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An Equilibrium Theory of Declining Reservation Wages and Learning

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Abstract

In this paper we consider learning from search as a mechanism to understand the relationship between unemployment duration and search outcomes as a labor market equilibrium. We rely on the assumption that workers do not have precise knowledge of their job finding probabilities and therefore, learn about them from their search histories. Embedding this assumption in a model of the labor market with directed search, we provide an equilibrium theory of declining reservation wages over unemployment spells. After each period of search, unemployed workers update their beliefs about the market matching efficiency. We characterize situations where reservation wages decline with unemployment duration. Consequently, the wage distribution is non-degenerate, despite the facts that matches are homogeneous and search is directed. Moreover, aggregate matching probability decreases with unemployment duration, in contrast to individual workers' matching probability, which increases over individual unemployment spells. The difficulty in establishing these results is that learning generates non-differentiable value functions and multiple solutions to a worker's optimization problem. We overcome this difficulty by exploiting a connection between convexity of a worker's value function and the property of supermodularity.

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1. Introduction

Workers with longer unemployment duration also have lower permanent incomes.[1] Motivated by this observation, we explore *learning from search* as the mechanism underlying the relationship between unemployment duration and search outcomes. The main assumption is that unemployed workers do not have precise knowledge of their job finding probabilities and, therefore, learn about them from their search histories. Embedding this assumption in a labor market with directed search, we construct an equilibrium theory of declining reservation wages over unemployment spells. This model allows us to consider jointly the search behavior of workers, the creation of jobs, and the wage distribution as functions of unemployment duration. The theory formalizes a notion akin to that of discouragement, as unemployed workers become more pessimistic about the probability of finding a job as they update their beliefs downward over their spell of unemployment.

To present the theory in the simplest format, we focus on the search behavior of ex ante identical workers and firms. The aggregate labor market consists of many markets, and we think of each single market in terms of the mix of physical characteristics of that particular labor market, such as geography and occupation. Accordingly, these characteristics are idiosyncratic to the workers and firms in that market, rather than economy-wide characteristics. In this context, unemployed workers' search conveys information about the matching efficiency and, therefore, about the worker's job finding probability. After an unemployed worker searches and fails to find employment, the worker views this search outcome as bad news and revises his beliefs about the matching probability downwards. In equilibrium, each market becomes segmented, containing workers with different unemployment durations and, thus, different beliefs about the matching efficiency in their market. Firms in these markets cater to the preferences of workers with different beliefs and supply the jobs that workers seek in all those markets, but with different terms of trade. In particular, as an unemployed worker searches for a job and fails to find a match, he becomes pessimistic about his chances to find a job and, hence, chooses to search for jobs which are easier to get. In a directed search equilibrium, those jobs necessarily come with lower wages, as the firms that provide the jobs also make the tradeoff between the matching probability and wages.[2] Thus, our theory provides an explanation for wage inequality among workers with

[1] See e.g. Devine and Kiefer (1991) and Machin and Manning (1999).

[2] In principle, the matching efficiency could also be affected by worker-specific characteristics, in which case workers may also be learning about themselves, that is, about their own ability to find and elicit a job offer. Here we disregard this latter case in the interest of simplicity — this would complicate the firms' learning problem, introducing further heterogeneity.

identical skills.[3]

Our theory is related in spirit to the work of Burdett and Vishwanath (1988), who propose a model of workers' learning about the distribution of wages as an explanation for the fact that reservation wages decline with the duration of unemployment spells. Their idea is intuitive and can be viewed as an attempt to understand a form of "discouragement" in the labor market. Wage offers convey information about the unknown distribution of wages. Accordingly, wage offers lower than expected lead to a reduction in the worker's reservation wage, as the worker revises his beliefs about the wage distribution downwards. This learning process generates endogenous selection, because workers with longer unemployment duration are precisely those who have drawn and rejected relatively lower wages and, therefore, they perceive the jobs available to them as jobs offering low wages.

However, Burdett and Vishwanath examine only one side of the market by assuming that the wage distribution is exogenous. If one considers an equilibrium, instead, learning by the market participants will affect firms' wage offers. Thus, the wage distribution itself needs to be explained rather than assumed, in order to address the connection between reservation wages, job creation and equilibrium wages. Unfortunately, endogenizing the wage distribution in the Burdett-Vishwanath model is not tractable, partly because the distribution acts as a state variable in an individual's decision problem. Moreover, it is more intuitive to formulate the learning process directly as one about fundamental characteristics of the market, such as the matching efficiency and productivity, rather than the wage distribution which is generated by these characteristics.

In this paper we provide such a formulation of search and learning in an equilibrium. The characteristic which workers try to learn about is associated with labor market frictions. As a concrete way to formalize this type of uncertainty, we assume that individuals do not know precisely the matching efficiency given by the exogenous matching technology. Faced with this uncertainty, workers learn about the probability of finding a job through their private search histories. Moreover, we formulate search as a directed process. That is, each firm offers a wage knowing that his offer will affect his matching probability, and each worker observes all wage offers in the economy before choosing to apply to one.[4] However, an individual's matching probability is still uncertain and, hence, learning is useful,

[3] A related possibility is that search intensity also varies with the duration of unemployment. For instance, one could add a participation decision in the present context. Intuitively, workers would quit search after a sufficiently long unemployment spell. In practice discouraged workers may rather switch to occupations where they are less productive. Our analysis may help understand this process as well.

[4] For an earlier formulation of directed search, see Peters (1984, 1991). Other examples include Moen (1997), Acemoglu and Shimer (1999), Shi (2001) and Burdett et al. (2001).

because the individual does not know which wage offers belong to his own local market. As opposed to undirected search (e.g., Burdett and Vishwanath, 1988), the directed search framework eliminates the direct dependence of an individual's decision on the wage distribution, which reduces the dimensionality of the state variables for an individual's decision problem, and hence makes the model tractable.

In addition to a tractable formulation of an equilibrium with learning, we provide an analytical procedure for resolving a main theoretical problem in the analysis of learning. This problem is caused by convexity of the value function. Because search outcomes generate variations in a worker's posterior beliefs about the matching efficiency, search is informative only if these variations in beliefs are valuable to the worker, i.e., if the worker's value function is strictly convex in beliefs. Although the literature (e.g., Easley and Kiefer, 1988) recognizes that such convexity is likely to lead to multiple solutions and to render the first-order conditions inapplicable, it has either ignored the difficulty or focused on corner solutions (e.g. Balvers and Cosimano, 1993). To establish the result of declining reservation wages, all solutions need to be characterized. We resolve this difficulty by exploring a connection between convexity of the value function and the property of *supermodularity*.

The connection is not obvious at the first glance. In our model, neither a worker's current payoff nor his objective function is supermodular as is often required in applications of supermodularity (see Topkis, 1998). Moreover, a worker's value function is convex, rather than concave as is required in applications of supermodularity to dynamic programming (e.g., Amir et al., 1991). However, we can transform a worker's objective function into a supermodular function, and this transformation relies heavily on convexity of the value function. Then, (weak) monotonicity of workers' reservation wages follows from standard results in Topkis (1998). In turn, monotone optimal choices imply that the workers' reservation wage declines with unemployment duration.

We then provide conditions under which optimal choices are interior and show that reservation wages are strictly declining with unemployment duration in this case. Under the same conditions, we show that the value function is differentiable in a limited sense, i.e., differentiable in future periods at those beliefs along the equilibrium path. In turn, this implies that the paths of equilibrium choices and induced beliefs are unique almost everywhere and that non-uniqueness can occur at most in the first period of search.

Our theory generates "true" positive relationship between an unemployed individual's transition to employment and his unemployment duration; at the same time, the theory is consistent with a negative cross-sectional (aggregate) relationship between unemployment duration and unemployment outflows to employment. True positive duration dependence arises because workers optimally search for jobs that are easier to get as their reservation wages decline. Consequently, each worker's job finding rate increases over his unemployment spell. However, negative duration dependence can arise simultaneously as a feature of the cross-sectional distribution of unemployed workers because workers with longer unemployment durations are precisely those who have failed to find a match previously and, at every duration, they are more likely to be in markets with lower matching efficiency. In practice, assessing the relationship between unemployment duration and unemployment outflows is problematic, as it is difficult to identify the extent to which it is true duration dependence or rather worker heterogeneity that underlies the data. Nonetheless, as discussed by Machin and Manning (1999), there is little evidence of true negative duration dependence after controlling for heterogeneity.

Other explanations for the observed relationship between unemployment duration and search outcomes have been suggested in the literature. One is that declines in wealth over unemployment spells induce both declining reservation wages and falling job finding probabilities (Burdett, 1977, Mortensen, 1977). Another is that the human capital of unemployed workers deteriorates with the duration of unemployment (Lazear, 1976). Alternatively, if workers differ in their (unobservable) productivities, unemployment duration may become a signal of low productivity (Lockwood, 1991).[5] These explanations cannot address the evidence adequately. For example, although the flow from unemployment to employment typically has a spike as the expiration of unemployment benefits approaches, the flow continues to decline after the expiration.[6] On skill depreciation, it is difficult to find direct evidence that workers with longer unemployment duration are less productive, whereas indirect evidence does not indicate an important role of skill depreciation. Despite the lack of evidence, the common perception that there is substantial unemployment "scarring" largely associated with skill depreciation has lent support to government sponsored training programs.[7] This is so even though such programs often seem to fail to increase the trainees' job finding rates.[8] Related to this subject, a voluminous literature on social

[5] Blanchard and Diamond (1994) explore the possibility that employers may base employment decisions on unemployment duration as an arbitrary ranking device. They note that the ranking scheme is not robust to directed search and homogeneous workers.

[6] Even after controling for wealth, Alexopoulos and Gladden (2006) have found that reservation wages still fall significantly over the unemployment duration.

[7] See e.g., Arulampalam, Gregg and Gregory (2001).

[8] For instance, Ham and LaLonde (1996) find that, controlling for sample selection, the National Supported Work Demonstration training program raised trainees' employment rates solely by lengthening their employment durations.

psychology emphasizes the effect of unemployment duration on psychological well-being.[9] However, the effect of unemployment on distress is often found to be short-lived, disappearing with re-employment (e.g., Kessler, Turner and House, 1989). Work in this area also suggests that young unemployed workers value employment more, rather than less, as the duration of their unemployment spell increases (see McFadyen and Thomas, 1997).

Our model also provides an alternative explanation for wage dispersion. In previous models of directed search with homogeneous workers and firms (see earlier citations), wage dispersion is very limited and, often, degenerate. Our model generates rich dispersion of equilibrium wages by turning ex ante identical agents into heterogeneous ones who differ in posterior beliefs about the market. This mechanism of wage dispersion can be useful for explaining the fact that about 70 percent of the variation in wages remains unexplained by observed worker characteristics (see Mortensen, 2003). In particular, the mechanism has the testable implication that differences in unemployment duration among homogeneous workers may be an important factor of wage dispersion among workers earning relatively low wages. Contrasting with other explanations that also build on search frictions (e.g. Butters, 1977, Burdett and Judd, 1983, and Burdett and Mortensen, 1998), our explanation features directed search and focuses on workers' learning from search.

The rest of the paper is organized as follows. The next section presents the model. Section 3 establishes existence of an equilibrium and characterizes the properties of the value of search. In Section 4 we show that reservation wages decline with unemployment duration. Section 5 provides conditions under which reservation wages are strictly declining and explores differentiability of the value function. Section 6 characterizes the steady state distribution of workers. Section 7 concludes and the Appendix collects all proofs.

2. The Model

2.1. Agents, Markets and Matching

Time is discrete and all agents discount the future at a rate $r > 0$. There are a large number of workers and firms. A worker is either employed or unemployed. When employed, a worker produces $y > 0$ units of goods. When unemployed, a worker searches for a job and the utility of leisure is normalized to zero.

The economy consists of a continuum of *local markets*, with mass 1. We think of each single local market in terms of the mix of physical characteristics of that particular labor market, such as geography and occupation. These characteristics are idiosyncratic to the

[9] See Darity and Goldsmith (1996) and McFadyen and Thomas (1997).

workers and the firms in that market, rather than economy-wide characteristics. Workers are assigned to one of these local markets at random, remaining in the same local market as long as they continue to search. There is free entry of firms in every local market. Local markets may differ in terms of their matching efficiency. A fraction $p \in (0,1)$ of all local markets have matching efficiency m_H. The remaining markets have matching efficiency $m_L \in (0, m_H)$. Thus, m denotes the common type of workers and firms in a given local market with matching efficiency m.[10]

The key feature of the model is that the true value of the matching efficiency m in every local market is unknown, which workers can learn about from their individual unemployment histories. We assume that individual search histories are private information. In addition, we suppose that agents observe all aggregates, but they do not observe the local labor market conditions specific to any given local market.

Each local market consists of a continuum of *submarkets* indexed by x. The matching probability of a worker in a submarket x is assumed to be mx. The domain of x is $X = [0, 1/m_H]$. In any given local market, a submarket x is characterized by a wage level, $W(x)$, and a tightness, $\lambda(x)$ (i.e., the vacancy-unemployment ratio). The functions $W(.)$ and $\lambda(.)$ are public information, but the matching probability in each submarket is unknown. Note that two submarkets can be indexed by the same x but they may belong to different local markets that differ in the matching efficiency. In the equilibrium analyzed below, the functions $W(.)$ and $\lambda(.)$ are the same in all local markets, and so observing the wage and tightness in a submarket does not reveal the matching efficiency of the local market in which the particular submarket is located.

Search is directed as follows. In each period, firms and workers in a given local market can choose which submarket to enter. We refer to this choice as an agent's search decision, because it affects the agent's matching probability. We also refer to $W(x)$ as the *reservation wage* of a worker who chooses to enter the submarket x. Search is directed in the sense that an agent's choice of a submarket involves a tradeoff between the wage and the tightness, because the two characteristics are negatively related to each other across submarkets. The equilibrium wage in a submarket "clears" the submarket in the sense that the induced entry of firms and workers is consistent with the tightness in that submarket. We will provide a formal definition of a competitive search equilibrium later.

To make precise the meaning of the matching efficiency, let us specify the matching function in a submarket x as $mF(u(x), v(x))$, where $u(x)$ is the number of unemployed

[10] Alternatively, one can think of m_i as a worker's characteristic. Then, the logic of the problem would be similar, but some details of the analysis would change, introducing unnecessary complications.

workers and $v(x)$ the number of vacancies in submarket x. Then, $x = F(u(x), v(x))/u(x)$ and $\lambda(x) = v(x)/u(x)$. It is important to emphasize that individuals do not observe $u(x)$ or $v(x)$ for each x, although they observe $\lambda(x)$, $F(x)$ and $W(x)$.

We impose the following standard assumption on the function F:

Assumption 1. *The function $F(u, v)$: (i) is strictly increasing, strictly concave and twice differentiable in each argument, (ii) is linearly homogeneous, and (iii) has $F(1, 0) = 0$ and $F(1, \infty) > 1/m_H$.*

Under this assumption, we can determine $\lambda(x)$ by $F(1, \lambda(x)) = x$. Moreover, $\lambda(x)$ has the following properties:

$$\lambda'(x) > \frac{\lambda(x)}{x} > 0,\ \lambda''(x) > 0, \text{ for all } x \in (0, 1/m_H). \tag{2.1}$$

The recruiting probability of a firm in submarket x is $mx/\lambda(x)$. The above properties of λ imply that a firm's recruiting probability decreases with x. That is, if it is easy for a worker to find a job at x, it must be difficult for a firm to recruit at x. Posting a vacancy for a period in any local market requires the firm to incur the vacancy cost $c \in (0, y)$. To ensure that equilibrium wages are positive we assume that

$$y > \frac{m_H}{m_L} rc\lambda\left(1/m_H\right). \tag{2.2}$$

The following examples of the matching function satisfy Assumption 1 and will be used in various parts below:

Example 2.1. *(i) One example of F is the CES function: $F(u, v) = \left[(1-\alpha)\, u^\rho + \alpha v^\rho\right]^{1/\rho}$, where $\rho < 1$ and $\alpha \in (0, 1)$. In this case, $\lambda(x) = \left(\frac{x^\rho - 1}{\alpha} + 1\right)^{1/\rho}$. Moreover,*

$$\lambda'(x) - \frac{\lambda(x)}{x} = \left(\frac{1}{\alpha} - 1\right) x^{-\rho} \left(\frac{1 - (1-\alpha)x^{-\rho}}{\alpha}\right)^{(1-\rho)/\rho}.$$

A special case of this example is the Cobb-Douglas function, where $\rho = 0$, which leads to $\lambda(x) = x^{1/\alpha}$ and $\lambda'(x) - \lambda(x)/x = \left(\frac{1}{\alpha} - 1\right) x^{(1-\alpha)/\alpha}$. (ii) Another example is the urn-ball matching function: $F(u, v) = v\left(1 - e^{-u/v}\right)$. In this case, $\lambda(x)$ is defined by $\lambda(x)\left[1 - e^{-1/\lambda(x)}\right] = x$. Moreover,

$$\lambda'(x) - \frac{\lambda(x)}{x} = \frac{\frac{1}{\lambda} e^{1/\lambda}}{\left(e^{1/\lambda} - 1\right)\left(e^{1/\lambda} - 1 - 1/\lambda\right)}.$$

To focus on search, we assume that employment is an absorbing state. In this environment, the steady state distribution of workers is non-trivial only if there is a flow into unemployment. For this reason, we assume that the labor force grows at the constant rate n. Thus, if L is the labor force at the beginning of period t, a mass nL of new workers enters the labor market in period t, joining the pool of unemployed workers that period.

2.2. Learning from Unemployment

Agents update their beliefs on m after observing whether or not they have a match. The updating depends on the particular submarket into which the agent just searched. To describe the updating process, it is convenient to express the beliefs in terms of their expected types. Let the initial prior expectation of m be $\mu_0 \in (m_L, m_H)$, for all agents (workers and firms). This prior mean belief can be calculated from the distribution of workers across local markets when they first enter the economy, and it is common to both workers and firms: $\mu_0 = pm_H + (1-p)m_L$, where $p \in (0,1)$.

Consider the updating process for a worker. It should be noted that public information about aggregate statistics does not reveal any valuable information to individual workers beyond what is already contained in the equilibrium functions of wages, $W(.)$, and tightness, $\lambda(.)$. Since these functions are identical across all local markets, and agents do not observe the behavior of others in their own local market, the only valuable information to a worker is his private history of search outcomes. Let $P(m_i)$ be the prior probability with which $m = m_i$, where $i \in \{H, L\}$. Let μ be the expected value of m according to this prior belief. Note that the prior distribution of m is Bernoulli, with $E(m) = \mu$ and $Var(m) = (m_H - \mu)(\mu - m_L)$. From the definition of μ, we can solve $P(m_i)$ in terms of μ:

$$P(m_H) = \frac{\mu - m_L}{m_H - m_L}, \qquad P(m_L) = \frac{m_H - \mu}{m_H - m_L}.$$

Let $k \in \{0,1\}$ be the matching outcome in the current period, where $k = 0$ indicates that the worker fails to get a match and $k = 1$ indicates that the worker succeeds in getting a match. Then,

$$P(m_i|x;\ k=1) = \frac{m_i}{\mu}P(m_i),\ P(m_i|x;\ k=0) = \frac{1-xm_i}{1-x\mu}P(m_i).$$

Because the conditional distribution of m is Bernoulli, then conditional on $k \in \{0,1\}$, the mean and variance of m are:

$$E(m|k) = m_H P(m_H|k) + m_L(1 - P(m_H|k))$$

$$Var(m|k) = (m_H - m_L)^2 P(m_H|k)(1 - P(m_H|k)).$$

Note that, if $x < 1/m_H$, we have $P(m_H|k = 0) > 0$ for all $\mu > m_L$. Thus, if the initial mean belief μ_0 exceeds m_L, then $E(m|k) > m_L$ for both $k = 0$ and $k = 1$.

This updating process has two preliminary properties. First, the sequence $\{E(m)\}$ is a Markov process. Second, a worker's mean beliefs $E(m)$ are a sufficient statistic for the worker's unemployment history.

Search in a market with a high x generates outcomes that are more informative than outcomes of search in a market with a low x. More precisely, a higher x causes a mean-preserving spread in the distribution of the posterior expectation $E(m|k)$. To see this, note that ex ante k is a random variable, and so is the posterior expectation $E(m|k)$. The mean of this posterior expectation is $E(E(m|k)) = E(m) = \mu$, which is unaffected by x. The variance of the posterior expectation is:

$$Var(E(m|k)) = (\mu - m_L)^2 \left[\frac{xm_H^2}{\mu} + \frac{(1 - xm_H)^2}{1 - x\mu} - 1\right].$$

This variance increases with x.

The informational content of x is asymmetric with respect to the matching outcome. For a worker who succeeds in finding a match, the posterior, $P(m|k = 1)$, is not a function of x. The posterior mean belief in this case is $E(m|k = 1) = m_H + m_L - m_H m_L/\mu$, which is also independent of x. Therefore, a worker's choice of submarket, x, does not affect the information contained in a successful search outcome. In contrast, for a worker who fails to find a match, the posterior, $P(m_H|k = 0)$, decreases with x. That is, the higher x of a submarket in which a worker searches for a job, the more the worker will reduce the posterior on the matching efficiency after he fails to find a match. This is because finding a match in a submarket with a higher x is supposed to be easier, and failure to find a match there should induce the worker to revise the beliefs downward more sharply.[11]

We will focus on unemployed workers' decisions. For this purpose, it is useful to write separately the updating process of a worker who fails to find a job. For such a worker, we refer to the mean of the beliefs, μ, simply as the beliefs. The posterior belief of an unemployed worker who searches and fails to find a job is given by

$$H(x, \mu) \equiv E(m|k = 0) = m_H - \frac{1 - xm_L}{1 - x\mu}(m_H - \mu). \tag{2.3}$$

This posterior expectation has the following properties, whose verification is straightforward and hence omitted here:

[11] This asymmetry of the role of x in the posterior holds more generally in the following way. Suppose that the job finding probability is $\varphi(x, m)$. If $\varphi(x, m_H)/\varphi(x, m_L)$ is independent of x, then $P(m|k = 1)$ is independent of x but $P(m_H|k = 0)$ decreases in x.

Lemma 2.2. *The function $H(x, \mu)$ satisfies: (i) $H_1 < 0$; (ii) $H_2 > 0$, (iii) $H_{11} = \frac{2\mu}{1-x\mu} H_1 < 0$ and $H_{22} = \frac{2x}{1-x\mu} H_2 > 0$; (iv) $\mu(1 - x\mu)H_{12} - H_1 - \mu^2 H_2 = -m_H m_L$.*

Property (i) states that a higher x reduces the worker's posterior beliefs after the worker fails to find a match, as discussed above. In particular, property (i) implies that $H(x, \mu) < \mu$ for all $x > 0$ and $\mu > m_L$. Thus, a worker's beliefs about the local market's matching efficiency decrease over time as the number of search failures increases. Of course, if a worker's beliefs have reached m_L, there is no further updating; that is, $H(x, m_L) = m_L$ for all x. Property (ii) states that, for any given x, a worker with higher prior beliefs will also have higher posterior beliefs. Properties (iii) and (iv) will be useful later.

2.3. The Value of Search

Consider an unemployed worker who enters a period with beliefs, μ. Let $V(\mu)$ be his value function. If he chooses to search in a submarket x, the expected probability of finding a match is $x\mu$. Suppose that the worker accepts the match, which will yield wage $W(x)$. Because employment is permanent, the present value of the job is $W(x)/r$. If the worker does not find a job in the current period, he will revise the beliefs to $H(x, \mu)$ and continue to search in the next period. In that case, the expected value from the next period onward will be $V(H(x, \mu))$. Thus, the expected payoff of searching in a submarket x is:

$$R(x, \mu) \equiv x\mu \frac{W(x)}{r} + (1 - x\mu) \frac{V(H(x, \mu))}{1 + r}. \tag{2.4}$$

The above calculation presumes that a worker accepts the offer, which may not be true in principle. Both workers and firms may have incentive to engage in a particular form of "experimentation", searching during a period solely to gather information and, thus, refusing to enter a match once they learn that a match has occurred. A worker that has searched in a submarket x and found a match will revise his beliefs to $E(m|k = 1) = m_H + m_L - m_H m_L/\mu$, as explained above. Suppose that the worker chooses to reject the match and continue to search in the next period. Then, his expected payoff of entering a submarket x to search today is:

$$R^e(x, \mu) \equiv x\mu \frac{V(m_H + m_L - m_H m_L/\mu)}{1 + r} + (1 - x\mu) \frac{V(H(x, \mu))}{1 + r}. \tag{2.5}$$

We do not think that this form of experimentation is important in practice, unless it is associated with heterogeneous matches. Thus, we rule out such experimentation by imposing two assumptions. First, we assume that firms commit to accepting all successful matches. This assumption can be viewed as a natural implication of the maintained assumption of directed search; that is, a firm is committed to accepting a worker at the posted wage as long as the worker has the specified productivity. As an additional justification, note that the main motivation for a firm to reject a match in reality is to search for a more productive match. This motivation does not exist in our model, because all matches are homogeneous. Second, we require that a worker should always accept a match which he searches for, as specified in the assumption below. A sufficient condition to validate this assumption will be provided in Lemma 3.1.

Assumption 2. $\max_{x \in X} R(x, \mu) \geq \max_{x \in X} R^e(x, \mu)$ *for all* $\mu \in M$.

It can be verified that the inequality stated in the assumption must hold for values of μ that are sufficiently close to m_L and for values of μ that are sufficiently close to m_H. Below we shall provide an intuitive sufficient condition for it to hold for all $\mu \in M$ (see Lemma 3.1). Under Assumption 2, the value of search under beliefs μ is given by:

$$V(\mu) = \max_{x \in X} R(x, \mu). \tag{2.6}$$

Denote the set of optimal decisions as $G(\mu) = \arg\max_{x \in X} R(x, \mu)$ and a selection from $G(\mu)$ as $g(\mu)$. The reservation wage can be written as $w(\mu) = W(g(\mu))$.

Before analyzing the solution to the above decision problem, consider the behavior of firms. Firms also choose which submarket to enter to post vacancies and, after observing the matching outcome, they update their beliefs. Their initial prior belief is the same as the workers', μ_0. The updating process of a firm is similar to that of the workers'. Let x^v describe the submarket which a firm enters, for a given local market with matching efficiency m. If the firm finds a match, its posterior expectation of m does not depend on x^v. If the firm fails to find a match, then the posterior expectation of m decreases in x^v. Because the firm's matching probability is $mx^v/\lambda(x^v)$, then the firm's expectation of m after failing to find a match in one period is $H(x^v/\lambda(x^v), \mu_0)$, where H is defined in (2.3). Note that $H(x^v/\lambda(x^v), \mu_0) < \mu_0$ for all $x^v \in (0, 1/m_H)$.

Let $J(\mu^v)$ be the value of a vacancy given that the firm's mean belief at the beginning of a period is μ^v. With free entry, $J(\mu_0) = 0$. Because $H(x^v/\lambda(x^v), \mu_0) < \mu_0$ for all $x^v < 1$, as explained above, continuing to post a vacancy under the beliefs $H(x^v/\lambda(x^v), \mu_0)$ yields a negative value. That is, a firm will always exit the market after one period of search if the search fails to find a match in that period.[12] This result allows us to simplify a firm's value function as follows:

$$J(\mu_0) = \max_{x^v \in X} \left[-c + \mu_0 \frac{x^v}{\lambda(x^v)} \frac{y - W(x^v)}{r} \right].$$

[12] A positive entry cost would explain why vacancies may last longer than one period.

The first-order condition of the above problem involves the wage function W and its derivative. This can be alternatively viewed as a differential equation for the wage function. Without an initial condition, this differential equation has a continuum of solutions. The indeterminacy simply says that there are many levels of x^v that are optimal for the firm. Put differently, a firm is willing to enter into any submarket, provided that the wage in the submarket is consistent with the free-entry condition, which we discuss next.

2.4. Free-Entry of Firms and the Equilibrium Definition

Free-entry of firms implies $J(\mu_0) = 0$. Together with the firm's value function, this condition yields the following wage function:

$$W(x) = y - \frac{rc}{\mu_0}\frac{\lambda(x)}{x}. \tag{2.7}$$

For future reference, it is useful to note that, for all $x \in X$, the function $W(x)$ is twice continuously differentiable and it has the following properties:

$$(i)\ 0 < W(x) \le y; \quad (ii)\ W'(x) < 0, \quad (iii)\ 2W'(x) + xW''(x) < 0. \tag{2.8}$$

Part (i) is ensured by (2.2) and other parts by (2.1). Part (ii) says that a higher employment probability comes together with a lower wage. This is necessary for directed search to be meaningful, as it provides a tradeoff between the wage and the tightness of the submarket. As such, it is a necessary condition for inducing firms to enter the submarket. Part (iii) is implied by $\lambda''(x) > 0$, and it says that the function $xW(x)$ is concave in x.

Focus on stationary equilibria. An equilibrium consists of workers' decision x, firms' decision x^v, and a wage function $W(x)$, that meet the following requirements. (i) Given the wage function and a worker's belief at the beginning of a period, the worker's choice of the submarket obeys the rule $x = g(\mu)$. (ii) Given the initial belief μ_0 and the wage function, a firm's choice is optimal; that is, $x^v = g^v(\mu)$. (iii) Conditional on unsuccessful search, a worker's beliefs are updated according to $H(g(\mu), \mu)$ and a firm's according to $H(g^v(\mu)/\lambda(g^v(\mu)), \mu)$. (iv) Consistency: for every x in every local market, the mass of all firms who choose $g^v(\mu) = x$ divided by the mass of all workers who choose $g(\mu) = x$ is equal to $\lambda(x)$. (v) Free-entry: for each local market, the wage function W satisfies (2.7).

In the above definition we have left out the steady-state conditions on worker flows and the wage distribution, which will be characterized in section 6. We deliberately do so in order to emphasize the feature of the model that individuals' decisions and matching probabilities can be analyzed without any reference to the wage distribution. Instead, all that is required for such an analysis is the wage function $W(.)$ and the tightness function $\lambda(.)$, which are determined by firms' free-entry condition and the matching function. This feature makes the analysis tractable by reducing the dimensionality of the state variables for individuals' decision problems significantly (i.e., by infinity).[13] As an implication of directed search, this feature is not possessed by models of undirected search such as Burdett and Vishwanath (1988). In the latter models, an individual's search decision depends on the wage distribution which, in turn, evolves as individuals learn about the market. Solving for these dynamics of the wage distribution, even quantitatively, is a daunting task.

To conclude the description of the model, let us emphasize the information structure. The economy consists of many local markets, and each local market has many submarkets, one for each duration of unemployment. The actual labor market outcomes are different across local markets with different matching efficiency. However, agents do not know the matching efficiency of their own local market. The distribution of labor market outcomes across local markets is common knowledge, but it conveys no useful information to the agents about their own local markets. As a result, each agent's optimal choice is a function of their beliefs and, therefore, of their unemployment duration only, but not of the matching efficiency of the agent's local market.

3. Equilibrium Learning with Search

Let us now analyze a worker's optimization problem, (2.6). When choosing a submarket x, the worker faces two considerations. One is the familiar tradeoff between wages and the matching probability in models of directed search. That is, a submarket with a higher x has a lower wage and a higher probability of finding a match. Another consideration is learning from the search outcome. As discussed earlier, search in a submarket with a high x (i.e., a low wage) is more informative about the matching efficiency in that market than search in a submarket with a low x. To see how the model captures the value of learning, we examine the value function.

It is easy to see that the mapping defined by the right-hand side of (2.6) is a contraction. Using the features in (2.8), standard arguments show that a unique value function V exists, which is positive, bounded and continuous on $M = [m_L, m_H]$ (see Theorem 4.6 in Stokey and Lucas, 1989, p.79). Moreover, the set of maximizers, G, is nonempty, closed, and upper-hemicontinuous. Existence of the optimal decision, together with the characterization of the steady-state distribution in section 6, establishes existence of an equilibrium.

[13]Shi (2006) also explores a similar feature in a directed search model of wage-tenure contracts.

Lemma 3.1. *Under Assumption 2, there exists an equilibrium where all successful matches are accepted. A sufficient condition for Assumption 2 to hold is that labor productivity satisfies:* $y \geq (1+r)\, c \frac{m_H}{m_L} \lambda\, (1/m_H)$.

The sufficient condition for existence, stated in the lemma, implies that a worker prefers getting the lowest feasible wage every period starting now to getting the full surplus from a match every period starting next period. Intuitively, because of discounting, increasing y acts as a higher search cost, making it more profitable to accept a job today rather than waiting. Note that, as in Burdett and Vishwanath (1988), we can reduce the restrictive force of this condition by introducing a constant cost of search that an unemployed worker must pay every period. By increasing a worker's cost of rejecting an offer, such a cost enlarges the parameter region in which a worker always accepts a match.[14]

The following lemma describes additional properties of the value function (see Appendix B for a proof):

Lemma 3.2. *V is strictly increasing and strictly convex. As a result, V is almost everywhere twice differentiable with almost everywhere continuous first derivative.*

Strict convexity of the value function captures the feature that a more informative search outcome is valuable to the worker. In a market with a higher x, whether search succeeds or fails generates relatively larger contrasts which allow the worker to update his beliefs more precisely. Convexity of the value function reflects the fact that such variations in the posterior are valuable to the worker. However, more informative search reduces the worker's future payoff in unemployment. This feature arises from the fact that a worker continues to search only after he fails to find a match. Learning is valuable to the worker because it enables the worker to deduce that the market is worse than he expected. With more informative search, the worker's posterior beliefs will deteriorate more rapidly if he does not find a match, in which case the value of continuing to search will fall by more.

These interesting and inevitable consequences of learning from search cause analytical difficulties. Mathematically, the future payoff $(1 - x\mu)V(H(x, \mu))$ is convex in x, which can make the objective function $R(x, \mu)$ convex in x. Hence, the optimal decision is not necessarily unique or interior. The possibility of multiple solutions implies that the value function may not be differentiable at some levels of beliefs, although it is twice differentiable almost everywhere. Because the objective function of the worker's optimization problem

[14] For simplicity, we have not included such a cost of search for the workers in the following analysis.

involves the value function in the future, it may not be differentiable either. Thus, the first-order condition may not be applicable to the workers' optimization problem.

The possibility of multiple solutions and non-differentiability is well known in the literature on optimal learning (e.g., Easley and Kiefer, 1988). However, this literature has either ignored these difficulties or focused on corner solutions (e.g., Balvers and Cosimano, 1993). We need to examine all solutions in order to establish the central result of declining reservation wages. In different modeling environments, there are techniques to generate smooth optimal choices and differentiable value functions, e.g., Santos (1991). However, those techniques require the value function to be concave, which is violated here.

4. Declining Reservation Wages

In this section, we establish the central result that a worker's reservation wage increases in the worker's beliefs; i.e., $w(\mu)$ is an increasing function. Because a worker's beliefs deteriorate with the duration of unemployment, monotonicity of $w(\mu)$ implies that reservations wages decline over the spell of unemployment. By definition, $w(\mu) = W(g(\mu))$, where $g(\mu)$ is a worker's optimal choice of the submarket in which he searches under beliefs μ. Thus, monotonicity of $w(\mu)$ is equivalent to the feature that $g(\mu)$ is a decreasing function, which reflects the fact that a higher wage always comes together with a low matching probability.

We will first use a heuristic approach to illustrate the desired result and then establish the result formally. Before carrying out these analyses, it is convenient to transform the worker's choice from x to $z \equiv -x$. The transformation will be useful in what follows, and it enables us to attach the label *monotone decisions* naturally to the feature that z increases in the beliefs. After the transformation, the objective function in (2.6) becomes $R(-z, \mu)$ and the feasible set of choices is $-X \ni z$. Denote $Z(\mu) = \arg\max_{z \in -X} R(-z, \mu)$ and $z(\mu) \in Z(\mu)$. Then, the set of optimal choices for x is $G(\mu) = -Z(\mu)$ and a typical selection is $g(\mu) = -z(\mu)$.

4.1. A Heuristic Illustration of Monotone Optimal Choices

We use the first-order condition as a heuristic illustration of monotonicity, although such a condition is not applicable in general. To do so, suppose furthermore that the value function is twice differentiable and the objective function $R(-z, \mu)$ is strictly concave in the first argument. In addition, suppose that the optimal choice is interior. Then, the optimal choice, $z(\mu)$, is unique and obeys the following first-order condition:

$$R_1(-z(\mu), \mu) = 0. \tag{4.1}$$

In what follows, we will use the notation R_1 and R_{11} to refer to the partial derivative of $R(-z,\mu)$ with respect to the first argument, rather than to z; and similarly for $H(-z,\mu)$.

Differentiating the previous equation, we obtain $z'(\mu) = R_{12}/R_{11}$. Because $R_{11} < 0$ as we suppose here, then the desired result, $z'(\mu) > 0$, holds if and only if $R_{12} < 0$. Writing (4.1) explicitly and using it to substitute for $(W + \lambda W')$, we can compute:

$$R_{12} = \frac{V'(H)}{1+r}\Big[\mu(1+\mu z)H_{12} - H_1 - \mu^2 H_2\Big] + (1+\mu z)\frac{V''(H)}{1+r}H_1 H_2,$$

where $H = H(-z(\mu),\mu)$. Because $V' > 0$, $V'' > 0$, $H_1 < 0$ and $H_2 > 0$, we can use part (iv) of Lemma 2.2 to verify $R_{12} < 0$. Thus, $z'(\mu) > 0$ indeed holds.

This illustration suggests that strict convexity of the value function should be important for the optimal choice $x = -z(\mu)$ to be decreasing with μ. The intuition is as follows. Searching in a market has the consequence of reducing the worker's posterior beliefs when search fails to generate a match. This is an implicit cost of search. Strict convexity of the (future) value function implies that, for the same choice of x, this cost is higher when beliefs are at high levels than when beliefs are at low levels. Because the reduction in the posterior beliefs increases with x, it is more costly to choose a high x when beliefs are high than when beliefs are low. Roughly speaking, getting bad news about the market is more damaging when the worker is optimistic than when the worker is pessimistic. Therefore, it is optimal to increase x to generate more information as beliefs deteriorate. This explains why $g(\mu)$ is decreasing and, hence, why $z(\mu)$ is increasing.

The illustration also suggests that monotonicity of $z(\mu)$ may depend only on the features of the value function and the updating function, H. In contrast, the wage function does not play any explicit role for the signs of R_{12} and $z'(\mu)$, provided that it induces the value function to be increasing and convex. In particular, monotone choices do not require the current payoff, $-\mu z W(-z)$, to have a positive cross partial derivative in (μ, z).

Both suggestions above hold true generally, even when the value function fails to be differentiable. To establish the general result, we need a different apparatus for the analysis.

4.2. Supermodularity and Monotone Optimal Choices

Supermodularity is a powerful method for conducting comparative statics. Topkis (1998) formulated the theory of supermodularity, which has been applied to dynamic programming (e.g., Amir et al., 1991). Milgrom and Shannon (1994) extended the theory from a cardinal one to an ordinal one. In our model, supermodularity is equivalent to the feature of increasing differences, because the variables under investigation, (z,μ), lie in closed intervals of the real line. Let $z \in Z$ and $\mu \in M$, where Z and M are partially ordered sets. A function $f(z, \mu)$ has increasing differences in (z, μ) if $f(z_1, \mu_1) - f(z_1, \mu_2) \geq f(z_2, \mu_1) - f(z_2, \mu_2)$ for all $z_1 > z_2$ and $\mu_1 > \mu_2$. If the inequality is strict, then f has strictly increasing differences. In our model, Z, M and $Z \times M$ (under the product order) are all lattices. In this case, the feature of increasing differences implies supermodularity (see Topkis, 1998, p.45).

It is far from obvious whether the concept of supermodularity can be usefully applied here. On the economic side, our model does not have the usual reasons for supermodularity, such as complementarity in consumption or production (see Topkis, 1998). On the technical side, there are two features of our model that can complicate the use of supermodularity. First, as an inseparable feature of dynamic programming, the objective function in (2.6) involves the future value function. A similar feature is present in models of optimal growth. In order to apply the method of supermodularity, those models assume that the current payoff function is supermodular (e.g., Amir et al., 1991). In our model, the current payoff is $-\mu z W(-z)/r$. Neither is this function supermodular, nor is such supermodularity necessary for monotone optimal choices. Second, the future value function is discounted with an endogenous factor, $(1 + \mu z)$. Stern (2006) uses supermodularity in optimal growth with endogenous discounting, but he assumes that the discount factor is a concave function (also see Becker and Boyd, 1997, pp. 277-284). It is easy to see that the discount factor in our model is not concave in (μ, z). Therefore, we cannot follow the well-trodden path to establish supermodularity in our model.

Complicating the matter further, the objective function $R(-z, \mu)$ is unlikely to be supermodular. To see this, note that we obtained the result $R_{12} < 0$ in the above illustration by substituting the first-order condition. This means that the cross partial derivative of R with respect to z and μ is positive locally at $z = z(\mu)$. The local property does not imply the global property of supermodularity.

Fortunately, monotonicity of optimal choice is invariant to transformations of the objective function that are monotone in the choice variables. Thus, it becomes possible to transform the objective function into a supermodular function. To that end, we transform the worker's maximization problem into

$$V(\mu) = \mu \max_{z \in -X} \hat{R}(z, \mu)$$

where $\hat{R}$ is defined as follows:

$$\hat{R}(z, \mu) \equiv \frac{1}{\mu} R(-z, \mu) = -z \frac{W(-z)}{r} + \left(z + \frac{1}{\mu} \right) \frac{V(H(-z, \mu))}{1 + r}.$$

Denote $Z(\mu) = \arg\max_{z \in -X} \hat{R}(z, \mu)$ and $z(\mu) \in Z(\mu)$. Clearly, the set of optimal choices for x is $G(\mu) = -Z(\mu)$ and a typical selection is $g(\mu) = -z(\mu)$. Denote the greatest selection of $Z(\mu)$ as $\bar{z}(\mu)$ and the least selection as $\underline{z}(\mu)$.

The following theorem states the result on monotonicity (see Appendix C for a proof):

Theorem 4.1. *Let $z \in -X$ and $\mu \in M$. The function $\hat{R}(z, \mu)$ is strictly supermodular in (z, μ). Thus, every selection $z(\mu)$ is an increasing function. Similarly, every selection $g(\mu)$ is a decreasing function, and the wage $w(\mu)$ is an increasing function.*

The main task in the proof of this theorem is to establish supermodularity of $\hat{R}$, after which monotonicity of $z(\mu)$ follows from Topkis (1998, p.79). Two aspects of the proof are worth noting, both of which extend the features in the above heuristic illustration from local properties to global ones. First, as expected, strict convexity of the value function plays an important role for supermodularity of $\hat{R}$ and, hence, for monotone optimal choices. Second, supermodularity of $\hat{R}$ relies only on the properties of the value function, V, and the updating function, H, not on those of the wage function, W. In particular, supermodularity of $\hat{R}$ does not require the current payoff function, $-\mu z W(-z)$, to be supermodular.

Remark 1. *There is another way to see why monotonicity of optimal choices does not rely on the properties of W or on supermodularity of the original objective function, $R(-z, \mu)$. As shown by Milgrom and Shannon (1994), monotone comparative static analysis requires not supermodularity, but rather a weaker property — the single-crossing property. In our model, $R(-z, \mu)$ has the strict single crossing property in (z, μ) if and only if $\hat{R}$ is supermodular. Because W depends only on z, it drops out of the condition for the single crossing property. In light of this remark, the result of monotone optimal choices also follows from Theorem $4'$ in Milgrom and Shannon (1994).*

The result of monotone optimal choices is a general one. It holds even when optimal choices are corner solutions and when there are multiple solutions to the worker's optimization problem. When multiple solutions exist, every solution for z is an increasing function of the beliefs. This strong result comes from the feature that $\hat{R}(z, \mu)$ is strictly supermodular. However, as a general result, the above theorem allows the possibility that a solution $z(\mu)$ is only a weakly increasing function. In the next section, we address strict monotonicity and other issues.

5. Strict Monotonicity and Uniqueness of the Optimal Path

In order to understand the conditions under which reservation wages decline over a worker's unemployment spell, there are two further questions that need to be answered. First, when are reservation wages *strictly* declining with unemployment duration? This stronger property holds if and only if optimal choices, $z(\mu)$, are strictly increasing functions of the beliefs. Second, if optimal choices are not unique, is there any discipline on the set of paths of optimal choices? To answer these questions, we impose the following assumption, which ensures optimal choices to be interior solutions:

Assumption 3. *Initial beliefs $\mu_0 \in (m_L, m_H)$ satisfy*

$$\mu_0 < \frac{c}{y}\left[\left(r + \frac{m_L}{m_H}\right)\lambda'\left(\frac{1}{m_H}\right) - m_L\lambda\left(\frac{1}{m_H}\right)\right]. \tag{5.1}$$

As shown in Appendix D, this assumption amounts to ensuring that $R_1(-1/m_H, m_L) > 0$, so even a worker with beliefs $\mu = m_L$ will find it optimal to choose $z > -1/m_H$. Since $z(\mu)$ is increasing, this assumption is sufficient for a workers' choices to be interior along the optimal path, starting at $\mu_0 \in (m_L, m_H)$.

It should be noted that Assumption 2 and Assumption 3 can hold simultaneously for some $\mu_0 > m_L$. For instance, a sufficient condition for this to be the case is that labor productivity satisfies:

$$\frac{c}{m_L}\left[\left(r + \frac{m_L}{m_H}\right)\lambda'\left(\frac{1}{m_H}\right) - m_L\lambda\left(\frac{1}{m_H}\right)\right] > y \geq (1+r)\left(\frac{m_H}{m_L}c\lambda\left(\frac{1}{m_H}\right)\right),$$

where the second inequality is the sufficient existence condition stated in Lemma 3.1. For this interval of y to be non-empty, it is sufficient that

$$\lambda'\left(\frac{1}{m_H}\right) - \frac{\lambda\left(\frac{1}{m_H}\right)}{\frac{1}{m_H}} > \left(\frac{1}{r + \frac{m_L}{m_H}}\right)\frac{\lambda\left(\frac{1}{m_H}\right)}{\frac{1}{m_H}}.$$

To see that this condition can be satisfied, consider the examples in Example 2.1. When the matching function is CES, the above condition holds provided $\rho < 0$. With $\rho = 0$ (i.e., the Cobb-Douglas function), the condition also holds with $\alpha = 1/2$ and $r + \frac{m_L}{m_H} > 1$. Similarly, for the urn-ball matching function, the above condition is satisfied when $r + 1 > \frac{e^a - 1}{a}\left(1 - m_L e^{-a}\right)$ where $a = \lambda(1/m_H)$.

In Appendix D, we establish the following lemma.

Lemma 5.1. *Under Assumption 3, an unemployed worker's optimal choices are interior. Moreover, the derivative $V'(H(-z(\mu),\mu))$ exists for all $z(\mu) \in Z(\mu)$. Thus, optimal choices obey the first-order condition, $\hat{R}_1(z(\mu),\mu) = 0$, where*

$$\begin{aligned}\hat{R}_1(z(\mu),\mu) = \quad & \tfrac{z(\mu)W'(-z(\mu))-W(-z(\mu))}{r} + \tfrac{V(H(-z(\mu),\mu))}{1+r} \\ & -\left(\tfrac{1}{\mu}+z(\mu)\right)\tfrac{V'(H(-z(\mu),\mu))}{1+r}H_1(-z(\mu),\mu).\end{aligned}$$

In addition to ensuring interior solutions, this lemma describes a limited sense of differentiability of the value function: the value function is differentiable in future periods at particular posterior beliefs induced by optimal choices, i.e., along the path of optimal choices. Despite the fact that the value function may still fail to be differentiable in the first period and at beliefs off the optimal paths, the limited sense of differentiability is enough for the first-order condition to be applicable in every period. In turn, the first-order condition enables us to establish strict monotonicity of optimal choices, as stated in the following theorem (see Appendix E for a proof):

Theorem 5.2. *Under Assumption 3, every selection of optimal choices, $z(\mu)$, is a strictly increasing function. Therefore, along every path of optimal choices, reservation wages are strictly declining with unemployment duration.*

As it is the case with supermodularity, strict monotonicity relies on the properties of the functions V and H, but not those of the wage function W directly. Not surprisingly, strict convexity of the value function plays a critical role for strict monotonicity of optimal choices. It is worth noting that Amir (1996) also establishes strict monotonicity of optimal choices, with the additional assumption that the value function is continuously differentiable. We do not rely on this assumption because it does not hold in our model.

Let us now turn to the question about the set of optimal paths. When there are multiple solutions, optimal choices can evolve over time in many ways. One case is that multiple choices occur in every period, in which case the path of optimal choices branches out. Another case is that multiplicity occurs only in the first period. Clearly, the path of optimal choices is more predictable in the second case than in the first case. To know more about the set of paths of optimal choices, we establish a link between multiplicity of optimal choices and differentiability of the value function at all possible beliefs. The following lemma states the link (see Appendix F for a proof):

Lemma 5.3. *Maintain Assumption 3. For each μ_a in the interior of (m_L, m_H), let μ_a^+ denote the limit to μ_a from the right (above) and μ_a^- the limit from the left (below). Then,*

$V'(\mu_a^+) = R_1(-\bar{z}(\mu_a), \mu_a)$ *and* $V'(\mu_a^-) = R_1(-\underline{z}(\mu_a), \mu_a)$. *Moreover,* $V'(\mu_a^+) \geq V'(\mu_a^-)$, *where the inequality is strict if and only if* $\bar{z}(\mu_a) > \underline{z}(\mu_a)$.

This lemma says that, at arbitrary beliefs $\mu \in (m_L, m_H)$, the value function is differentiable if and only if the beliefs induce a unique choice to be optimal. If multiple choices are optimal at particular beliefs, then the right derivative of the value function is strictly greater than the left derivative. Denote the set of such beliefs as

$$N = \{\mu \in (m_L, m_H) : \bar{z}(\mu) > \underline{z}(\mu)\}.$$

Because V is almost everywhere twice differentiable, the set N has measure zero in M.

For any μ_0 in the interior of M, let $\{\mu_n\}_{n=0}^{\infty}$ be a path of beliefs generated by optimal choices; i.e., $\mu_n = H(-z(\mu_{n-1}), \mu_{n-1})$ with $z(\mu_{n-1}) \in Z(\mu_{n-1})$, for $n = 1, 2, ...$ For arbitrary initial beliefs, μ_0, the following theorem characterizes the entire set of paths of beliefs and optimal choices (see Appendix F for a proof):

Theorem 5.4. *For any* $\mu_0 \in (m_L, m_H)$, *let* $z(\mu_0)$ *be an arbitrary selection from* $Z(\mu_0)$ *and let* $\mu_1 = H(-z(\mu_0), \mu_0)$ *be the posterior beliefs induced by* $z(\mu_0)$. *Given* μ_1, μ_n *is unique,* $Z(\mu_n)$ *is a singleton, and* $V'(\mu_n)$ *exists for all* $n = 1, 2, ...$ *If* $\mu_0 \notin N$, *then* $Z(\mu_0)$ *is also a singleton, in which case the entire path* $\{\mu_n\}_{n=0}^{\infty}$ *is unique and* $V'(\mu_n)$ *exists for all* $n = 0, 1, 2, ...$ *If* $\mu_0 \in N$, *then* $\bar{z}(\mu_0) > \underline{z}(\mu_0)$, $H(-\bar{z}(\mu_0), \mu_0) > H(-\underline{z}(\mu_0), \mu_0)$ *and* $V'(\mu_0^+) > V'(\mu_0^-)$.

This theorem states that the paths of optimal choices and induced beliefs are unique almost everywhere. The only case of non-uniqueness is when the worker's initial prior lies in the set N, which has measure zero. Even in this case, non-uniqueness occurs only in the first period of search. Given any optimal choice in the first period and the induced posterior, the future paths of optimal choices and induced beliefs are unique from that point onward. Thus, no matter where initial beliefs lie, the worker will choose search decisions optimally to keep the beliefs out of the set N from the second period onward. More precisely, whenever the search decision will induce the posterior beliefs to be close to a particular level in the set N, it is optimal to modify the decision so as to keep the posterior beliefs above that level. This result is a consequence of the value of learning, as captured by strict convexity of the value function.

To understand why a worker chooses optimally to avoid the set N in future periods, suppose counterfactually that the worker's choice in some period n induces the posterior beliefs to lie in N; that is, $\mu_{n+1} = H(-z_n, \mu_n) \in N$ for some $n \geq 0$, where $z_n = z(\mu_n)$. In this case, multiple choices will be optimal in period $(n+1)$, which induce the left derivative of $V(\mu_{n+1})$ to be lower than its right derivative. Recall that the derivative of the future value function captures an implicit (opportunity) cost of learning bad news. Thus, the discrete fall in $V'(\mu_{n+1})$ from the right side of μ_{n+1} to the left side implies that learning slightly more about the market in the current period increases the cost of learning by a discrete amount. The worker can avoid this discretely larger cost by choosing z_n slightly above $z(\mu_n)$, which will keep the posterior slightly above μ_{n+1}. In contrast to this discrete increase in the benefit, the increase in the cost of z_n is a marginal reduction in the matching probability. Thus, the net gain from increasing z_n slightly above $z(\mu_n)$ is positive. This contradicts the optimality of z_n.

6. Steady State Distributions and Duration Dependence

We now analyze the aggregate characteristics of the market. One purpose of this analysis is to illustrate that the learning process in previous sections is consistent with aggregation. The other purpose is to distinguish the duration dependence at an individual's level from the aggregate dependence.

Let $\hat{U}$, E and L denote the economy-wide unemployment, employment and labor force at the beginning of period t. The aggregate number of searchers in period t, denoted as U, includes both $\hat{U}$ and the number of newborns, nL. Let f denote the average job finding rate in the economy. Denoting next period's variables with a prime, we have:

$$U = \hat{U} + nL, \quad L = \hat{U} + E, \quad E' = E + fU, \quad \hat{U}' = (1 - f)U.$$

Use lowercase letters to denote the ratios of these variables to the labor force. Then,

$$u = \hat{u} + n, \quad 1 = \hat{u} + e, \quad (1 + n)e = e + fu, \quad (1 + n)\hat{u} = (1 - f)u.$$

In turn, this implies that

$$u = \left[\frac{1 + n}{n + f}\right] n, \quad e = \frac{(1 + n)f}{n + f} \quad \hat{u} = \left[\frac{1 - f}{n + f}\right] n.$$

Now consider the distribution of unemployment durations in every local market with matching efficiency m_i. Define

$$Q_i(\tau) = \prod_{s=0}^{\tau-1} \left[1 - m_i x\left(\mu\left(s\right)\right)\right].$$

At the beginning of period t, the mass of unemployed workers who have already searched for τ periods in a local market with matching efficiency m_i, denoted by $\hat{U}_{i,t}(\tau)$, is:

$$\hat{U}_{i,t}(\tau) = nL_{i,t-\tau}Q_i(\tau)$$

where $L_{i,t-\tau}$ is the proportion of workers, among all those who were born τ periods before t, who were allocated to a local market with matching efficiency m_i. Because newborns are allocated randomly to the local market, $L_{i,t-\tau} = L_{t-\tau}$ for all i, t, τ. Since all local markets with the same matching efficiency have the same distribution of unemployment durations, aggregating over the local markets yields the following mass of economy-wide unemployed workers at the beginning of period t whose unemployment duration is equal to τ:

$$\hat{U}_t(\tau) = p\hat{U}_{H,t}(\tau) + (1-p)\hat{U}_{L,t}(\tau).$$

Total unemployment in period t is

$$U_t = \sum_{\tau=1}^{\infty} \hat{U}_t(\tau) + nL_t,$$

and the aggregate unemployment rate is

$$u_t = \frac{U_t}{L_t} = \sum_{\tau=1}^{\infty} \frac{\hat{U}_t(\tau)}{L_t} + n.$$

Hence, the economy-wide job finding rate solves

$$\sum_{\tau=1}^{\infty} \frac{\hat{U}_t(\tau)}{L_t} + n = \left[\frac{1+n}{n+f}\right] n.$$

Note that individuals know $(U, L, E', \hat{U}', f)$ and hence they know $(u, n, e, \hat{u})$. They also know $\hat{U}_t(\tau)$ and $[pQ_H(\tau) + (1-p)Q_L(\tau)]$, for each τ. However, they do not know $Q_i(\tau)$ or $\hat{U}_{i,t}(\tau)$ for any τ. Since the functions W and λ are identical across local markets, individuals cannot infer the distributions of unemployment durations and wages in their own local markets and thus, they cannot infer the matching efficiency in their local markets.

The implications of the model for duration dependence are, in principle, ambiguous. On the one hand, the matching probability of each unemployed worker rises with the duration of his unemployment spell, for a given matching efficiency. Note, however, that the workers' permanent incomes, as described by the value of search V fall with the duration of unemployment, even though each unemployed worker's history exhibits positive duration dependence. On the other hand, the ratio of unemployed workers in m_L-type local markets to total unemployment increases with τ. To see this, compute the ratio as follows:

$$\frac{\hat{U}_{L,t}(\tau)}{\hat{U}_t(\tau)} = \frac{nL_{t-\tau}Q_L(\tau)}{pnL_{t-\tau}Q_H(\tau) + (1-p)nL_{t-\tau}Q_L(\tau)} = \frac{1}{1-p+p\frac{Q_H(\tau)}{Q_L(\tau)}}.$$

This ratio is increasing in τ, since the ratio $Q_H(\tau)/Q_L(\tau)$ falls with τ:

$$\frac{Q_H(\tau)}{Q_L(\tau)} = \left[\frac{Q_H(\tau-1)}{Q_L(\tau-1)}\right]\left[\frac{1-m_H x\left(\mu\left(\tau-1\right)\right)}{1-m_L x\left(\mu\left(\tau-1\right)\right)}\right] < \frac{Q_H(\tau-1)}{Q_L(\tau-1)}.$$

Accordingly, a cross-section of all workers at any point in time may well be such that, on average, workers who have been unemployed longer have lower probabilities of finding a job in the current period. Again, this is so even though each unemployed worker's history exhibits positive duration dependence. A sufficient condition for the cross sectional distribution of unemployed workers to exhibit negative duration dependence is $m_L g(m_L) \leq m_H g(m_H)$. In turn, using the first order conditions for $g(m_L)$ and $g(m_H)$, it can be verified that $\hat{R}_1(g(m_L), m_L) = \hat{R}_1(g(m_H), m_H) = 0$ implies that

$$m_H g(m_H) = \left[\frac{\lambda'\left(g(m_L)\right) - \lambda'\left(g(m_H)\right)}{\lambda'\left(g(m_H)\right) - \frac{\lambda(g(m_H))}{g(m_H)}}\right] r + \left[\frac{\lambda'\left(g(m_L)\right) - \frac{\lambda(g(m_L))}{g(m_L)}}{\lambda'\left(g(m_H)\right) - \frac{\lambda(g(m_H))}{g(m_H)}}\right] m_L g(m_L).$$

Since the first term is positive, a sufficient condition for $m_H g(m_H) \geq m_L g(m_L)$ is:

$$\lambda'\left(g(m_L)\right) - \frac{\lambda\left(g(m_L)\right)}{g(m_L)} \geq \lambda'\left(g(m_H)\right) - \frac{\lambda\left(g(m_H)\right)}{g(m_H)},$$

which is satisfied if $[\lambda'(x) - \lambda(x)/x]$ is an increasing function of x.

To see that the above condition can be satisfied, consider first the CES matching function in Example 2.1. Then, the function $[\lambda'(x)-\lambda(x)/x]$ increases in x iff $(1-\alpha)x^{-\rho} > \rho$. A sufficient condition is $\rho \leq 0$. The condition is also satisfied for some positive values of ρ that are close to 0. However, the condition is violated when ρ is sufficiently close to 1. Next, consider the urn-ball matching function in Example 2.1. Then, the function $[\lambda'(x) - \lambda(x)/x]$ is always an increasing function.

7. Conclusions

In this paper we have proposed an equilibrium theory of declining reservation wages, in which unemployed workers are faced with uncertainty that is associated directly with labor market frictions. As a concrete way to formalize this type of uncertainty, we have assumed that individuals do not know precisely the matching efficiency given by the exogenous matching technology. Faced with this uncertainty, workers learn about the probability of finding a job through their private search histories. We examine this possibility in a labor market equilibrium in which search is directed, workers know which wages are being offered, but they do not know the relevant wage distribution — that in their own local markets. However, by severing the direct dependence of search behavior on the wage distribution, the directed search framework simplifies the task of determining the equilibrium wage distribution and thus, the task of addressing jointly the workers' search behavior, the incentives to create jobs and the wage distribution.

In this context, we have shown how declining reservation wages over unemployment spells arise as workers update their beliefs about the matching efficiency downwards with the duration of unemployment. This formalizes a notion akin to that of discouragement, as workers become more pessimistic about the probability of finding a job over their spell of unemployment. Consequently, the wage distribution is non-degenerate, despite the facts that matches are homogeneous and search is directed. Moreover, aggregate matching probability decreases with unemployment duration, in contrast to individual workers' matching probability, which increases over individual unemployment spells.

The difficulty in establishing these results is that learning generates non-differentiable value functions and multiple solutions to a worker's optimization problem. We have overcome this difficulty by exploring a connection between convexity of a worker's value function and the property of supermodularity. A contribution of this paper is to establish such a connection, which is likely to be usefully exploited in many other learning problems. Our analysis differs from previous applications of supermodularity, which often emphasize the presence of complementarity in consumption or production. It is also different from other applications of supermodularity to dynamic problems, where the current payoff function is supermodular and the corresponding value is concave. In the present context neither the worker's current payoff nor his objective function is supermodular and furthermore the value function is convex, rather than concave.

Extensions of our model may consider workers' labor force participation, job destruction and on-the-job-search. These theoretical extensions do not change the nature of our analysis, but they may provide a useful structural framework for empirical studies of the wage distribution and the distribution of unemployment durations.

Appendix

A. Proof of Lemma 3.1

Given the analysis leading to Lemma 3.1, it suffices to show that the condition stated in the lemma is indeed sufficient for Assumption 2. To see this, note that a sufficient condition for $V(\mu) = \max_{x\in X} R(x,\mu) \geq \max_{x\in X} R^e(x,\mu)$ for all $\mu \in M$, where $M = [m_L, m_H]$ and $X = [0, 1/m_H]$, is that

$$\frac{W\left(1/m_H\right)}{r} \geq \frac{y/r}{1+r}.$$

Using the definition of W, this condition can be written as

$$y \geq (1+r)\left(\frac{m_H}{\mu_0} c\lambda\left(\frac{1}{m_H}\right)\right).$$

A sufficient condition for this inequality to hold for all $\mu_0 \in M$ is that it holds when $\mu_0 = m_L$, which gives the condition stated in the lemma. **QED**

B. Proof of Lemma 3.2

Let $TV(\mu)$ denote the right-hand side of (2.6). The value function, V, is a fixed point of the mapping T. Let $C_1(M)$ be the set containing all bounded, continuous and increasing functions on M. Let $C_1^s(M)$ be the subset of $C_1(M)$ which contains strictly increasing functions. Similarly, let $C_2(M)$ be the subset of $C_1(M)$ which contains convex functions, and $C_2^s(M)$ be the subset of $C_2(M)$ which contains strictly convex functions. We need to show that $V \in C_1^s(M) \cap C_2^s(M)$.

To show that $V \in C_1^s(M)$, it suffices to show that $T : C_1(M) \to C_1^s(M)$, which will be accomplished by Lemma B.1 below. By the argument of contraction mapping, the fixed point of T is strictly increasing. Similarly, to prove $V \in C_2^s(M)$, it suffices to show that $T : C_2(M) \to C_2^s(M)$, which will be accomplished by the last two lemmas in this proof. Because a convex function is almost everywhere twice differentiable with almost everywhere continuous first derivative (see Lemma 3.2 in Rader, 1973), then V has these properties.

Let $G(\mu) = \arg\max_{x\in X} R(x,\mu)$ and $G^e(\mu) = \arg\max_{x\in X} R^e(x,\mu)$, where R is defined by (2.4) and R^e by (2.5). Let $g(\mu) \in G(\mu)$ and $g^e(\mu) \in G^e(\mu)$. We next establish the monotonicity of V.

Lemma B.1. $T : C_1(M) \to C_1^s(M)$.

Proof. We show first that $T : C_1(M) \to C_1(M)$, which implies $V \in C_1(M)$ by the contraction mapping theorem. Then we show that $V = TV \in C_1^s(M)$. To establish these results, pick an arbitrary $V_0 \in C_1(M)$ and replace V with V_0 in the definitions of R and R^e. Pick any $\mu_a, \mu_b \in M$ with $\mu_a > \mu_b$. Denote $g_i = g(\mu_i)$ and $g_i^e = g^e(\mu_i)$, where $i \in \{a, b\}$.

We begin by showing that $T : C_1(M) \to C_1(M)$. Note that

$$\begin{array}{ll} R^e(g_b^e, \mu_a) & \geq \frac{g_b^e \mu_a}{1+r} V_0(m_H + m_L - \frac{m_H m_L}{\mu_a}) + \frac{1-g_b^e \mu_a}{1+r} V_0(H(g_b^e, \mu_b)) \\ R^e(g_b^e, \mu_b) & \leq \frac{g_b^e \mu_b}{1+r} V_0(m_H + m_L - \frac{m_H m_L}{\mu_a}) + \frac{1-g_b^e \mu_b}{1+r} V_0(H(g_b^e, \mu_b)). \end{array}$$

These results and monotonicity of V_0 imply $R^e(g_b^e, \mu_a) \geq R^e(g_b^e, \mu_b)$. Consider the case where $R(g_b, \mu_b) = R^e(g_b^e, \mu_b)$. Then monotonicity of TV_0 is established as follows:

$$TV_0(\mu_a) \geq R^e(g_a^e, \mu_a) \geq R^e(g_b^e, \mu_a) \geq R^e(g_b^e, \mu_b) = TV_0(\mu_b).$$

The first inequality comes from Assumption 2, the second inequality from the fact that $g_a^e = \arg\max_x R^e(x, \mu_a)$, the third inequality from the result established above, and the last equality from the hypothesis in the current case.

Consider the case where $R(g_b, \mu_b) > R^e(g_b^e, \mu_b)$, the only remaining case that is consistent with Assumption 2. Then, we have:

$$\begin{aligned} 0 & < R(g_b, \mu_b) - R^e(g_b^e, \mu_b) \\ & \leq R(g_b, \mu_b) - R^e(g_b, \mu_b) = \mu_b g_b \left[\frac{W(g_b)}{r} - \frac{V_0\left(m_H + m_L - \frac{m_H m_L}{\mu_b}\right)}{1+r} \right]. \end{aligned} \tag{B.1}$$

The first inequality is the hypothesis in the current case; the second inequality comes from the fact that $g_b^e \in \arg\max_x R^e(x, \mu_b)$; and the ensuing equality comes from the definitions of $R(x, \mu)$ and $R^e(x^e, \mu)$ evaluated at (g_b, μ_b). Note that for any μ such that $R(g(\mu), \mu) > R^e(g^e(\mu), \mu)$, we have $g(\mu) > 0$: if $g(\mu) = 0$, instead, then $R(g(\mu), \mu) = H(\mu)/(1 + r) = R^e(0, \mu) \leq R^e(g^e(\mu), \mu)$, which is a contradiction. Thus, $g_b > 0$ in the current case. Therefore, (B.1) yields:

$$\frac{W(g_b)}{r} > \frac{V_0\left(m_H + m_L - \frac{m_H m_L}{\mu_b}\right)}{1+r}. \tag{B.2}$$

Now, the procedure below establishes the desired result $TV_0(\mu_a) > TV_0(\mu_b)$:

$$\begin{array}{rcl} R(g_a, \mu_a) - R(g_b, \mu_b) & \geq & R(g_b, \mu_a) - R(g_b, \mu_b) \\ & \geq & g_b(\mu_a - \mu_b) \left[\frac{W(g_b)}{r} - \frac{1}{1+r} V_0(H(g_b, \mu_b))\right] \\ & > & \frac{g_b(\mu_a - \mu_b)}{1+r} \left[V_0\left(m_H + m_L - \frac{m_H m_L}{\mu_b}\right) - V_0(H(g_b, \mu_b))\right] \\ & \geq & 0. \end{array} \tag{B.3}$$

The first inequality comes from the fact that $g_a \in \arg\max_x R(x, \mu_a)$, the second inequality from the fact $V_0(H(g_b, \mu_a)) \geq V_0(H(g_b, \mu_b))$, the third inequality from the above result on $W(g_b)$, and the last inequality from $V_0 \in C_1(M)$. This completes the proof of the statement that $T : C_1(M) \to C_1(M)$.

Now we prove that $V \in C_1^s(M)$, where V is the fixed point of T. We need to show that $TV(\mu_a) > TV(\mu_b)$. If $R(g_b, \mu_b) > R^e(g_b^e, \mu_b)$, replacing V_0 with V in the above proof establishes $TV(\mu_a) > TV(\mu_b)$. The only other case that is consistent with Assumption 2 is $R(g_b, \mu_b) = R^e(g_b^e, \mu_b)$. Because $V(m_H + m_L - \frac{m_H m_L}{\mu_b}) \geq V(\mu_b)$, we divide the proof in this case further into the following two cases:

Case 1. $V(m_H + m_L - \frac{m_H m_L}{\mu_b}) = V(\mu_b)$. In this case, we have:

$$R^e(g_b^e, \mu_b) = \frac{g_b^e \mu_b}{1+r} V(\mu_b) + \frac{1 - g_b^e \mu_b}{1+r} V(H(g_b^e, \mu_b)) \le \frac{1}{1+r} V(\mu_b).$$

The equality follows from the hypothesis in the current case and the inequality from the fact that $V(H(x, \mu_b)) \le V(\mu_b)$. Since $V(\mu_b) = R(g_b, \mu_b) = R^e(g_b^e, \mu_b)$, the above inequality would imply $V(\mu_b) \le 0$. This would not be optimal for the worker because there exists $x > 0$ such that $R(x, \mu_b) > 0$. Therefore, it must be true that $V(\mu_b) = R(g_b, \mu_b) > R^e(g_b^e, \mu_b)$, in which case $V(\mu_a) > V(\mu_b)$, as shown above.

Case 2. $V(m_H + m_L - \frac{m_H m_L}{\mu_b}) > V(\mu_b)$. Because $V(\mu_b) \ge V(H(g_b, \mu_b))$ and $g_b > 0$ (otherwise, $V(\mu_b) \le 0$, which contradicts the fact that $V(\mu_b)$ is a maximized value), the current hypothesis implies that the last inequality in B.3 is strict, where V_0 is replaced with V. Hence, $TV(\mu_a) > TV(\mu_b)$.

This completes the proof of Lemma B.1. **QED**

Lemma B.2. *If $V \in C_2(M)$, then $R(x, \mu)$ defined by (2.4) is convex in μ for any given x. If $V \in C_2^s(M)$, then $R(x, \mu)$ is strictly convex in μ.*

Proof. We prove the second part of the lemma first. Let V be a strictly convex function. Let μ_a and μ_b be two arbitrarily values in M, with $\mu_a > \mu_b$. Let $\theta \in (0, 1)$ be a number. Denote $\mu_\theta = \theta\mu_a + (1-\theta)\mu_b$. We show that

$$R(x, \mu_\theta) < \theta R(x, \mu_a) + (1-\theta) R(x, \mu_b).$$

Denote $H_i = H(x, \mu_i)$, where $i \in \{a, b, \theta\}$. Since $\partial H / \partial \mu > 0$, then $H_a > H_\theta > H_b$. Let

$$\sigma = \frac{H_\theta - H_b}{H_a - H_b}.$$

Note that $\sigma \in (0, 1)$ and $\sigma H_a + (1-\sigma) H_b = H_\theta$. If V is strictly convex, then

$$V(H_\theta) < \sigma V(H_a) + (1-\sigma) V(H_b). \tag{B.4}$$

By the definition of R in (2.4), we have:

$$\begin{aligned} R(x, \mu_\theta) &< \mu_\theta x \tfrac{W(x)}{r} + \tfrac{1-\mu_\theta x}{1+r} \left[\sigma V(H_a) + (1-\sigma) V(H_b)\right] \\ &= \theta R(x, \mu_a) + (1-\theta) R(x, \mu_b) + \tfrac{V(H_a)}{1+r} \Delta_a + \tfrac{V(H_b)}{1+r} \Delta_b, \end{aligned}$$

where

$$\Delta_a = (1-\mu_\theta x)\sigma - \theta(1-\mu_a x),$$
$$\Delta_b = (1-\mu_\theta x)(1-\sigma) - (1-\theta)(1-\mu_b x).$$

For $i, j \in \{a, b, \theta\}$, we use (2.3) to compute:

$$\sigma = \frac{(\mu_\theta - \mu_b)(1-\mu_a x)}{(\mu_a - \mu_b)(1-\mu_\theta x)} = \frac{\theta(1-\mu_a x)}{(1-\mu_\theta x)}.$$

Now it is easy to see that $\Delta_a = 0 = \Delta_b$. Therefore, R is strictly convex.

If V is convex rather than strictly convex, then (B.4) holds as " $\le$ " instead of " $<$ ". The rest of the proof can be adapted easily to show that $R(x, \mu)$ is convex, rather than strictly convex. **QED**

Lemma B.3. $T : C_2(M) \to C_2^s(M)$.

Proof. Pick any $V_0 \in C_2(M)$. Denote $V_1(\mu) = TV_0(\mu)$. Let μ_a and μ_b be two arbitrarily values in M, with $\mu_a > \mu_b$. Let $\theta \in (0, 1)$ be a number. Denote $\mu_\theta = \theta\mu_a + (1 - \theta)\mu_b$. We need to show that

$$V_1(\mu_\theta) < \theta V_1(\mu_a) + (1 - \theta)V_1(\mu_b).$$

We divide the proof in two cases: the case where V_0 is strictly convex and the case where V_0 has linear segments.

Case 1: $V_0 \in C_2^s(M)$. In this case, the previous lemma implies that $R(x, \mu)$ is strictly convex in μ for any given x. Denote $x_i^* = max_x R(x, \mu_i)$, $i \in \{a, b, \theta\}$. That is, $V_1(\mu_i) = R(x_i^*, \mu_i)$, with V in (2.4) being replaced with V_0. Strict convexity of V is established below:

$$\begin{aligned} V_1(\mu_\theta) &= R(x_\theta^*, \mu_\theta) \\ &< \theta R(x_\theta^*, \mu_a) + (1 - \theta)R(x_\theta^*, \mu_b) \\ &\leq \theta R(x_a^*, \mu_a) + (1 - \theta)R(x_b^*, \mu_b) \\ &= \theta V_1(\mu_a) + (1 - \theta)V_1(\mu_b). \end{aligned} \tag{B.5}$$

The first inequality comes from the fact that R is strictly convex in μ and the second inequality from the fact that $R(x, \mu_i) \leq R(x_i^*, \mu_i)$ for all x.

Case 2: V_0 is convex and has some linear segments. If any two of the elements, $V_0(H(x_\theta^*, \mu_\theta))$, $V_0(H(x_\theta^*, \mu_a))$ and $V_0(H(x_\theta^*, \mu_b))$, do not lie on the same linear segment of V_0, then the first inequality in (B.5) is still strict and V_1 is strictly convex. Suppose that all three elements lie on the same linear segment of V_0. Temporarily denote this linear segment as $V_0(H) = A + BH$, with $B > 0$ (because V is strictly increasing). Using (2.3), we can compute:

$$(1 - \mu x)V_0(H) = (1 - \mu x)(A + Bm_H) - B(1 - m_L x)(m_H - \mu).$$

This is linear and differentiable in (μ, x). Restrict μ to be such that $V_0(H(x, \mu))$ lies on the linear segment described above. Using (2.8), we can verify that $R(x, \mu)$ is strictly concave in x. Thus, the solution x^* is unique and satisfies the following first-order condition:

$$0 = R_1(x, \mu) = \mu \left[\frac{W + \lambda W'}{r} - A - Bm_H \right] + Bm_L(m_H - \mu).$$

Differentiating this first-order condition, we find that the solution, $x^* = g(\mu)$, satisfies:

$$g'(\mu) = -\frac{m_H}{m_H - \mu} < 0.$$

Thus, $x_a^* \neq x_\theta^*$ and $x_b^* \neq x_\theta^*$. Because the solutions are unique, then $R(x_\theta^*, \mu_b) < R(x_b^*, \mu_b)$ and $R(x_\theta^*, \mu_a) < R(x_a^*, \mu_a)$. The second inequality in (B.5) is strict, and so V_1 is strictly convex. This completes the proofs of the current lemma and Lemma 3.2. **QED**

C. Proof of Theorem 4.1

Take arbitrary z_a,$z_b \in -X$ and arbitrary μ_a,$\mu_b \in M$, with $z_a > z_b$ and $\mu_a > \mu_b$. Denote:

$$D = \left[\hat{R}(z_a, \mu_a) - \hat{R}(z_a, \mu_b)\right] - \left[\hat{R}(z_b, \mu_a) - \hat{R}(z_b, \mu_b)\right].$$

We need to show $D > 0$. Temporarily denote $H_{ij} = H(-z_i, \mu_j)$ and $V_{ij} = V(H_{ij})$, where $i, j \in \{a, b\}$. Computing D, we have:

$$\frac{(1+r)D}{z_a - z_b} = \left(\frac{1}{\mu_a} + z_b\right)\frac{V_{aa} - V_{ba}}{z_a - z_b} - \left(\frac{1}{\mu_b} + z_b\right)\frac{V_{ab} - V_{bb}}{z_a - z_b} + (V_{aa} - V_{ab}).$$

There are two cases to consider: $\mu_b = m_L$ and $\mu_b > m_L$. First, suppose that $\mu_b = m_L$. Then $V_{aa} - V_{ba} \geq 0$, with equality if and only if $\mu_a = m_H$; $V_{ab} - V_{bb} = 0$; and $V_{aa} - V_{ab} > 0$. Hence, $D > 0$ in this case. Next, consider the second case, where $\mu_b > m_L$. Here there are also two cases to consider: $\mu_a = m_H$ and $\mu_a < m_H$. We start with the second case. Suppose that $\mu_a < m_H$. Because $H_1(-z, \mu) < 0$ and $H_2(-z, \mu) > 0$, then $H_{aa} > \max\{H_{ab}, H_{ba}\} \geq \min\{H_{ab}, H_{ba}\} > H_{bb}$. Strict convexity of $V(H)$ implies (see Royden, 1988, p.113):

$$\frac{V_{aa} - V_{ba}}{H_{aa} - H_{ba}} > \frac{V_{ab} - V_{bb}}{H_{ab} - H_{bb}} \text{ and } \frac{V_{aa} - V_{ab}}{H_{aa} - H_{ab}} > \frac{V_{ab} - V_{bb}}{H_{ab} - H_{bb}}. \tag{C.1}$$

Thus, the following (strict) inequality holds:

$$\begin{array}{ll} & \frac{H_{ab}-H_{bb}}{V_{ab}-V_{bb}}\left(\frac{(1+r)D}{z_a-z_b}\right) \\ > & \left(\frac{1}{\mu_a} + z_b\right)\frac{H_{aa}-H_{ba}}{z_a-z_b} - \left(\frac{1}{\mu_b} + z_b\right)\frac{H_{ab}-H_{bb}}{z_a-z_b} + (H_{aa} - H_{ab}). \end{array}$$

Substituting H_{ij}, we get:

$$\begin{array}{ll} & \frac{H_{ab}-H_{bb}}{V_{ab}-V_{bb}}\left(\frac{(1+r)D}{z_a-z_b}\right) \\ > & \left(\frac{1}{\mu_a} + z_b\right)\frac{m_H-\mu_a}{z_a-z_b}\left(\frac{1+z_b m_L}{1+z_b\mu_a} - \frac{1+z_a m_L}{1+z_a\mu_a}\right) \\ & -\left(\frac{1}{\mu_b} + z_b\right)\frac{m_H-\mu_b}{z_a-z_b}\left(\frac{1+z_b m_L}{1+z_b\mu_b} - \frac{1+z_a m_L}{1+z_a\mu_b}\right) + (1 + z_a m_L)\left(\frac{m_H-\mu_b}{1+z_a\mu_b} - \frac{m_H-\mu_a}{1+z_a\mu_a}\right) \\ = & \frac{(m_H-\mu_a)(\mu_a-m_L)}{\mu_a(1+z_a\mu_a)} - \frac{(m_H-\mu_b)(\mu_b-m_L)}{\mu_b(1+z_a\mu_b)} + (1 + z_a m_L)\left(\frac{m_H-\mu_b}{1+z_a\mu_b} - \frac{m_H-\mu_a}{1+z_a\mu_a}\right) \\ = & m_H m_L \frac{\mu_a-\mu_b}{\mu_a\mu_b} > 0. \end{array}$$

The second equality comes from collecting terms according to $(m_H - \mu_i)$. Hence, $D > 0$ in this case as well. Finally, if $\mu_b > m_L$ and $\mu_a = m_H$, the last string of inequalities becomes

$$\begin{array}{ll} & \frac{H_{ab}-H_{bb}}{V_{ab}-V_{bb}}\left(\frac{(1+r)D}{z_a-z_b}\right) \\ > & -\left(\frac{1}{\mu_b} + z_b\right)\frac{m_H-\mu_b}{z_a-z_b}\left(\frac{1+z_b m_L}{1+z_b\mu_b} - \frac{1+z_a m_L}{1+z_a\mu_b}\right) + (1 + z_a m_L)\left(\frac{m_H-\mu_b}{1+z_a\mu_b}\right) \\ = & -\frac{(m_H-\mu_b)(\mu_b-m_L)}{\mu_b(1+z_a\mu_b)} + (1 + z_a m_L)\left(\frac{m_H-\mu_b}{1+z_a\mu_b}\right) \\ = & \frac{m_L(m_H-\mu_b)}{\mu_b} > 0. \end{array}$$

Thus, the function $\hat{R}(z, \mu)$ is strictly supermodular. Because $-X$ is a lattice, the monotone selection theorem in Topkis (1998, Theorem 2.8.4, p.79) implies that every selection from $Z(\mu)$ is increasing. As a result, every selection $g(\mu)$ from $G(\mu)$ is decreasing, and $w(\mu) = W(g(\mu))$ is increasing. **QED**

D. Proof of Lemma 5.1

First, we show that optimal choices are interior under Assumption 3. Consider the corner, $z = 0$. For any prior beliefs, μ, the choice $z = 0$ yields zero expected wage in the period and the posterior beliefs $H(0, \mu) = \mu$. The value of this choice is $R(0, \mu) = 0$, which can be increased by any choice $z < 0$. Thus, the choice $z = 0$ is never optimal.

Now consider the other corner, $z = -1/m_H$. Since optimal choices are such that $z = -g(\mu)$, and $g(\mu)$ is decreasing with μ, a sufficient condition for $z > -1/m_H$ and, equivalently, for $g(\mu) < 1/m_H$, is that $g(m_L) < 1/m_H$. Note that $g(m_L)$ solves:

$$V(m_L) = \max_{z \in -X} \left[-m_L z \frac{W(-z)}{r} + (1 + m_L z) \frac{V(m_L)}{1+r} \right].$$

Substituting the optimal choice and rearranging terms, we have:

$$\frac{V(m_L)}{1+r} = \left(\frac{g(m_L) m_L}{r + g(m_L) m_L} \right) \frac{W(g(m_L))}{r},$$

where the wage function W is given by (2.7). To ensure that $g(m_L) < 1/m_H$, it is sufficient that the objective function $R(-z, m_L)$ has a strictly positive derivative with respect to z at $z = -1/m_H$. After computing the derivative and substituting $V(m_L)$ from above and W from (2.7), we can write this condition as

$$\mu_0 < \frac{c}{y} \left[\left(r + \frac{m_L}{m_H} \right) \lambda' \left(\frac{1}{m_H} \right) - m_L \lambda \left(\frac{1}{m_H} \right) \right],$$

where μ_0 are the firms' initial beliefs, which enter through W. This is the condition (5.1) stated in Assumption 3.

Next, we show that $V'(H(-z(\mu), \mu))$ exists. For any real number r, define $r^- = \lim_{\varepsilon \downarrow 0}(r - \varepsilon)$ and $r^+ = \lim_{\varepsilon \downarrow 0}(r + \varepsilon)$. Fix $\mu \in (m_L, m_H)$. Under Assumption 3, the optimal choice $z(\mu)$ is interior. Such a solution satisfies $\hat{R}_1(z^-(\mu), \mu) \geq \hat{R}_1(z^+(\mu), \mu)$. Note that a continuous, convex function has left and right derivatives. Because $W(-z)$ is continuous, V is continuous and convex, and H is continuously differentiable, then

$$\begin{aligned} \hat{R}_1(z^+(\mu), \mu) = \; & \tfrac{z(\mu) W'(-z(\mu)) - W(-z(\mu))}{r} + \tfrac{V(H(-z(\mu), \mu))}{1+r} \\ & - \left(\tfrac{1}{\mu} + z(\mu) \right) \tfrac{V'(H^+(-z(\mu), \mu))}{1+r} H_1(-z(\mu), \mu), \\ \hat{R}_1(z^-(\mu), \mu) = \; & \tfrac{z(\mu) W'(-z(\mu)) - W(-z(\mu))}{r} + \tfrac{V(H(-z(\mu), \mu))}{1+r} \\ & - \left(\tfrac{1}{\mu} + z(\mu) \right) \tfrac{V'(H^-(-z(\mu), \mu))}{1+r} H_1(-z(\mu), \mu). \end{aligned}$$

Here we have used the fact that $H_1(-z, \mu) < 0$ — recall that H_1 denotes the derivative of H with respect to the first argument, rather than z. Since $H_1 < 0$, then the feature $\hat{R}_1(z^-(\mu), \mu) \geq \hat{R}_1(z^+(\mu), \mu)$ implies

$$V'(H^-(-z(\mu), \mu)) \geq V'(H^+(-z(\mu), \mu)).$$

Because V is convex, the reversed inequality holds. Thus,

$$V'(H^-(-z(\mu), \mu)) = V'(H^+(-z(\mu), \mu)) = V'(H(-z(\mu), \mu)).$$

In turn, this implies that optimal choices in every period satisfy the first-order conditions. **QED**

E. Proof of Theorem 5.2

It suffices to show that the case $z(\mu_a) \neq z(\mu_b)$ cannot occur for any pair (μ_a, μ_b) with $\mu_a > \mu_b$. Suppose to the contrary that $z(\mu_a) = z(\mu_b)$. Denote this common value as z^*. By Lemma 5.1, $z(\mu_a)$ and $z(\mu_b)$ are interior and satisfy first-order conditions. That is,

$$\hat{R}_1(z^*, \mu_a) = 0 = \hat{R}_1(z^*, \mu_b).$$

Shorten the notation $H(-z^*, \mu_i)$ to H_i, where $i \in \{a, b\}$. Substituting $\hat{R}_1(z^*, \mu_i)$, we have:

$$\begin{aligned} & (1+r)\left[\hat{R}_1(z^*, \mu_a) - \hat{R}_1(z^*, \mu_b)\right] \\ = \quad & V(H_a) - V(H_b) - \left(\tfrac{1}{\mu_a} + z^*\right) V'(H_a) H_1(-z^*, \mu_a) + \left(\tfrac{1}{\mu_b} + z^*\right) V'(H_b) H_1(-z^*, \mu_b). \end{aligned}$$

Because $V(H)$ is continuous and strictly convex, and because $V'(H_b)$ exists by Lemma 5.1, we have: $V(H_a) - V(H_b) > V'(H_b)(H_a - H_b)$. Then,

$$\begin{aligned} & (1+r)\left[\tfrac{\hat{R}_1(z^*, \mu_a) - \hat{R}_1(z^*, \mu_b)}{V'(H_b)}\right] \\ \geq \quad & (H_a - H_b) - \left(\tfrac{1}{\mu_a} + z^*\right) \tfrac{V'(H_a)}{V'(H_b)} H_1(-z^*, \mu_a) + \left(\tfrac{1}{\mu_b} + z^*\right) H_1(-z^*, \mu_b) \\ > \quad & (H_a - H_b) - \left(\tfrac{1}{\mu_a} + z^*\right) H_1(-z^*, \mu_a) + \left(\tfrac{1}{\mu_b} + z^*\right) H_1(-z^*, \mu_b). \end{aligned}$$

The first inequality comes from substituting the inequality between the V's and the fact that $V' > 0$; the second (strict) inequality comes from the facts that V is strictly convex, $H_a > H_b$, and $H_1 < 0$. Denote the last expression temporarily as $f(\mu_a)$. Because f is differentiable, we can compute:

$$\begin{aligned} f'(\mu_a) &= H_1(-z^*, \mu_a) + \tfrac{1}{(\mu_a)^2} H_1(-z^*, \mu_a) - \left(\tfrac{1}{\mu_a} + z^*\right) H_{12}(-z^*, \mu_a) \\ &= \tfrac{m_H m_L}{(\mu_a)^2} > 0. \end{aligned}$$

The second equality comes from property (iv) in Lemma 2.2. Because $\mu_a > \mu_b$, then $f(\mu_a) > f(\mu_b) = 0$. That is, $\hat{R}_1(z^*, \mu_a) > \hat{R}_1(z^*, \mu_b)$. This result contradicts the supposition that $z(\mu_a) = z(\mu_b)$. **QED**

F. Proofs of Lemma 5.3 and Theorem 5.4

First, we prove the following lemma (which does require optimal choices to be interior):

Lemma F.1. *$\bar{z}(\mu)$ is right-continuous and $\underline{z}(\mu)$ is left-continuous at each $\mu \in M$.*

Proof. Pick an arbitrary $\mu \in M$. Let $\{\mu_n\}$ be a sequence with $\mu_n \to \mu$ and $\mu_n \geq \mu_{n+1} \geq \mu$ for all n. Because $\bar{z}(\mu)$ is an increasing function, then $\{\bar{z}(\mu_n)\}$ is a decreasing sequence and $\bar{z}(\mu_n) \geq \bar{z}(\mu)$ for all n. Thus, $\bar{z}(\mu_n) \downarrow A$ for some $A \geq \bar{z}(\mu)$. On the other hand, the Theorem of the Maximum (see Stokey and Lucas, 1989) implies that the correspondence $Z(\mu)$ is upper hemicontinuous (uhc). Because $\mu_n \to \mu$, and $\bar{z}(\mu_n) \in Z(\mu_n)$ for each n, uhc of Z implies that there is a subsequence of $\{\bar{z}(\mu_n)\}$ that converges to an element in $Z(\mu)$. This element must be A, because all convergent subsequences of a convergent sequence must have the same limit. Thus, $A \in Z(\mu)$, and so $A \leq \max Z(\mu) = \bar{z}(\mu)$. Therefore, $\bar{z}(\mu_n) \downarrow A = \bar{z}(\mu)$, which shows that $\bar{z}(\mu)$ is right-continuous.

Similarly, by examining the sequence $\{\mu_n\}$ with $\mu_n \to \mu$ and $\mu \geq \mu_{n+1} \geq \mu_n$ for all n, we can show that $\underline{z}$ is left-continuous. This completes the proof of Lemma F.1.

Next, we prove Lemma 5.3. Fix $\mu_a \in (m_L, m_H)$. Because $\bar{z}(\mu)$ maximizes $R(-z, \mu)$ for each given μ, then

$$\begin{aligned} V(\mu) &= R(-\bar{z}(\mu), \mu) \geq R(-\bar{z}(\mu_a), \mu) \\ V(\mu_a) &= R(-\bar{z}(\mu_a), \mu_a) \geq R(-\bar{z}(\mu), \mu_a). \end{aligned}$$

Taking $\mu > \mu_a$, where $\mu_a < m_H$, and dividing the above inequalities by $(\mu - \mu_a)$, we obtain:

$$\frac{R(-\bar{z}(\mu_a), \mu) - R(-\bar{z}(\mu_a), \mu_a)}{\mu - \mu_a} \leq \frac{V(\mu) - V(\mu_a)}{\mu - \mu_a} \leq \frac{R(-\bar{z}(\mu), \mu) - R(-\bar{z}(\mu), \mu_a)}{\mu - \mu_a}.$$

Take the limit $\mu \downarrow \mu_a$. Under Assumption 3, $V'(H(-\bar{z}(\mu_a), \mu_a))$ exists for each μ (see Lemma 5.1). Because $\bar{z}(\mu)$ is right-continuous, then $R_1(-\bar{z}(\mu_a), \mu_a)$ exists. The limits of the first and last ratios are both $R_1(-\bar{z}(\mu_a), \mu_a)$. Thus, $V'(\mu_a^+) = R_1(-\bar{z}(\mu_a), \mu_a)$.

Now conduct the above exercise with $\underline{z}$ replacing $\bar{z}$. For $\mu < \mu_a$ and $\mu_a > m_L$, we have:

$$\frac{R(-\underline{z}(\mu_a), \mu) - R(-\underline{z}(\mu_a), \mu_a)}{\mu - \mu_a} \geq \frac{V(\mu) - V(\mu_a)}{\mu - \mu_a} \geq \frac{R(-\underline{z}(\mu), \mu) - R(-\underline{z}(\mu), \mu_a)}{\mu - \mu_a}.$$

Take the limit $\mu \uparrow \mu_a$. Because $\underline{z}(\mu)$ is left-continuous and interior, then $V'(\mu_a^-) = R_1(-\underline{z}(\mu_a), \mu_a)$.

To establish the inequality between the left- and right- derivatives of V, use the definition $R(-z, \mu) = \mu \hat{U}(z, \mu)$ to compute:

$$R_1(-z(\mu), \mu) = \hat{R}(z(\mu), \mu) + \mu \hat{U}_2(z(\mu), \mu) = V(\mu)/\mu + \mu \hat{U}_2(z(\mu), \mu).$$

Because $\hat{R}(z, \mu)$ is strictly supermodular, $\hat{R}_2(\bar{z}(\mu_a), \mu_a) \geq \hat{R}_2(\underline{z}(\mu_a), \mu_a)$, where the inequality is strict if and only if $\bar{z}(\mu_a) > \underline{z}(\mu_a)$. Therefore, $V'(\mu_a^+) \geq V'(\mu_a^-)$, where the inequality is strict if and only if $\bar{z}(\mu_a) > \underline{z}(\mu_a)$. This completes the proof of Lemma 5.3.

Finally, we prove Theorem 5.4. Given any selection $z(\mu_0) \in Z(\mu_0)$ and the induced beliefs $\mu_1 = H(-z(\mu_0), \mu_0)$, Lemma 5.1 implies that $V'(\mu_1)$ exists. Then, Lemma 5.3 implies $\bar{z}(\mu_1) = \underline{z}(\mu_1)$. That is, $Z(\mu_1) = \{z(\mu_1)\}$ is a singleton. So, the posterior beliefs induced by $Z(\mu_1)$ are unique and are given by $\mu_2 = H(-z(\mu_1), \mu_1)$. Again, Lemma 5.1 implies that $V'(\mu_2)$ exists and Lemma 5.3 implies that $\bar{z}(\mu_2) = \underline{z}(\mu_2)$. Repeating this argument shows that μ_n is unique, $Z(\mu_n)$ is a singleton, and $V'(\mu_n)$ exists for all $n = 1, 2, ...$

If $\mu_0 \notin N$, then $\bar{z}(\mu_0) = \underline{z}(\mu_0)$ by the definition of N. In this case, the posterior beliefs $\mu_1 = H(z(\mu_0), \mu_0)$ are unique. Also, Lemma 5.3 implies that $V'(\mu_0)$ exists. If $\mu_0 \in N$, again, the results stated in Theorem 5.4 follow from Lemma 5.3. **QED**

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