire information coincides with the social optimum when $r = \mu + \delta$. Therefore, the bidders' incentive to acquire information is socially excessive when $r \in (0, \mu + \delta)$. For the one-bidder model with the Gaussian specification studied in Section 4, $\delta = \mu$.

5.5 Discussion

Rotation order. The rotation order ranks different information structures by comparing the distributions of the posterior estimates. In contrast, most existing information orders (for example, Lehmann (1988)) impose restrictions on the prior or posterior distributions of underlying states and signals. One can show that a weaker version of Lehmann's order, the MIO-ND order in Athey and Levin (2001), generates a family of distributions $\{H_{\alpha_i}\}$ ordered in terms of second-order stochastic dominance. But second-order stochastic dominance is not strong enough for our analysis.

Assumption 1 restricts the rotation point to be the mean valuation. However, if the rotation point v^+ is different from μ and does not move as level of information varies, then all our results remain valid as long as we replace μ in the statements of the results by v^+. If the rotation point moves as more information is acquired, or the rotation order assumption fails (for example, two distributions of the posterior estimate cross each other more than once), then some of our results (for instance, Theorem 1) still hold locally.

First order approach. In Appendix B, we provide several sets of sufficient conditions for the first-order approach to be valid. First, it is satisfied if the cost function is sufficiently convex. Second, if the support of H_{α_i} is invariant with respect to α_i, then a condition analogous to the CDFC condition in the principal-agent literature (Mirrlees (1999), and Rogerson (1985)) is sufficient. Third, we present sufficient conditions for the case of the Gaussian specification and the truth-or-noise technology, respectively.

As pointed out in Bolton and Dewatripoint (2005), the requirement that the bidders' first-order condition be necessary and sufficient is too strong. All we need is that the replacement of the (IA) constraint by the first-order condition can generate necessary conditions for the seller's original maximization problem. Thus, our analysis may remain valid even when the second-order condition of the bidders' maximization problem fails.

[25] This is consistant with the results in Bergemann and Välimäki (2002): the individual incentives to acquire information coincide with the social optimum for efficient mechanisms in the private value setting.

Private value. Our model focuses on the independent private value framework, but it can be immediately extended to a setting with a common component. Suppose buyer i's true valuation θ_i has two components:

$$\theta_i = \omega_i + y.$$

The first term ω_i represents the individual idiosyncratic valuation and is unknown ex-ante. Buyer i can acquire costly information about ω_i. The second term y is the common value component, and both the buyers and the seller learn it for free.[26] In this setting, all our analysis remain valid because the common component only shifts the distribution but does not affect the buyers' incentives.

Pre-auction investment. Finally, our model framework can easily modified to analyze pre-auction investment that stochastically shifts bidders' valuation upwards.[27]

6 Conclusion

The mechanism design literature studies how carefully designed mechanisms can be used to elicit agents' private information in order to achieve a desired goal. Most of the papers in the literature, however, ignore the influence of the proposed mechanisms on agents' incentives to gather information. In particular, with endogenous information acquisition, the optimal selling mechanism should take into account the bidders' information decision as a response to the proposed mechanism. We show that under some conditions standard auctions with a reserve price remain optimal but the reserve price has to be adjusted in order to incorporate the buyers' incentives to acquire information.

Relative to the existing literature, our model has three distinctive features. First, we study the optimal mechanism that maximizes revenue in the presence of information acquisition. This distinguishes our model from papers studying information acquisition in fixed auction formats. Second, we study private and decentralized information acquisition, thus differing from previous studies on the seller's optimal disclosure policy and various entry models. Finally, the information structure required for our results is more general than most of the existing literature on mechanism design: we require only that the distributions of the posterior estimates be rotation-ordered.

[26]For example, a firm typically has two types of assets: liquid and illiquid. All potential buyers of the firm may agree on the value liquid assets reflected in the financial statement, but they may value the illiquid assets differently.

[27]See Obara (2007) for an analysis of optimal auctions with hidden actions and correlated signals.

7 Appendix A: Omitted Proofs

Proof of Proposition 1: Under mechanism $\{q_i(v_i, v_{-i}), t_i(v_i, v_{-i})\}$, a bidder's expected payoffs (information rent) with information structure α_i' and α_i'' are, respectively[28]

$$\begin{aligned}
\mathbb{E}u\left(v_i;\alpha_i'\right) &= \mathbb{E}_{v_{-i}}\left[\int_{\underline{\omega}}^{\overline{\omega}}\left(1-H_{\alpha_i'}(v_i)\right)q_i(v_i,v_{-i})\,dv_i\right],\\
\mathbb{E}u\left(v_i;\alpha_i''\right) &= \mathbb{E}_{v_{-i}}\left[\int_{\underline{\omega}}^{\overline{\omega}}\left(1-H_{\alpha_i''}(v_i)\right)q_i(v_i,v_{-i})\,dv_i\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}_v\left[u\left(v_i;\alpha_i'\right)\right]-\mathbb{E}_v\left[u\left(v_i;\alpha_i''\right)\right]\\
=\;&\mathbb{E}_{v_{-i}}\left[\int_{\underline{\omega}}^{\overline{\omega}}\left(H_{\alpha_i''}(v_i)-H_{\alpha_i'}(v_i)(v_i)\right)q_i(v_i,v_{-i})\,dv_i\right]\\
=\;&-\mathbb{E}_{v_{-i}}\left[\int_{\underline{\omega}}^{\overline{\omega}}\left(\int_{\underline{\omega}}^{\omega_i}q_i(x,v_{-i})\,dx\right)\left(h_{\alpha_i''}(v_i)-h_{\alpha_i'}(v_i)\right)dv_i\right]\quad\text{(integration by part)}\\
=\;&-\int_{\underline{\omega}}^{\overline{\omega}}\left(\int_{\underline{\omega}}^{\omega_i}Q_i(x)\,dx\right)\left(h_{\alpha_i''}(v_i)-h_{\alpha_i'}(v_i)\right)dv_i,
\end{aligned}$$

where $Q_i(x)=\mathbb{E}_{v_{-i}}[q_i(x,v_{-i})]$. Since $Q_i(x)$ is increasing in x, $\int_{\underline{\omega}}^{\omega_i}Q_i(x)\,dx$ is convex. By Lemma 1, $H_{\alpha_i'}$ SOSD $H_{\alpha_i''}$. Therefore, $\mathbb{E}_v u\left[(v_i;\alpha_i')\right]-\mathbb{E}_v\left[u\left(v_i;\alpha_i''\right)\right]>0.\blacksquare$

Proof of Proposition 3: We can write the Lagrangian of the seller's optimization problem as follows:

$$\mathcal{L}(r,\alpha^*)=r\left(1-H_{\alpha^*}(r)\right)+\lambda\left(-\int_r^\infty\frac{\partial H_{\alpha^*}(v_i)}{\partial\alpha^*}dv_i-c\right).$$

The first-order conditions are

$$\frac{\partial\mathcal{L}}{\partial r} = 1-H_{\alpha^*}(r^*)-r^*h_{\alpha^*}(r^*)+\lambda\frac{\partial H_{\alpha^*}(r^*)}{\partial\alpha^*}=0, \tag{16}$$

$$\frac{\partial\mathcal{L}}{\partial\alpha^*} = -r^*\frac{\partial H_{\alpha^*}(r^*)}{\partial\alpha^*}+\lambda\left(-\int_{r^*}^\infty\frac{\partial^2H_{\alpha^*}(v_i)}{\partial\alpha^{*2}}dv_i\right)=0. \tag{17}$$

Note that the second-order condition of the buyer's optimization problem implies

$$-\int_{r^*}^\infty\frac{\partial^2H_{\alpha^*}(v_i)}{\partial\alpha^{*2}}dv_i<0.$$

It follows from the rotation order property (3) of $\{H_{\alpha^*}\}$ and (17) that

$$r^*\gtreqless\mu\Leftrightarrow\lambda\gtreqless 0. \tag{18}$$

That is, a seller prefers a more informed buyer if and only if $r^*>\mu$.

[28] If the support of the posterior estimates varies with respect to information choice α_i, we can redefine the distribution as follows. Suppose under information structure α_i, the support is $\left[\underline{\omega}_{\alpha_i},\overline{\omega}_{\alpha_i}\right]$. Then define $H_{\alpha_i}(v_i)=0$ for $v_i\in\left[\underline{\omega},\underline{\omega}_{\alpha_i}\right]$ and $H_{\alpha_i}(v_i)=1$ for $v_i\in\left[\overline{\omega}_{\alpha_i},\overline{\omega}\right]$.

Next, we will show that

$$\begin{cases} \mu < r^* < r_{\alpha^*} & \text{if} \quad r^* > \mu \\ r^* = r_{\alpha^*} = \mu & \text{if} \quad r^* = \mu \\ r^* < r_{\alpha^*} < \mu & \text{if} \quad r^* < \mu \end{cases}.$$

First consider the case: $r^* > \mu$, and suppose the opposite is true: $r^* \geq r_{\alpha^*}$. By (18), $\lambda > 0$. Then it follows from the definition of r_{α^*} and monotone hazard rate property and rotation order property of H_{α^*},

$$\frac{\partial \mathcal{L}}{\partial r}|_{r=r^*} \geq 1 - H_{\alpha^*}(r_{\alpha^*}) - r_{\alpha^*} h_{\alpha^*}(r_{\alpha^*}) + \lambda \frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} = \lambda \frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} > 0,$$

a contradiction of the optimality of r^*. Thus, $\mu < r^* < r_{\alpha^*}$. The other two cases can be proved analogously.

To complete the proof, we need to show that for a fixed β, there exists a $\widehat{\mu}$ such that

$$r^* \gtreqless \mu \Leftrightarrow \mu \lesseqgtr \widehat{\mu}.$$

We can rewrite the first-order condition for the buyer's maximization problem as

$$0 = -\int_r^\infty \frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i} dv_i - c = \frac{1}{2\sqrt{2\pi}} \sqrt{\frac{\beta}{\alpha(\alpha+\beta)^3}} \exp\left(-\frac{(r-\mu)^2}{2\sigma^2}\right) - c \tag{19}$$

Applying implicit function theorem to (19), we can show that

$$\frac{\partial \alpha}{\partial r} \begin{cases} > 0 & \text{if} \quad r < \mu \\ = 0 & \text{if} \quad r = \mu \\ < 0 & \text{if} \quad r > \mu \end{cases} \quad \text{and} \ \frac{\partial^2 \alpha}{\partial r \partial \mu} > 0. \tag{20}$$

We can also write the necessary first-order condition the seller's maximization problem as

$$\frac{d\pi_s}{dr}|_{r=r^*} = 1 - H_{\alpha^*}(r^*) - r^* h_{\alpha^*}(r^*) - r^* \frac{\partial H_\alpha(r^*)}{\partial \alpha^*} \frac{\partial \alpha^*}{\partial r} = 0 \tag{21}$$

Define the function $\Gamma(r^*, \mu)$ as

$$\Gamma(r^*, \mu) = 1 - H_{\alpha^*}(r^*) - r^* h_{\alpha^*}(r^*) - r^* \frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{\partial \alpha^*}{\partial r}.$$

Then applying implicit function theorem to $\Gamma(r^*, \mu) = 0$, we have

$$\frac{dr^*}{d\mu} = -\frac{\frac{\partial \Gamma(r^*,\mu)}{\partial \mu}}{\frac{\partial \Gamma(r^*,\mu)}{\partial r}}.$$

Notice that

$$\frac{\partial \Gamma(r^*, \mu)}{\partial \mu} = \frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{\partial \alpha^*}{\partial r^*} + r^* \frac{\partial h_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{\partial \alpha^*}{\partial r} + r \frac{\partial^2 H_{\alpha^*}(r)}{\partial \alpha^{*2}} \left(\frac{\partial \alpha^*}{\partial r^*}\right)^2 - r^* \frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{\partial^2 \alpha^*}{\partial r^* \partial \mu}$$

and

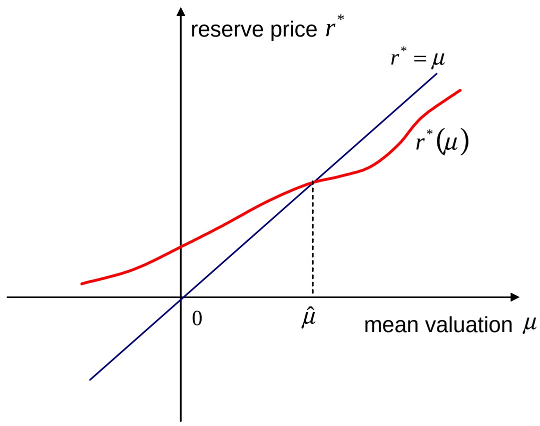
$$\frac{\partial \Gamma(r^*, \mu)}{\partial r^*} = -2h_{\alpha^*}(r^*) - 2\frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{\partial \alpha^*}{\partial r^*} - r^* \frac{\partial h_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{\partial \alpha^*}{\partial r^*} - r^* \frac{\partial^2 H_{\alpha^*}(r^*)}{\partial \alpha^{*2}} \left(\frac{\partial \alpha^*}{\partial r^*}\right)^2 + r^* \frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{\partial^2 \alpha^*}{\partial r^* \partial \mu}$$

Given (20), it is easy to show that

$$\begin{aligned}\text{for } r^* &> \mu : \frac{\partial\Gamma(r^*,\mu)}{\partial\mu} > 0, \frac{\partial\Gamma(r^*,\mu)}{\partial r} < 0, \text{ and } \frac{\partial\Gamma(r^*,\mu)}{\partial\mu} < -\frac{\partial\Gamma(r^*,\mu)}{\partial r} \\ \text{for } r^* &= \mu : \frac{\partial\Gamma(r^*,\mu)}{\partial\mu} = 0 \text{ and } \frac{\partial\Gamma(r^*,\mu)}{\partial r} < 0.\end{aligned}$$

Therefore, for $r^* > \mu$, $\frac{dr^*}{d\mu} \in (0,1)$, and for $r^* = \mu$, $\frac{dr^*}{d\mu} = 0$.

Note that $r^*(\mu) > \mu$ for $\mu < 0$. Hence, there must exists a $\widehat{\mu}$ such that $r^*(\widehat{\mu}) = \widehat{\mu}$.



Moreover, because $\frac{dr^*}{d\mu} \in (0,1)$ for $r^* > \mu$, and $\frac{dr^*}{d\mu} = 0$ for $r^* = \mu$, $\widehat{\mu}$ is unique and $r^* \gtreqless \mu \Leftrightarrow \mu \lesseqgtr \widehat{\mu}$.
■

Proof of Proposition 4: The second-order condition for the buyer's optimization problem is

$$-\int_r^\infty \frac{\partial^2 H_{\alpha_i}(v)}{\partial\alpha_i^2} dv_i < 0.$$

With some algebra, we can show

$$\frac{\partial^2 H_{\alpha_i}(v_i)}{\partial\alpha_i^2} = \frac{4\alpha_i + 3\beta}{2\alpha_i(\alpha_i+\beta)} \frac{(v_i-\mu)}{2\sqrt{2\pi}} \sqrt{\frac{\beta^3}{\alpha_i^3(\alpha_i+\beta)}} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right)\left(1 - \frac{\alpha_i+\beta}{4\alpha_i+3\beta}\frac{\beta^2}{\alpha_i}(v_i-\mu)^2\right).$$

Therefore, we can rewrite the second-order condition as

$$\int_r^\infty \frac{(v_i-\mu)}{2\sqrt{2\pi}} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right)\left(1 - \frac{\alpha_i+\beta}{4\alpha_i+3\beta}\frac{\beta^2}{\alpha_i}(v_i-\mu)^2\right) dv_i > 0.$$

By a change of variable with $y = \frac{v_i-\mu}{\sigma}$, we can obtain

$$\int_x^\infty y \exp\left(-\frac{1}{2}y^2\right)\left(1 - ky^2\right) dy > 0, \tag{22}$$

where

$$x = \frac{r-\mu}{\sigma} = (r-\mu)\Big/\sqrt{\frac{\alpha_i}{(\alpha_i+\beta)\beta}}, \text{ and } k = \frac{\alpha_i+\beta}{4\alpha_i+3\beta}\frac{\beta^2}{\alpha_i}\sigma^2 = \frac{\beta}{4\alpha_i+3\beta}.$$

The inequality (22) can be simplified into

$$-e^{-\frac{1}{2}x^2}\left(kx^2+2k-1\right)>0 \Leftrightarrow k<\frac{1}{2+x^2}.$$

Substitute the expression of k and x and we can obtain

$$\frac{4\alpha_i+\beta}{\alpha_i+\beta}\frac{\alpha_i}{\beta^2}>(r-\mu)^2. \tag{23}$$

Now if $r\in[\mu-2\sigma(\underline{\alpha}),\mu+2\sigma(\underline{\alpha})]$, then $r\in[\mu-2\sigma,\mu+2\sigma]$ because $\sigma(\alpha_i)\geq\sigma(\underline{\alpha})$ for all α_i. Therefore, a sufficient condition for (23) is

$$\frac{4\alpha_i+\beta}{\alpha_i+\beta}\frac{\alpha_i}{\beta^2}>4\sigma^2,$$

which is equivalent to $\alpha_i>3\beta/4$. Since $\alpha_i>\underline{\alpha}$ for all i, the second-order condition is satisfied whenever $\underline{\alpha}>\beta$. Hence, the first-order approach is valid if $r\in[\mu-2\sigma(\underline{\alpha}),\mu+2\sigma(\underline{\alpha})]$ and $\underline{\alpha}>\beta$.■

Proof of Lemma 3: For the Gaussian specification, we know from the text that

$$H_{\alpha_i}(v_i)=\int_{-\infty}^{v_i}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx \quad \text{where } \sigma^2=\frac{\alpha_i}{(\alpha_i+\beta)\beta}.$$

Since H_α is normal, it has an increasing hazard rate and the regularity assumption is satisfied. Recall equation (3)

$$\frac{\partial H_{\alpha_i}(v_i)}{\partial\alpha_i}=-\frac{(v_i-\mu)}{2\sqrt{2\pi}}\exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right)\sqrt{\frac{\beta^3}{\alpha_i^3(\alpha_i+\beta)}}.$$

In addition,

$$\frac{\partial H_{\alpha_i}(v_i)}{\partial\alpha_i}\frac{1}{h_{\alpha_i}(v_i)}=-\frac{\beta(v_i-\mu)}{2\alpha_i(\alpha_i+\beta)}.$$

It is easy to see that the other two assumptions are satisfied as well.

For the truth-or-noise technology, a buyer who observes a realization s_i with precision α_i will revise his posterior estimate as follows:

$$v_i(s_i,\alpha_i)=\mathbb{E}(\omega_i|s_i,\alpha_i)=\alpha_i s_i+(1-\alpha_i)\mu.$$

The distribution and density of the posterior estimate are, respectively

$$H_{\alpha_i}(v_i)=F\left(\frac{v_i-(1-\alpha_i)\mu}{\alpha_i}\right);\quad h_{\alpha_i}(v_i)=\frac{1}{\alpha_i}f\left(\frac{v_i-(1-\alpha_i)\mu}{\alpha_i}\right).$$

Simple calculations lead to

$$\frac{\partial H_{\alpha_i}(v_i)}{\partial\alpha_i} = f\left(\frac{v_i-(1-\alpha_i)\mu}{\alpha_i}\right)\frac{(\mu-v_i)}{\alpha_i^2}, \tag{24}$$

$$\frac{\partial H_{\alpha_i}(v_i)}{\partial\alpha_i}\frac{1}{h_{\alpha_i}(v_i)} = -\frac{v_i-\mu}{\alpha_i}, \tag{25}$$

$$\frac{h_{\alpha_i}(v_i)}{1-H_{\alpha_i}(v_i)} = \frac{1}{\alpha_i}\frac{f\left(\frac{v_i-(1-\alpha_i)\mu}{\alpha_i}\right)}{1-F\left(\frac{v_i-(1-\alpha_i)\mu}{\alpha_i}\right)}. \tag{26}$$

Equation (24) shows that the family of distributions $\{H_{\alpha_i}(\cdot)\}$ is rotation-ordered with rotation point equal to μ. The monotonicity assumption follows from equation (25). Finally, $H_{\alpha_i}(\cdot)$ has an increasing hazard rate, because, by assumption, the underlying distribution $F(\cdot)$ has an increasing hazard rate. Therefore, the family of distributions $\{H_{\alpha_i}(\cdot)\}$ generated by the "truth-or-noise" technology satisfies all assumptions. ■

Proof of Lemma 4: Let α^* denote the equilibrium information choice of bidders in the symmetric equilibrium. We prove the lemma by establishing the following two claims.

Claim 1: The seller's revenue in standard auctions with reserve price r is increasing in α^* if (1) $r \geq \mu$; or (2) Gaussian specification or truth-or-noise information technology, and n is large.

Proof: Let $V_{k,n}$ denote the k-th order statistic from n random variables independently drawn from H_{α^*}. The seller's revenue in standard auctions with a reserve price r is:

$$\begin{aligned}
\pi_s(\alpha^*, r) &= r \Pr(V_{n-1,n} < r \leq V_{n,n}) + \mathbb{E}[V_{n-1,n} | V_{n-1,n} \geq r] \Pr(V_{n-1,n} \geq r) \\
&= r[H_{n-1,n}(r) - H_{n,n}(r)] + \int_r^{\overline{\omega}_{\alpha^*}} v_i h_{n-1,n}(v_i)\, dv_i \\
&= r[1 - H_{\alpha^*}(r)^n] + \int_r^{\overline{\omega}_{\alpha^*}} \left[1 - nH_{\alpha^*}(v_i)^{n-1} + (n-1) H_{\alpha^*}(v_i)^n\right] dv_i.
\end{aligned}$$

Thus,

$$\frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} = -rnH_{\alpha^*}(r)^{n-1} \frac{\partial H_{\alpha^*}(r)}{\partial \alpha^*} - n(n-1) \int_r^{\overline{\omega}_{\alpha^*}} H_{\alpha^*}(v_i)^{n-2} [1 - H_{\alpha^*}(v_i)] \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} dv_i.$$

If $r \geq \mu$, then $\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} \leq 0$ for all $v_i \geq r$. As a result, $\frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} > 0$. Thus we have proved (1).

For Gaussian specification or truth-or-noise information technology, we only need to consider the case of $r < \mu$ given the result in (1). For Gaussian specification, we have

$$\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} = -\frac{\beta(v_i - \mu)}{2\alpha^*(\alpha^* + \beta)} h_{\alpha^*}(v_i).$$

Therefore,

$$\begin{aligned}
\frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} &= -nrH_{\alpha^*}(r)^{n-1} \frac{\partial H_{\alpha^*}(r)}{\partial \alpha^*} - n(n-1) \int_r^{\infty} H_{\alpha^*}(v_i)^{n-2} [1 - H_{\alpha^*}(v_i)] \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} dv_i \\
&= nrH_{\alpha^*}(r)^{n-1} \frac{\beta(r - \mu)}{2\alpha^*(\alpha^* + \beta)} h_{\alpha^*}(r) + n(n-1) \int_r^{\infty} \frac{H_{\alpha^*}(v_i)^{n-2} [1 - H_{\alpha^*}(v_i)] \beta(v_i - \mu)}{2\alpha^*(\alpha^* + \beta)} h_{\alpha^*}(v_i)\, dv_i \\
&= \frac{\beta}{2\alpha^*(\alpha^* + \beta)} \left\{ \begin{array}{c} nr(r-\mu) H_{\alpha^*}(r)^{n-1} h_{\alpha^*}(r) + n\left[(1 - H_{\alpha^*}(v_i))(v_i - \mu) H_{\alpha^*}(v_i)^{n-1}\right]_r^{\infty} \\ -n\int_r^{\infty} (1 - H_{\alpha^*}(v_i) - h_{\alpha^*}(v_i)(v_i - \mu)) H_{\alpha^*}(v_i)^{n-1} dv_i \end{array} \right\} \\
&= \frac{\beta}{2\alpha^*(\alpha^* + \beta)} \left\{ \begin{array}{c} n(r-\mu) H_{\alpha^*}(r)^{n-1} h_{\alpha^*}(r) \left(r - \frac{1 - H_{\alpha^*}(r)}{h_{\alpha^*}(r)}\right) \\ +n\int_r^{\infty} \left(v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right) H_{\alpha^*}(v_i)^{n-1} h_{\alpha^*}(v_i)\, dv_i - \mu(1 - H_{\alpha^*}(r)^n) \end{array} \right\}
\end{aligned}$$

If $r \leq r_{\alpha^*}$, then

$$r - \frac{1 - H_{\alpha^*}(r)}{h_{\alpha^*}(r)} \leq 0, \quad \text{and} \quad n(r-\mu) H_{\alpha^*}(r)^{n-1} h_{\alpha^*}(r) \left(r - \frac{1 - H_{\alpha^*}(r)}{h_{\alpha^*}(r)} \right) \geq 0$$

In addition, when n is relatively large, the following inequality holds:

$$\mathbb{E}[V_{n-1,n}] - \mu > 0.$$

Therefore,

$$\begin{aligned} \frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} &> n \int_{-\infty}^{\infty} \left(v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} \right) H_{\alpha^*}(v_i)^{n-1} h_{\alpha^*}(v_i) \, dv_i - \mu \\ &= \mathbb{E}[V_{n-1,n}] - \mu > 0. \end{aligned}$$

If $r > r_{\alpha^*}$, then

$$r - \frac{1 - H_{\alpha^*}(r)}{h_{\alpha^*}(r)} > 0 \quad \text{and} \quad n(r-\mu) H_{\alpha^*}(r)^{n-1} h_{\alpha^*}(r) \left(r - \frac{1 - H_{\alpha^*}(r)}{h_{\alpha^*}(r)} \right) > 0.$$

Hence,

$$\begin{aligned} \frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} &> \frac{\beta}{2\alpha^*(\alpha^* + \beta)} \left[\int_r^{\infty} \left(v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} \right) d\left[H_{\alpha^*}^n(v_i) \right] - \mu \right] \\ &> \frac{\beta}{2\alpha^*(\alpha^* + \beta)} \left[\int_\mu^{\infty} \left(v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} \right) d\left[H_{\alpha^*}^n(v_i) \right] - \mu \right]. \end{aligned}$$

The last inequality follows from the assumption that $r \leq \mu$. When n is relatively large, the seller's revenue with n bidders and reserve price μ will be higher than μ. Therefore, $\frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} > 0$.

Similarly, for truth-or-noise technology, we have

$$\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} = -\frac{v_i - \mu}{\alpha^*} h_{\alpha^*}(v_i).$$

Therefore,

$$\begin{aligned} \frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} &= -rnH_{\alpha^*}(r)^{n-1} \frac{\partial H_{\alpha^*}(r)}{\partial \alpha^*} - n(n-1) \int_r^{\overline{\omega}_{\alpha^*}} H_{\alpha^*}(v_i)^{n-2} \left[1 - H_{\alpha^*}(v_i) \right] \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} dv_i \\ &= nrH_{\alpha^*}(r)^{n-1} \frac{(r-\mu)}{\alpha^*} h_{\alpha^*}(r) + n(n-1) \int_r^{\overline{\omega}_{\alpha^*}} \frac{H_{\alpha^*}(v_i)^{n-2} \left[1 - H_{\alpha^*}(v_i) \right] (v_i - \mu)}{\alpha^*} h_{\alpha^*}(v_i) \, dv_i \\ &= \left\{ \begin{array}{c} \frac{n}{\alpha^*} (r-\mu) H_{\alpha^*}(r)^{n-1} h_{\alpha^*}(r) \left(r - \frac{1 - H_{\alpha^*}(r)}{h_{\alpha^*}(r)} \right) \\ + \frac{n}{\alpha^*} \int_r^{\overline{\omega}_{\alpha^*}} \left(v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} \right) H_{\alpha^*}(v_i)^{n-1} h_{\alpha^*}(v_i) \, dv_i - \mu \left(1 - H_{\alpha^*}(r)^n \right) \end{array} \right\} \end{aligned}$$

If $r \leq r_{\alpha^*}$, then

$$r - \frac{1 - H_{\alpha^*}(r)}{h_{\alpha^*}(r)} \leq 0, \quad \text{and} \quad \frac{n}{\alpha^*} (r-\mu) H_{\alpha^*}(r)^{n-1} h_{\alpha^*}(r) \left(r - \frac{1 - H_{\alpha^*}(r)}{h_{\alpha^*}(r)} \right) \geq 0$$

Note that $\alpha^* \leq 1$, so we have

$$\begin{aligned}\frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} &> \frac{n}{\alpha^*}\int_{-\infty}^{\infty}\left(v_i - \frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right) H_{\alpha^*}(v_i)^{n-1} h_{\alpha^*}(v_i)\, dv_i - \mu \\ &= \mathbb{E}[V_{n-1,n}] - \mu > 0.\end{aligned}$$

If $r > r_{\alpha^*}$, then

$$r - \frac{1-H_{\alpha^*}(r)}{h_{\alpha^*}(r)} > 0 \quad \text{and} \quad \frac{n}{\alpha^*}(r-\mu) H_{\alpha^*}(r)^{n-1} h_{\alpha^*}(r)\left(r - \frac{1-H_{\alpha^*}(r)}{h_{\alpha^*}(r)}\right) > 0.$$

Hence,

$$\begin{aligned}\frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} &> \int_{r}^{\overline{\omega}_{\alpha^*}}\left(v_i - \frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right) d\left[H_{\alpha^*}^n(v_i)\right] - \mu \\ &\geq \int_{\mu}^{\overline{\omega}_{\alpha^*}}\left(v_i - \frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right) d\left[H_{\alpha^*}^n(v_i)\right] - \mu\end{aligned}$$

The last inequality uses the fact that $r^* \leq \mu$. Again, as n is relatively large, the seller's revenue with n bidders and reserve price μ is higher than μ. So $\frac{\partial \pi_s(\alpha^*, r)}{\partial \alpha^*} > 0$.

Claim 2: If the seller's revenue is increasing in α^* in standard auctions with reserve price r, then $\lambda > 0$.

Proof: Recall the seller's maximization problem is

$$\max_{q_i, u(\underline{\omega}), \alpha^*} \left\{\mathbb{E}_{v,\alpha^*}\sum_{i=1}^{n}\left[\left(v_i - \frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right) q_i(v_i, v_{-i})\right] - nu(\underline{\omega})\right\}$$

$$\begin{aligned} s.t. \quad &: \quad 0 \leq q_i(v_i, v_{-i}) \leq 1; \sum_{i=1}^{n} q_i(v_i, v_{-i}) \leq 1, && \text{(Regularity)} \\ &: \quad Q_i(v_i) \text{ is increasing in } v_i, && \text{(Monotonicity)} \\ &: \quad u(\underline{\omega}) \geq 0, && \text{(IR)} \\ &: \quad -\mathbb{E}_{v,\alpha^*}\left[\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} q_i(v_i, v_{-i})\right] - C'(\alpha^*) = 0. && \text{(IA)}\end{aligned}$$

Note that the expectation term is independent of $u(\underline{\omega})$, and $u(\underline{\omega})$ is nonnegative, so the (IR) constraint must be binding. Ignore the regularity constraint and monotonicity constraint for the moment.

We adopt the same strategy of Rogerson (1985) by weakening the equality (IA) constraint to the following inequality constraint.

$$-\mathbb{E}_{v,\alpha^*}\left[\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} q_i(v_i, v_{-i})\right] - C'(\alpha^*) \geq 0.$$

With the inequality constraint, the corresponding Lagrangian multiplier λ is always nonnegative. If we can show that $\lambda > 0$ at the optimal solution of the relaxed program, then the constraint is binding in equilibrium. Then, the optimal solution of relaxed program is also an optimal solution of the original program. Hence $\lambda > 0$ for the original program.

We can write and simplify the Lagrangian for the relaxed program as

$$L=\mathbb{E}_{v,\alpha^*}\sum_{i=1}^{n}\left[\left(v_i-\frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}-\frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial\alpha^*}\frac{1}{h_{\alpha^*}(v_i)}\right)q_i(v_i,v_{-i})\right]-\lambda C'(\alpha^*)$$

The necessary first-order condition is

$$0=\frac{\partial L}{\partial\alpha^*}=\left\{\begin{array}{c}\frac{\partial\left\{\mathbb{E}_{v,\alpha^*}\sum_{i=1}^{n}\left[\left(v_i-\frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right)q_i(v_i,v_{-i})\right]\right\}}{\partial\alpha^*}\\ +\lambda\frac{\partial\left[-\mathbb{E}_{v,\alpha^*}\left[\frac{\partial H_{\alpha^*}(v_i)}{\partial\alpha^*}\frac{1}{h_{\alpha^*}(v_i)}q_i(v_i,v_{-i})\right]-C'(\alpha^*)\right]}{\partial\alpha^*}\end{array}\right\}. \tag{27}$$

Since $\lambda\geq 0$, standard auctions are optimal by theorem 3. Therefore, we can restrict attention to standard auctions, and the seller's revenue in standard auctions is

$$\mathbb{E}_{v,\alpha^*}\sum_{i=1}^{n}\left[\left(v_i-\frac{1-H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)}\right)q_i(v_i,v_{-i})\right].$$

By assumption, the seller's revenue is increasing in α^*, so the first term in the big bracket of (27) is positive. In order to show $\lambda>0$, we need to show that the second term is negative. Note that in a standard auction with reserve price r,

$$-\mathbb{E}_{v,\alpha^*}\left[\frac{\partial H_{\alpha^*}(v_i)}{\partial\alpha^*}\frac{1}{h_{\alpha^*}(v_i)}q_i(v_i,v_{-i})\right]-C'(\alpha^*)=\int_r^{\overline{\omega}_{\alpha^*}}(1-H_{\alpha^*}(v_i))\,H_{\alpha^*}(v_i)^{n-1}\,dv_i-C'(\alpha^*).$$

Thus,

$$\begin{aligned}&\frac{\partial\left[-\mathbb{E}_{v,\alpha^*}\left[\frac{\partial H_{\alpha^*}(v_i)}{\partial\alpha^*}\frac{1}{h_{\alpha^*}(v_i)}q_i(v_i,v_{-i})\right]-C'(\alpha^*)\right]}{\partial\alpha^*}\\ =\;&\underbrace{-\int_r^{\overline{\omega}_{\alpha^*}}\frac{\partial^2 H_{\alpha^*}(v_i)}{\partial\alpha^{*2}}H_{\alpha^*}(v_i)^{n-1}\,dv_i+\frac{\partial^2 H_{\alpha^*}(\overline{\omega}_{\alpha^*})}{\partial\alpha^{*2}}\frac{\partial\overline{\omega}_{\alpha^*}}{\partial\alpha^*}-C''(\alpha^*)}_{A}\\ &\underbrace{-\int_r^{\overline{\omega}_{\alpha^*}}\left(\frac{\partial H_{\alpha^*}(v_i)}{\partial\alpha^*}\right)^2(n-1)\,H_{\alpha^*}(v_i)^{n-2}\,dv_i}_{B}\end{aligned}$$

Since α^* maximize a bidder's expected payoff, the local second-order condition of the bidder's maximization problem holds. As a result, term A is negative. Since term B is also negative, the partial derivative is negative. Therefore, $\lambda>0$ at the optimal solution (α^*,q^*). The solution to the relaxed program is the same as the one for the original program, and the maximum of the relaxed program can be achieved by the original program. Hence, the Lagrangian multiplier λ for the original problem must be strictly positive.

To complete the proof, we need to show that $r_{\underline{\alpha}}>\mu$ implies $r_{\alpha^*}>\mu$. Note that by definition of $r_{\underline{\alpha}}$ and Assumption 3, $r_{\underline{\alpha}}>\mu$ is equivalent to

$$\mu-\frac{1-H_{\underline{\alpha}}(\mu)}{h_{\underline{\alpha}}(\mu)}<0.$$

Since $\alpha^* \geq \underline{\alpha}$, by Assumption 1,

$$H_{\underline{\alpha}}(\mu) = H_{\alpha^*}(\mu) \text{ and } h_{\underline{\alpha}}(\mu) \geq h_{\alpha^*}(\mu)$$

It follows that

$$\mu - \frac{1 - H_{\alpha^*}(\mu)}{h_{\alpha^*}(\mu)} \leq \mu - \frac{1 - H_{\underline{\alpha}}(\mu)}{h_{\underline{\alpha}}(\mu)} < 0$$

Thus, $r_{\alpha^*} > \mu$. Finally, from Theorem 2, for $\lambda > 0$, $r_{\alpha^*} > \mu \Leftrightarrow r^* \geq \mu$.

The Lemma now follows from the results of Claim 1 and Claim 2. ■

Proof of Theorem 2: Recall the virtual surplus function is

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n} \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} \frac{1}{h_{\alpha^*}(v_i)}.$$

The optimal reserve price r^* has to satisfy

$$q_i(v_i, v_{-i}) > 0 \Rightarrow v_i \geq r^*,$$

and

$$r^* \leq \min\{r : J^*(v_i) \geq 0 \text{ for all } v_i \geq r\}. \tag{28}$$

The last condition says that the seller will sell the object as long as the marginal revenue is nonnegative.

Case 1: $\lambda > 0$ and $r_{\alpha^*} > \mu$. First we show $r^* < r_{\alpha^*}$. By definition of r_{α^*},

$$r_{\alpha^*} - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} = 0.$$

Then for all $v_i \geq r_{\alpha^*} > \mu$,

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n} \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} \frac{1}{h_{\alpha^*}(v_i)} = -\frac{\lambda}{n} \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} \frac{1}{h_{\alpha^*}(v_i)} > 0.$$

The last inequality follows from the fact that $\{H_{\alpha^*}\}$ is rotation-ordered. Therefore, there exists $\varepsilon > 0$, such that

$$J^*(r_{\alpha^*} - \varepsilon) \geq 0.$$

Therefore, by (28), the optimal reserve price $r^* < r_{\alpha^*}$.

Next, we show $r^* \geq \mu$. Suppose $r^* < \mu$ by contradiction. Then

$$J^*(r^*) = r^* - \frac{1 - H_{\alpha^*}(r^*)}{h_{\alpha^*}(r^*)} - \frac{\lambda}{n} \frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{1}{h_{\alpha^*}(r^*)} < -\frac{\lambda}{n} \frac{\partial H_{\alpha^*}(r^*)}{\partial \alpha^*} \frac{1}{h_{\alpha^*}(r^*)} < 0.$$

The first inequality follows because $r^* < r_{\alpha^*}$, and the second inequality follows from the rotation order. This contradicts the fact the $J^*(r^*) \geq 0$. Thus, we have shown $\mu \leq r^* < r_{\alpha^*}$

Case 2: $\lambda > 0$ and $r_{\alpha^*} = \mu$. Then for all $v_i > \mu$,

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n} \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} \frac{1}{h_{\alpha^*}(v_i)} \geq -\frac{\lambda}{n} \frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*} \frac{1}{h_{\alpha^*}(v_i)} > 0.$$

Therefore, r^* cannot be higher than μ. On the other hand, for all $v_i < \mu$

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} \leq -\frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} < 0.$$

Therefore, r^* cannot be lower than μ. Therefore, $r^* = r_{\alpha^*} = \mu$.

Case 3: $\lambda > 0$ and $r_{\alpha^*} < \mu$. Note that for all $v_i < r_{\alpha^*}$,

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} \leq -\frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} < 0.$$

Therefore, $r^* > r_{\alpha^*}$. Furthermore, for all $v_i > \mu$,

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} \geq -\frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} > 0.$$

Thus, $r^* \leq \mu$. As a result, $r_{\alpha^*} < r \leq \mu$.

Case 4: $\lambda < 0$ and $r_{\alpha^*} < \mu$. Note that for all $v_i \in [r_{\alpha^*}, \mu]$

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} \geq -\frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)} > 0.$$

In addition, r^* cannot be higher than μ, otherwise $\lambda > 0$. Therefore, $r^* < r_{\alpha^*} < \mu$.

From the proof of Lemma 4, we know $r_{\alpha^*} \geq \mu$ implies $\lambda > 0$, the above four cases include all possible cases, and our proof is complete.■

Proof of Theorem 3: Under Assumption 2 and 3,

$$J^*(v_i) = v_i - \frac{1 - H_{\alpha^*}(v_i)}{h_{\alpha^*}(v_i)} - \frac{\lambda}{n}\frac{\partial H_{\alpha^*}(v_i)}{\partial \alpha^*}\frac{1}{h_{\alpha^*}(v_i)}$$

is increasing in v_i. In this case, we can define the reserve price as

$$r^* = \inf\{r : J^*(r) \geq 0\}.$$

Therefore the optimal auctions will assign the object to the bidder with highest posterior estimate provided his estimate is higher than r^*. So standard auctions with reserve price r^* are optimal, where r^* is set according to Theorem 2.■

8 Appendix B: Sufficient Conditions for Validity of the First Order Approach

This Appendix will provide several sets of sufficient conditions under which the first-order approach is valid. Recall that bidder i chooses α_i to maximize his payoff given other bidders choose α_j $(j \neq i)$. Bidder i's payoff under mechanism $\{q_i(v), t_i(v)\}$ is,

$$\pi_b(\alpha_i) = \mathbb{E}_{v_{-i}}\left\{\int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}} [1 - H_{\alpha_i}(v_i)]\, q_i(v_i, v_{-i})\, dv_i - C(\alpha_i)\right\}.$$

Then with some algebra, one can show that

$$\frac{\partial^2\pi_b}{\partial\alpha_i^2}=-\mathbb{E}_{v_{-i}}\left\{\begin{array}{c}\frac{\partial q_i\left(\underline{\omega}_{\alpha_i},v_{-i}\right)}{\partial v_i}\left(\frac{\partial\underline{\omega}_{\alpha_i}}{\partial\alpha_i}\right)^2+q_i\left(\underline{\omega}_{\alpha_i},v_{-i}\right)\frac{\partial^2\underline{\omega}_{\alpha_i}}{\partial\alpha_i^2}-\frac{\partial H_{\alpha_i}\left(\underline{\omega}_{\alpha_i}\right)}{\partial\alpha_i}q_i\left(\underline{\omega}_{\alpha_i},v_{-i}\right)\frac{\partial\underline{\omega}_{\alpha_i}}{\partial\alpha_i}\\+\frac{\partial H_{\alpha_i}\left(\overline{\omega}_{\alpha_i}\right)}{\partial\alpha_i}q_i\left(\overline{\omega}_{\alpha_i},v_{-i}\right)\frac{\partial\overline{\omega}_{\alpha_i}}{\partial\alpha_i}+\int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}}\frac{\partial^2 H_{\alpha_i}(v_i)}{\partial\alpha_i^2}q_i\left(v_i,v_{-i}\right)dv_i\end{array}\right\}-C''\left(\alpha_i\right)$$

The first-order approach is valid if $\partial^2\pi_b/\partial\alpha_i^2<0$, which holds as long as the cost function is sufficient convex.[29]

If the support of the posterior estimates is independent of information choice α_i, all terms except the last one in the expectation are zero. Therefore, the first-order approach is valid if the last term in the expectation is nonnegative. A sufficient condition for the last term to be nonnegative is

$$\frac{\partial^2 H_{\alpha_i}\left(v_i\right)}{\partial\alpha_i^2}\geq 0 \text{ for all } v_i. \tag{29}$$

That is, the distribution of the posterior estimates is convex in the bidder's information choice. This condition is analogous to the CDFC (convexity of the distribution function condition) in the principal-agent literature, which requires that the distribution function of output be convex in the action the agent takes (Mirrlees (1999), and Rogerson (1985)).[30]

For our two leading information structures, we will provide sufficient conditions under which the first-order approach is valid.

Proposition 6

For the truth-or-noise technology, if $C''\left(\alpha_i\right)\alpha_i\geq f\left(\overline{\omega}\right)\left(\overline{\omega}-\mu\right)^2$ for all α_i, the second-order condition of buyers' maximization problem is satisfied either (1) $F\left(x\right)$ is convex, or (2) $F\left(x\right)=x^b\left(b>0\right)$ with support $[0,1]$. For the Gaussian specification, the second-order condition is satisfied if, for all α_i,

$$\sqrt{\frac{\beta^3}{\alpha_i^3\left(\alpha_i+\beta\right)^5}}<2\sqrt{2\pi}C''\left(\alpha_i\right).$$

Proof: For the truth-or-noise technology,

$$\begin{aligned}\frac{\partial^2\pi_b}{\partial\alpha_i^2} &= -\mathbb{E}_{v_{-i}}\left\{\begin{array}{c}\frac{\partial q_i\left(\underline{\omega}_{\alpha_i},v_{-i}\right)}{\partial v_i}\left(\frac{\partial\underline{\omega}_{\alpha_i}}{\partial\alpha_i}\right)^2-\frac{\partial H_{\alpha_i}\left(\underline{\omega}_{\alpha_i}\right)}{\partial\alpha_i}q_i\left(\underline{\omega}_{\alpha_i},v_{-i}\right)\frac{\partial\underline{\omega}_{\alpha_i}}{\partial\alpha_i}\\+\frac{\partial H_{\alpha_i}\left(\overline{\omega}_{\alpha_i}\right)}{\partial\alpha_i}q_i\left(\overline{\omega}_{\alpha_i},v_{-i}\right)\frac{\partial\overline{\omega}_{\alpha_i}}{\partial\alpha_i}+\int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}}\frac{\partial^2 H_{\alpha_i}(v_i)}{\partial\alpha_i^2}q_i\left(v_i,v_{-i}\right)dv_i\end{array}\right\}-C''\left(\alpha_i\right)\\
&< -\frac{\partial H_{\alpha_i}\left(\overline{\omega}_{\alpha_i}\right)}{\partial\alpha_i}Q_i\left(\overline{\omega}_{\alpha_i}\right)\frac{\partial\overline{\omega}_{\alpha_i}}{\partial\alpha_i}+\frac{\partial H_{\alpha_i}\left(\underline{\omega}_{\alpha_i}\right)}{\partial\alpha_i}Q_i\left(\underline{\omega}_{\alpha_i}\right)\frac{\partial\underline{\omega}_{\alpha_i}}{\partial\alpha_i}-C''\left(\alpha_i\right)-\int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}}\frac{\partial^2 H_{\alpha_i}\left(v_i\right)}{\partial\alpha_i^2}Q_i\left(v_i\right)dv_i\\
&= f\left(\overline{\omega}\right)\frac{\left(\overline{\omega}-\mu\right)^2}{\alpha_i}Q_i\left(\overline{\omega}_{\alpha_i}\right)-f\left(\underline{\omega}\right)\frac{\left(\mu-\underline{\omega}\right)^2}{\alpha_i}Q_i\left(\underline{\omega}_{\alpha_i}\right)-C''\left(\alpha_i\right)-\int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}}\frac{\partial^2 H_{\alpha_i}\left(v_i\right)}{\partial\alpha_i^2}Q_i\left(v_i\right)dv_i\\
&\leq f\left(\overline{\omega}\right)\frac{\left(\overline{\omega}-\mu\right)^2}{\alpha_i}-C''\left(\alpha_i\right)-\int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}}\frac{\partial^2 H_{\alpha_i}\left(v_i\right)}{\partial\alpha_i^2}Q_i\left(v_i\right)dv_i\\
&\leq -\int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}}\frac{\partial^2 H_{\alpha_i}\left(v_i\right)}{\partial\alpha_i^2}Q_i\left(v_i\right)dv_i\end{aligned}$$

[29] Persico (2000) makes such an assumption in his example of information acquisition.

[30] See also Jewitt (1988).

The last equality follows from equation (24) and the fact that

$$\partial\overline{\omega}_{\alpha_i}/\partial\alpha_i = \overline{\omega} - \mu, \quad \partial\underline{\omega}_{\alpha_i}/\partial\alpha_i = (\mu - \underline{\omega}).$$

and the last inequality follows from the assumption that $C''(\alpha_i)\alpha_i \geq f(\overline{\omega})(\overline{\omega}-\mu)^2$.

Note that

$$\begin{aligned}
& -\int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}} \frac{\partial^2 H_{\alpha_i}(v_i)}{\partial\alpha_i^2} Q_i(v_i)\,dv_i \\
= & -\int_{\underline{\omega}_{\alpha_i}}^{\overline{\omega}_{\alpha_i}} \left\{ f'\left(\frac{v_i-(1-\alpha_i)\mu}{\alpha_i}\right)\frac{(\mu-v_i)^2}{\alpha_i^4} - f\left(\frac{v_i-(1-\alpha_i)\mu}{\alpha_i}\right)\frac{2(\mu-v_i)}{\alpha_i^3}\right\} Q_i(v_i)\,dv_i \\
= & -\int_{\underline{\omega}}^{\overline{\omega}} \left\{ f'(s_i)\frac{(s_i-\mu)^2}{\alpha_i} + f(s_i)\frac{2(s_i-\mu)}{\alpha_i}\right\} Q_i(\alpha_i s_i + (1-\alpha_i)\mu)\,ds_i
\end{aligned}$$

If $F(\cdot)$ is convex, then

$$\begin{aligned}
\frac{\partial^2\pi_b}{\partial\alpha_i^2} &< -\frac{2}{\alpha_i}\int_{\underline{\omega}}^{\overline{\omega}} (s_i-\mu) f(s_i) Q_i(\alpha_i s_i + (1-\alpha_i)\mu)\,ds_i \\
&< -\frac{2}{\alpha_i}\int_{\underline{\omega}}^{\mu} (s_i-\mu) f(s_i) Q_i(\mu)\,ds_i - \frac{2}{\alpha_i}\int_{\mu}^{\overline{\omega}} (s_i-\mu) f(s_i) Q_i(\mu)\,ds_i \\
&= -\frac{2}{\alpha_i} Q_i(\mu)\int_{\underline{\omega}}^{\overline{\omega}} (s_i-\mu) f(s_i)\,ds_i \\
&= 0
\end{aligned}$$

If $F(x) = x^b\,(0 < b \leq 1)$ with support $[0,1]$, then

$$\begin{aligned}
\frac{\partial^2\pi_b}{\partial\alpha_i^2} &< -\int_{\underline{\omega}}^{\overline{\omega}} \left\{ f'(s_i)\frac{(s_i-\mu)^2}{\alpha_i} + f(s_i)\frac{2(s_i-\mu)}{\alpha_i}\right\} Q_i(\alpha_i s_i + (1-\alpha_i)\mu)\,ds_i \\
&= -\frac{1}{\alpha_i}\int_0^1 [(b+1)s + (1-b)\mu]\, b s^{b-2}(s-\mu) Q_i(\alpha_i s + (1-\alpha_i)\mu)\,ds \\
&< -\frac{1}{\alpha_i} Q_i(\mu)\int_0^1 ((b+1)s + (1-b)\mu)\, b s^{b-2}(s-\mu)\,ds \\
&= -\frac{1}{\alpha_i} Q_i(\mu)(b+1)\int_0^1 b s^{b-1}(s-\mu)\,ds - \frac{1}{\alpha_i} Q_i(\mu)(1-b)\mu b\int_0^1 s^{b-2}(s-\mu)\,ds \\
&= -\frac{1}{\alpha_i} Q_i(\mu)\frac{b}{(1+b)^2} \\
&< 0.
\end{aligned}$$

For the Gaussian specification,

$$\frac{\partial^2\pi_b}{\partial\alpha_i^2} = -\mathbb{E}_{v_{-i}}\left\{\int_{-\infty}^{\infty} \frac{\partial^2 H_{\alpha_i}(v_i)}{\partial\alpha_i^2} q_i(v_i, v_{-i})\,dv_i\right\} - C''(\alpha_i).$$

With some algebra, we can obtain

$$\frac{\partial^2 H_{\alpha_i}(v_i)}{\partial\alpha_i^2} = \frac{4\alpha_i + 3\beta}{2\alpha_i(\alpha_i+\beta)}\frac{(v_i-\mu)}{2\sqrt{2\pi}}\sqrt{\frac{\beta^3}{\alpha_i^3(\alpha_i+\beta)}}\exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right)\left(1 - \frac{\alpha_i+\beta}{4\alpha_i+3\beta}\frac{\beta^2}{\alpha_i}(v_i-\mu)^2\right).$$

Thus, we can write the second derivative as

$$\frac{\partial^2 \pi_b}{\partial \alpha_i^2} = \begin{pmatrix} -\int_{-\infty}^{\infty} \frac{4\alpha_i+3\beta}{2\alpha_i(\alpha_i+\beta)} \frac{(v_i-\mu)}{2\sqrt{2\pi}} \sqrt{\frac{\beta^3}{\alpha_i(\alpha_i+\beta)}} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right) Q_i(v_i)\, dv_i \\ +\int_{-\infty}^{\infty} \frac{\beta^2}{4\alpha_i^2} \frac{(v_i-\mu)^3}{\sqrt{2\pi}} \sqrt{\frac{\beta^3}{\alpha_i^3(\alpha_i+\beta)}} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right) Q_i(v_i)\, dv_i \end{pmatrix} - C''(\alpha_i)$$

$$= \begin{pmatrix} -\frac{4\alpha_i+3\beta}{2\alpha_i(\alpha_i+\beta)} \int_{-\infty}^{\infty} \left(-\frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i}\right) Q_i(v_i)\, dv_i \\ +\int_{-\infty}^{\infty} \frac{\beta^2}{4\alpha_i^2} \frac{(v_i-\mu)^3}{\sqrt{2\pi}\sigma} \frac{\beta}{\alpha_i(\alpha_i+\beta)} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right) Q_i(v_i)\, dv_i \end{pmatrix} - C''(\alpha_i).$$

By Proposition 1, bidders always prefer higher α_i, which implies

$$\int_{-\infty}^{\infty} \left(-\frac{\partial H_{\alpha_i}(v_i)}{\partial \alpha_i}\right) Q_i(v_i)\, dv_i > 0.$$

Thus, a sufficient condition for the second-order condition is

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\beta^2}{4\alpha_i^2} \frac{\beta}{\alpha_i(\alpha_i+\beta)} \frac{(v_i-\mu)^3}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right) Q_i(v_i)\, dv_i &< C''(\alpha_i) \Leftrightarrow \\ \frac{\beta^3}{4\alpha_i^3(\alpha_i+\beta)} \int_{-\infty}^{\infty} \frac{(v_i-\mu)^3}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right) Q_i(v_i)\, dv_i &< C''(\alpha_i). \end{aligned}$$

A sufficient condition for the above inequality is,

$$\begin{aligned} \frac{\beta^3}{4\alpha_i^3(\alpha_i+\beta)} \int_{\mu}^{\infty} \frac{(v_i-\mu)^3}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(v_i-\mu)^2}{2\sigma^2}\right) dv_i &< C''(\alpha_i) \Leftrightarrow \\ \frac{1}{2\sqrt{2\pi}} \sqrt{\frac{\beta^3}{\alpha_i^3(\alpha_i+\beta)^5}} &< C''(\alpha_i). \end{aligned}$$

Note that if β/α_i is small, the above sufficient condition is easy to be satisfied. Therefore, if $\underline{\alpha}/\beta$ is sufficiently large, the second-order condition is satisfied. ■

For the truth-or-noise technology, the condition, $C''(\alpha_i)\alpha_i \geq f(\overline{\omega})(\overline{\omega}-\mu)^2$, is to ensure that the relative gain from information acquisition is not too high so that bidders will not pursue extreme information choice $\overline{\alpha}$. The convexity of F is not necessary. For example, $F(x) = x^b$ may not be convex but the second-order condition is still satisfied.

For the Gaussian specification, if β is small and $\underline{\alpha}$ is large relative to β, then the second-order condition is satisfied. This is quite intuitive. Small β implies the prior distribution is quite spread out, so the potential gain from information acquisition is high. If $\underline{\alpha}$ is large relative to β, then the signal will be informative, which again implies information acquisition is profitable.

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