Nominal Bonds and Interest Rates: The Case of One-Period Bonds*

Shouyong Shi
University of Toronto

2002

Abstract

This paper is the first step in the integration of the (search-theoretic) microfoundation of monetary theory into the fruitful analysis by Lucas (1990). I construct two search models, in which fiat money coexists in equilibrium with default-free nominal bonds issued by the government, and then use the models to analyze the effects of money growth and open market operations. In the first model, matured bonds circulate in the goods market as perfect substitutes for money and there are a continuum of stationary equilibria of this sort. In the second model, matured bonds do not circulate in the goods market and there is a unique stationary equilibrium. In both models, newly issued bonds are sold at a discount for money and thus they bear positive interest. The effects of monetary policies differ in these two economies.

Keywords: Search; Money; Bonds; Interest Rates.

* Correspondence: Shouyong Shi, Department of Economics, University of Toronto, 150 St. George Street, Toronto, Ontario, Canada, M5S 3G7 (email: shouyong@economics.utoronto.ca). I have benefited from conversations with and comments from V. Chari, Eric Leeper, Guillaume Rocheteau, Neil Wallace, Randall Wright and the participants of the conference on monetary theory jointly organized by the Swiss National Bank and the Federal Reserve Bank of Cleveland.

1. Introduction

The primary question that I try to address here is: Why do government-issued, nominal bonds not circulate in the goods market as a medium of exchange, while money does? A related question is: Why are such bonds sold for money at a discount, even if they bear no default risk? By answering these questions, I take the first step to integrate the (search-theoretic) microfoundation of monetary theory with the influential work of Lucas (1990).

The questions raised here are fundamental in monetary theory. As Wallace (2001) forcefully argued, monetary theory is supposed to answer what objects perform the role of media of exchange and why. The questions are also important for understanding how monetary policies, such as open market operations, affect the price level and interest rates. When examining monetary policy, existing models have made strong assumptions to preclude nominal bonds from circulating in the goods market, such as putting money into the utility function or imposing the constraint of cash in advance (e.g., Lucas (1990)). To see whether the policy effects obtained in those models are robust, it is necessary to relax the assumptions to allow nominal bonds to compete against money as a medium of exchange.

I use the search framework to analyze the coexistence of money and one-period nominal bonds. In this framework, money and nominal bonds are both fiat objects, in the sense that they do not yield direct utility or facilitate production. Rather, decentralized exchanges (modelled as bilateral matches) and the lack of public record-keeping induce fiat objects to be valuable media of exchange in equilibrium. Nominal bonds compete against money to serve as such media of exchange. The only exogenous difference between bonds and money assumed here is that the government accepts money, but not bonds, as the means of payments in all transactions it encounters. Private agents can choose to use money and bonds for exchanges among themselves.

Two search models are constructed. In the first model, the government sells and redeems nominal bonds, but does not participate in the goods market. In the second model, the government also trades in the goods market. In both models, the bonds market is centralized, in contrast to decentralized trading in the goods market.

The two models generate positive discounting on newly issued bonds, provided that the money growth rate exceeds the discount factor. This result holds regardless of whether such bonds, when matured, will circulate as perfect substitutes for money in the goods market. The positive nominal interest rate is an outcome of the temporary separation between the bonds market and the goods market, as in Lucas (1990). The temporary separation implies that newly issued bonds cannot be used in the goods market in the same period, and hence they are not perfect substitutes for money or matured bonds that are circulating in the goods market. Positive discounting on newly issued bonds is a necessary compensation for this one-period loss of liquidity.

Whether matured bonds circulate as a medium of exchange depends on the model. When the government does not participate in the goods market (i.e., in the first model), matured bonds circulate in the goods market as perfect substitutes for money. Agents are indifferent about how large a fraction of matured bonds to redeem. This indeterminacy makes the price level, the ratio of matured bonds to money, and the asset values all indeterminate. However, all these equilibria have real output/consumption and the same nominal interest rate.

When the government participates in the goods market and refuses to accept bonds as payments (i.e., in the second model), matured bonds do not circulate in the goods market. The government's restriction drives bonds out of the goods market because of the decentralized nature of exchanges. With decentralized exchanges, a buyer in a match cannot exchange bonds for money instantaneously, nor switch costlessly from a match with a government seller to a match with a private seller. Given the government's restriction, holding money gives a buyer a higher chance to trade than holding bonds. This wedge induces buyers to use only money to buy goods. The wedge need not be high – Even an arbitrarily small measure of government sellers is enough to prevent bonds from circulating in the goods market.

It is important to emphasize that the same legal restriction by the government does not drive bonds out of circulation if the goods market is Walrasian. Rather, the legal restriction simply shifts money from the market for private goods to the market for government goods.

The effects of monetary policies differ in the two models. For example, an increase in the money growth rate, through lump-sum monetary transfers, reduces real output in the first model, but it can increase output for low money growth rates in the second model. Also, an increased issuing of bonds increases the price level and has no effect on real output in the first model, but it reduces the price level and increases real output in the second model.

As stated earlier, this paper is an attempt to combine the (search-theoretic) microfoundation of monetary theory with the fruitful analysis by Lucas (1990). The models constructed here share some important features of Lucas's model, such as a positive discount on newly issued bonds. The key distinctions from Lucas's analysis are that the value of money is supported by the description of the trading environment and that nominal bonds are not precluded from circulation as a medium of exchange. For the microfoundation of money, this paper builds onto two previous papers (Shi (1997, 1999)). I introduce nominal bonds into the search theory, and hence eliminate a major limitation of the theory for policy analysis.[1]

Aiyagari et al. (1996) present the first attempt to analyze the coexistence of money and government bonds in a search model. They show that there exist two types of equilibria where money and interest-bearing bonds coexist. In one type, matured bonds and money are perfect

[1]The original Kiyotaki-Wright search model assumed that goods and money are both indivisible. Shi (1995) and Trejos and Wright (1995) eliminated the assumption of indivisible money, Green and Zhou (1998) eliminated the assumption of indivisible money, and Molico (1997) and Shi (1997, 1999) eliminated both. Moreover, some search models have allowed for limited forms of competition between money and other means of exchange, such as bilateral credits (Shi, 1996) and middlemen (Li, 1998).

substitutes in the goods market and in the other, matured bonds are traded at a discount in trades among private agents. These results have some resemblances to the ones I obtain here.[2] However, there are significant differences. First, I eliminate their assumption that money and bonds are indivisible, as indivisibility itself may generate spurious results. Second, the models here are tractable for analyzing standard monetary policies, such as money growth and open market operations. Third, I assume that the bonds market is centralized. In contrast, Aiyagari et al. (1996) assume that exchanges in the bonds market are decentralized, and so an agent who wants to redeem bonds may fail to do so with positive probability. By assuming centralized issuing and redemption of bonds, I allow for a greater degree of competition between bonds and money, and hence makes the results more robust.[3]

I will restrict attention to one-period nominal bonds, leaving bonds of longer maturities for future research. With this restriction, the issue is whether agents will choose not to redeem bonds at maturity and instead use them to buy goods in the future. One may wonder why any bond holder would choose to miss out on the payment from redemption. This understanding is misguided by models where money is assumed to have an intrinsic value. In any model where money is intrinsically worthless, redemption of a nominal bond is a swap of one fiat object for another. Missing out on the redemption is not costly at all to a bond holder if bonds perform the role of a medium of exchange as well as money does.

2. A Search Economy with Nominal Bonds

2.1. Households, Matches, and Timing

Consider a discrete-time economy with many types of households. The number of households in each type is large and normalized to one. The households in each type are specialized in producing a specific good, which they do not consume, and exchange for consumption goods in the market. Goods are perishable between periods.[4] The utility of consumption is $u(.)$ for consumption goods and 0 otherwise. The cost (disutility) of production is $\psi(.)$. The utility function satisfies $u' > 0$, and $u'' \leq 0$. The cost function satisfies $\psi(0) = 0$, $\psi' > 0$ and $\psi'' > 0$. Moreover, $u'(0) > \psi'(0)$, $u'(\infty) < \psi'(\infty)$, and $q\psi'(q)/\psi(q)$ is a non-decreasing function.

Agents meet their trading partners bilaterally and randomly in the market, as in Kiyotaki and Wright (1991, 1993). In addition, there is no chance for a double coincidence of wants in a meeting to support barter or public record-keeping of transactions to support credit trades. As

[2]In particular, for matured bonds to be perfect substitutes for money and yet newly issued bonds to be sold at a discount, Aiyagari et al. (1996) require that the government reject unmatured bonds with positive probability in trades.

[3]Another related model is by Kocherlakota (2001). He assumes that some bonds cannot be used in the goods market and shows that, in the presence of taste shocks, such illiquid bonds can improve welfare by allowing heterogeneous agents to smooth consumption intertemporally. In my paper, why bonds are not liquid is the very issue to be examined. For this, I abstract from the taste shocks.

[4]See Shi (1999, 2001) for search models with capital accumulation and an endogenous fraction of sellers.

a result, every trade entails a medium of exchange. The medium can be money or default-free, nominal bonds issued by the government. These objects have no intrinsic value and can be stored without cost. Each unit of nominal bonds can be redeemed for one unit of money at (and only at) maturity. In this paper, I restrict attention to one-period bonds. I also assume that an agent can bring money and bonds separately into matches but not simultaneously into each match. This assumption is not critical, as shown in section 4.3.

To specify the matching technology, call an agent in the goods market a *buyer* if he holds money or bonds, and a *seller* if he holds neither money nor bonds. Let σ be the (fixed) fraction of sellers and $(1-\sigma)$ the fraction of buyers in the market. Of interest are the meetings of a single coincidence of wants, i.e., meetings in which one and only one agent can produce the partner's consumption goods. Call such a meeting a *trade match*. A buyer encounters a trade match in a period with probability $\alpha\sigma$ and a seller with probability $\alpha(1-\sigma)$, where $\alpha > 0$ is a constant.

Random matching generates non-degenerate distributions of agents' money holdings and consumption. To maintain tractability, I assume that each household consists of a large number of members who share consumption each period and regard the household's utility function as the objective. This assumption makes the distribution of money holdings degenerate across households and hence allows me to focus on equilibria that are symmetric across households.[5]

Pick an arbitrary household as the representative household. Normalize the measure of members in the household to one. Lower-case letters denote the decisions of this household, while capital-case letters denote other households' decisions or aggregate variables. Also, suppress the generic time subscript t, denoting $t+j$ as $+j$ and $t-j$ as $-j$ for $j \geq 1$.

A household consists of sellers and buyers. A seller produces and sells goods. The measure of sellers in the household is fixed at $\sigma \in (0,1)$. A buyer purchases consumption goods, using money or bonds. Those who use money are called *money holders* and those who use bonds are *bond holders*. Let n be the measure of money holders in the household and $(1-\sigma-n)$ of bond holders, where $n \in [0, 1-\sigma]$. This division n is a choice of the household.

In contrast to the goods market, the bonds market is centralized and has a much lower transaction cost. To simplify, I abstract from such a transaction cost altogether by assuming that trades in the bonds market take zero measure of members. Also, following Lucas (1990), I assume that the government sells bonds only for money. Each newly issued bond is sold at a market price S and can be redeemed for one unit of money at (and only at) maturity. Thus, $1/S$ is the gross nominal interest rate.

The events in an arbitrary period t unfold as in Figure 1, where the subscript t is suppressed. At the beginning of the period the household has an amount m of money and an amount b of

[5]The assumption of large households, used by Shi (1997, 1999), is a modelling device extended from Lucas (1990). Lagos and Wright (2001) use a different set of assumptions to achieve essentially the same purpose of risk smoothing.

matured bonds. These bonds matured in previous periods but were not redeemed (and hence they cannot be redeemed in the future). The household divides money into a fraction a, which the buyers will carry into the goods market, and a fraction $(1-a)$, which the household will take to the bonds market. The household also divides buyers into money holders (of a fraction n) and bond holders (of a fraction $1-\sigma-n$). Thus, each money holder holds an amount am/n of money and each bond holder holds $b/(1-\sigma-n)$ matured bonds. The household does not carry any matured bonds into the bonds market because new bonds are sold only for money.

t	portfolio (a,n)	markets open	markets closed	redemption decision r	$t+1$
$\vert\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\vert\rightarrow$
(m,b) measured	decisions on goods trade (q,x)	trades: d and (q,x)	pooling, consume	money transfers	

Figure 1 Timing of events in a period

At the time of choosing the portfolio divisions (a,n), the household also chooses the quantities of trade (q,x), which I will describe later.

Next, the goods market and the bonds market open simultaneously. In the bonds market, the household purchases an amount d of new bonds, using the money allocated to the bonds market. In the goods market, the agents trade according to the quantities (q,x) prescribed by the household. Then, the goods market closes. The household pools the receipts from the trades and allocates the same amount of consumption to every member.

After consumption, the bonds purchased at the beginning of the current period mature and the household chooses the fraction of such bonds to be redeemed (in the centralized market).[6] Let $r\in[0,1]$ denote this *redemption fraction*. Bonds that are not redeemed at maturity cannot be redeemed in the future. After redemption, the household receives a lump-sum monetary transfer L and time proceeds to the next period.

This timing sequence highlights the temporary separation of the bonds market and the goods market, as in Lucas (1990). A household cannot shift resources between the two markets within each period, although across periods it can reallocate resources between the two markets. This is a critical feature, because it makes it costly to bring money into the bonds market.

There are two main differences between this model and Lucas's. First, the goods market is decentralized here, in contrast to the centralized goods market. With decentralized trades, fiat objects like money and nominal bonds can have positive values in equilibrium, even when the current model is extended to allow agents to barter. This endogenous role for a medium of exchange is absent when the goods market is Walrasian. Second, I allow households to use bonds,

[6] It makes no difference in the current model to assume alternatively that the bonds issued in the current period mature at the beginning of the next period before other activities take place.

as well as money, to buy goods, while Lucas assumes that money is the only medium of exchange. Lucas's assumption amounts to imposing $r = 1$ a priori.

2.2. Quantities of Trade in the Goods Market

The household chooses the quantities of money and goods to be traded in each trade match. To describe these choices, let $\beta \in (0, 1)$ be the discount factor and $v(m, b)$ the household's value function, where I suppressed the dependence of v on aggregate variables. Let ω^i be the household's (shadow) value of asset i $(= m, b)$ in the next period, discounted to the current period. That is,

$$\omega^m \equiv \beta v_1(m_{+1}, b_{+1}), \qquad \omega^b \equiv \beta v_2 \left(m_{+1}, b_{+1}\right), \tag{2.1}$$

where the subscripts of v indicate partial derivatives and the subscript $+1$ indicates "the next period". Other households' values of the two assets are Ω^m and Ω^b, respectively.

To simplify the analysis, I assume that the buyer in a trade match makes a take-it-or-leave-it offer.[7] Consider a buyer holding asset i, where $i = m, b$. The offer specifies the quantity of goods that the buyer wants the seller to supply, q^i, and the quantity of asset i that the buyer gives, x^i. If a match is not a trade match, the household instructs its members to not trade.[8]

When choosing the quantities (q^m, x^m), the household anticipates the following constraints:

$$x^m \leq am/n, \tag{2.2}$$

$$\psi(q^m) \leq \Omega^m x^m. \tag{2.3}$$

The first constraint says that the amount of money traded cannot exceed the amount the buyer carries into the trade. This is necessary because the matches are separated from each other. The second constraint says that the offer cannot make the seller worse off than not trading, where $\Omega^m x^m$ is the value of money that the seller's household gets by agreeing to the trade.

Similarly, the following constraints apply when the buyer holds bonds:

$$x^b \leq b/(1 - \sigma - n), \tag{2.4}$$

$$\psi(q^b) \leq \Omega^b x^b. \tag{2.5}$$

I will call (2.2) and (2.4) the *trading constraints* in the goods market. When (2.2) binds, I say that money yields *liquidity* or *service* in the goods market. Similarly, matured bonds yield liquidity or service in the goods market when (2.4) binds.

[7] More general bargaining schemes are analyzed in a similar environment elsewhere (Shi, 2001).

[8] I omit the possible trades between a money holder and a bond holder in the goods market, because such trades are immaterial in the current model. As it will become clear later, whenever there are matured bonds in the goods market, they are traded at par with money.

2.3. A Household's Decision Problem

The household's choices in each period are the portfolio divisions (a, n), the quantities of trade (q^m, x^m, q^b, x^b), the amount of new bonds to purchase d, consumption c, the redemption fraction r, and future asset holdings (m_{+1}, b_{+1}). Taking other households' choices and aggregate variables (i.e., the capital-case letters) as given, the household solves the following problem:

$$(PH) \quad v(m,b) = \max \left\{ u(c) - \alpha\sigma \left[N\psi(Q^m) + (1-\sigma-N)\psi(Q^b) \right] + \beta v(m_{+1}, b_{+1}) \right\}.$$

The constraints are as follows:

(i) the constraints in the goods market, (2.2)–(2.5), and

$$c = \alpha\sigma \left[nq^m + (1-\sigma-n)\, q^b \right]; \tag{2.6}$$

(ii) the money constraint in the bonds market:

$$Sd \leq (1-a)m; \tag{2.7}$$

(iii) the laws of motion of asset holdings:

$$m_{+1} = m + (r-S)d + \alpha\sigma \left(NX^m - nx^m \right) + L, \tag{2.8}$$

$$b_{+1} = b + (1-r)\, d + \alpha\sigma \left[(1-\sigma-N)\, X^b - (1-\sigma-n)\, x^b \right]; \tag{2.9}$$

(iv) and other constraints:

$$0 \leq a \leq 1, \ 0 \leq n \leq 1-\sigma, \ 0 \leq r \leq 1. \tag{2.10}$$

The disutility of production incurred by the household is calculated as follows. The total number of trades which involve the household's sellers is $\alpha\sigma$. In a fraction N of such trades the trading partners are money holders, who ask for Q^m units of goods; in the remaining fraction of trades the trading partners are bond holders, who ask for Q^b units of goods. The amount of consumption given by (2.6) can be explained similarly and the constraint (2.7) in the bonds market is self-explanatory.

To explain the law of motion of money, (2.8), note that the household's money holdings can change between the current period and next period for three reasons: purchasing and redeeming newly issued bonds, trading in the goods market, and receiving monetary transfers. In particular, the household gets a net amount $(r-S)d$ from purchasing d units of money and redeeming a fraction r of them. Adding these three changes to the household's money holdings at the beginning of the current period gives the household's money holdings at the beginning of the next period.

The law of motion of matured bonds, given by (2.9), can be explained similarly. Note that bonds acquired in the goods market are matured bonds that have already passed their maturity, and so they cannot be redeemed for money.

To characterize optimal decisions, let ρ be the Lagrangian multiplier of the money constraint in the bonds market, (2.7), λ^m of the money constraint in the goods trade, (2.2), and λ^b of the bond constraint in the goods trade, (2.4). To simplify the equations, multiply λ^m by the number of trades that involve the household's money holders, $\alpha\sigma n$. Similarly, multiply λ^b by $\alpha\sigma(1-\sigma-n)$. Incorporating these constraints, I can rewrite the objective function in (PH) as follows:

$$\begin{aligned} v(m,b) &= \max\Big\{\beta v(m_{+1},b_{+1}) + u\left(\alpha\sigma\left[nq^m + (1-\sigma-n)\,q^b\right]\right) \\ &\quad -\alpha\sigma\left[N\psi(Q^m) + (1-\sigma-N)\psi(Q^b)\right] + \rho\left[(1-a)\,m - Sd\right] \\ &\quad +\alpha\sigma n\lambda^m\left(\tfrac{am}{n} - x^m\right) + \alpha\sigma\,(1-\sigma-n)\,\lambda^b\left(\tfrac{b}{1-\sigma-n} - x^b\right)\Big\}, \end{aligned}$$

where the quantities x^m and x^b satisfy (2.3) and (2.5) with equality (provided $\omega^m, \omega^b > 0$):

$$x^i = \psi(q^i)/\Omega^i, \text{ for } i = m, b. \tag{2.11}$$

The following conditions are necessary for the decisions to be optimal:

(i) For q^i:

$$u'(c) = \left(\omega^i + \lambda^i\right)\psi'(q^i)/\Omega^i,\ i = m, b. \tag{2.12}$$

(ii) For (r, a, d, n):

$$\omega^m = \omega^b \quad \text{ if } r \in (0,1), \tag{2.13}$$

$$\alpha\sigma\lambda^m = \rho \quad \text{ if } a \in (0,1), \tag{2.14}$$

$$r\omega^m + (1-r)\omega^b = (\omega^m + \rho)\,S \quad \text{ if } d \in (0,\infty), \tag{2.15}$$

$$q^m = q^b \quad \text{ if } n \in (0, 1-\sigma). \tag{2.16}$$

In each of these conditions, the variable attains the lowest value in the specified domain if the equality is replaced by "<", and the highest value if ">".

(iii) For (m, b) (envelope conditions):

$$\omega^m_{-1}/\beta = \omega^m + (1-a)\rho + a\alpha\sigma\lambda^m, \tag{2.17}$$

$$\omega^b_{-1}/\beta = \omega^b + \alpha\sigma\lambda^b. \tag{2.18}$$

The condition (2.12) requires that the net gain to the buyer's household from asking for an additional amount of goods be zero. By getting an additional unit of goods, the household increases utility by $u'(c)$. The cost is to pay an additional amount $\psi'(q^i)/\Omega^i$ of asset i in order to induce the seller to trade (see (2.11)). By giving an additional unit of asset i, the buyer foregoes the discounted future value of the asset, ω^i, and causes the asset constraint in the trade to be more binding. Thus, $(\omega^i + \lambda^i)$ is the shadow cost of each additional unit of asset i to the buyer's household and the right-hand side of (2.12) is the cost of getting an additional unit of goods from the seller.

In (ii), (2.13) says that matured bonds must have the same value as money in the next period if the household chooses not to redeem all bonds at maturity. The condition (2.14) says that for the household to allocate money to both the goods market and the bonds market, money must generate the same marginal "service" or liquidity in the two markets by relaxing the trading constraints in these markets. The condition (2.15) says that the expected future value of newly issued bonds must be equal to the cost of money that is used to acquire such bonds. Because the household may not redeem all bonds at the end of the period, the expected future value of newly issued bonds is the weighted average of the future values of money and matured bonds, where the weights are r and $(1-r)$.

The condition (2.16) says that for the household to send both money holders and bond holders to the goods market, the two types of buyers must obtain the same quantity of goods in a trade match. To explain this condition, note that a buyer of type i $(= m, b)$ obtains a net gain or surplus from a trade, $\left[u'(c)q^i - \left(\omega^i + \lambda^i\right) x^i\right]$, where $(\omega^i + \lambda^i)$ is the marginal cost of asset i to the buyer (as explained above). Since $(\omega^i + \lambda^i)x^i = u'\psi(q^i)/\psi'(q^i)$ by (2.11) and (2.12), a type-i buyer's surplus from trade is $u'(c)\left[q^i - \psi(q^i)/\psi'(q^i)\right]$. For the choice of n to be interior, the two types of buyers must obtain the same surplus in a trade. This requires $q^m = q^b$, because $[q - \psi(q)/\psi'(q)]$ is a strictly increasing function.

Finally, the envelope conditions require the current value of each asset to be equal to the sum of the future value of the asset and the expected service generated by the asset in the current markets. Take money for example. The current value of money is given by the left-hand side of (2.17), where ω_{-1}^m is divided by β because ω_{-1}^m is defined as the value of money discounted to one period earlier. The right-hand side of (2.17) consists of the future value of money, ω^m, the service generated by money in the current bond market, ρ, and the service generated by money in the current goods market, $\alpha\sigma\lambda^m$. The services in the two markets are weighted by the division of money into the two markets.

3. Symmetric Equilibrium

3.1. Definition, Focus, and the Nominal Interest Rate

A (symmetric) monetary equilibrium consists of a sequence of a representative household's choices

$$(a_t, n_t, q_t^m, x_t^m, q_t^b, x_t^b, d_t, c_t, r_t, m_{t+1}, b_{t+1})_{t=0}^{\infty},$$

the implied shadow prices $\left(\omega_t^m, \omega_t^b, \rho_t, \lambda_t^m, \lambda_t^b\right)_{t=0}^{\infty}$, and other households' choices such that the following requirements are met:

(i) Optimality: given other households' choices, the household's choices solve (PH) with given initial holdings (m_0, b_0);

(ii) Symmetry: the choices (and shadow prices) are the same across households;

(iii) Clearing in the bonds market: $d_t = zM_t$, where M_t is money holdings per household at the beginning of period t and zM_t is the amount of new bonds issued in period t, with $z > 0$;

(iv) Constant money growth: L_t ensures that $M_{t+1} = \gamma M_t$, where $\gamma > 0$ is a constant;

(v) Positive and finite values of assets: $0 < \omega_t^m m_t < \infty$ and $0 < \omega_t^b b_t < \infty$ for all t if $m, b > 0$.

In this definition I have restricted bond issuing to be a constant fraction of the money stock and money growth to be constant. I have also restricted the total value of each asset to be positive and finite, in order to examine the coexistence of money and bonds.[9] For the issues in this paper, I restrict attention further to equilibria that have the following features. First, money serves as a medium of exchange. This requires $a > 0$ and $n > 0$. (Note also that $a < 1$ as a result of the bonds market clearing condition, and so $\rho = \alpha\sigma\lambda^m$.) Second, the household redeems a positive fraction of matured bonds, i.e., $r > 0$. Third, the equilibrium is stationary and, in particular, the bond-money ratio (b/m) and the value of the total stock of money $(\omega_{-1}^m m/\beta)$ are constant.

Under these restrictions, the equilibrium is one of the following two types:

(a) $0 < r < 1$ and $0 < n < 1 - \sigma$: In this case $\omega^m = \omega^b$, $q^m = q^b$, and bonds circulate in the goods market as perfect substitutes for money;

(b) $(1 - r)(1 - \sigma - n) = 0$: In this case $q^m > q^b$ and bonds do not circulate.

In both cases, the gross nominal interest rate, $1/S$, can be derived from (2.15) as

$$1/S = 1 + \alpha\sigma\lambda^m/\omega^m. \tag{3.1}$$

Note from (2.17) that $\alpha\sigma\lambda^m/\omega^m = \omega_{-1}^m/(\beta\omega^m) - 1$. Since $\omega_{-1}^m/\omega^m = \gamma$ under the restriction that $\omega_{-1}^m m/\beta$ is constant over time, the above equation yields:

$$1/S = \gamma/\beta. \tag{3.2}$$

Newly issued bonds are sold at a discount and hence the net nominal interest rate is positive, provided $\gamma > \beta$. This is true regardless of whether matured bonds will circulate in the goods market as a medium of exchange. Even if bonds can become perfect substitutes for money one

[9] The bounded value of each asset is necessary for the household's optimal decisions to be indeed characterized by the first-order conditions obtained in the previous section.

period after they are issued, they are not perfect substitutes for money at the time of issuing. This is because the temporary separation between the bonds market and the goods market prevents newly issued bonds from being used to purchase goods in the *same* period as they are acquired. Thus, there is a one-period loss of liquidity to the amount of money allocated to purchase newly issued bonds. To compensate for this loss of liquidity, a positive discount on newly issued bonds is necessary; otherwise the household will not allocate any money to the bonds market.

The liquidity value of money is the extent to which money relaxes the trading constraints in the goods market, as captured by the term $\alpha\sigma\lambda^m$ in (3.1). This liquidity value is positive only if $\gamma > \beta$. When $\gamma = \beta$, money generates a rate of return equal to the rate of time preference, in which case a household is indifferent between spending a marginal unit of money on goods and keeping it for its store of value. In the remainder of this paper, I will impose $\gamma > \beta$ throughout this paper.

3.2. Equilibria with Bonds Circulating in the Goods Market

Matured bonds can circulate in the goods market as perfect substitutes for money. In fact, there are a continuum of such equilibria, as described below.

Proposition 3.1. *Assume $z \in (0, \gamma/\beta)$. If $\gamma > 1$, the redemption fraction r is indeterminate. For each $r \in (0, 1)$, there exists a stationary equilibrium in which matured bonds circulate in the goods market as perfect substitutes for money. In these equilibria, the price level, the values of assets, and the division n depend on the redemption fraction r. However, the quantity of goods traded in a match, the level of consumption, the division of money a and the nominal interest rate are all independent of r. If $\gamma \leq 1$, the only equilibrium that satisfies the requirement $b/m \in (0, \infty)$ is one in which bonds do not circulate in the goods market.*

It is useful to prove the proposition by constructing the continuum of equilibria in the proposition. Fix r at any arbitrary level in $(0, 1)$. For this interior choice to be optimal, matured bonds and money must have the same value, i.e., $\omega^b = \omega^m$ (see (2.13)). For matured bonds to circulate, the household must allocate a positive measure of members to trade with such bonds. However, for the choice $n \in (0, 1 - \sigma)$ to be optimal, matured bonds and money must exchange for the same quantity of goods in a trade, i.e., $q^m = q^b$ (see (2.16)). Since the two assets have the same value and trade for the same quantity of goods, they are perfect substitutes in the goods market.

The quantities q^m and q^b are independent of r. To show this, denote the common value of the qs as q^c, where the superscript c indicates that bonds circulate in such equilibria. Then, (2.12) yields $\lambda^m = \lambda^b$ and $\lambda^m/\omega^m = u'(c)/\psi'(q^c) - 1$, where $c = \alpha\sigma(1-\sigma)q^c$. Substituting these results into (3.1) and using (3.2), I get

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma \left[\frac{u'\left(\alpha\sigma(1-\sigma)q^c\right)}{\psi'(q^c)} - 1 \right]. \tag{3.3}$$

There is a unique solution for q^c for all $\gamma \geq \beta$ and the solution is independent of r. Thus, real consumption and real output are independent of r.

Also independent of r are the price of newly issued bonds, the nominal interest rate, and the division of money between the two markets, a. The price of newly issued bonds is $S = \beta/\gamma$, as shown before, and the fraction a is given by

$$a = 1 - Sd/m = 1 - z\beta/\gamma. \tag{3.4}$$

Clearly, $a \in (0,1)$ if and only if $0 < z < \gamma/\beta$, a condition imposed in Proposition 3.1. Notice that $S < 1$, even though matured bonds are perfect substitutes for money.

The redemption fraction affects other variables. First, the bond-money ratio, obtained from the law of motion of bonds (2.9), is

$$\frac{b}{m} = \frac{(1-r)\,z}{\gamma - 1}. \tag{3.5}$$

Given $r \in (0,1)$, this ratio is positive and finite if and only if $\gamma > 1$. Second, to keep the money stock growing at the constant rate γ, (2.8) requires monetary transfers to satisfy:

$$\frac{L}{m} = \gamma - 1 + z\left(\frac{\beta}{\gamma} - r\right). \tag{3.6}$$

Third, because $\gamma > \beta$, (3.3) implies $u'(c) > \psi'(q^c)$, and so $\lambda^m = \lambda^b > 0$. That is, $x^m = am/n$ and $x^b = b/\left(1 - \sigma - n\right)$. Dividing these two equations and using $x^i = \psi(q^c)/\omega^i$, I get

$$n = (1-\sigma) \Big/ \left[1 + \left(\frac{1-r}{\gamma - 1}\right)\left(\frac{z}{1 - z\beta/\gamma}\right)\right]. \tag{3.7}$$

This fraction indeed lies in $(0, 1-\sigma)$ for $\gamma > 1$. Fourth, the values of money and bonds are:

$$\begin{aligned} \omega^m &= \psi\left(q^c\right)/x^m = n\psi\left(q^c\right)/\left(am\right); \\ \omega^b &= \psi\left(q^c\right)/x^b = \left(1-\sigma-n\right)\psi\left(q^c\right)/b. \end{aligned} \tag{3.8}$$

Finally, nominal prices of goods are:

$$p^m \equiv x^m/q^c = x^b/q^c \equiv p^b. \tag{3.9}$$

This completes the construction of the equilibrium for an arbitrary $r \in (0,1)$.

The redemption fraction is indeterminate because matured bonds are perfect substitutes for money in the goods market. By choosing not to redeem a unit of matured bond, the household misses out on one unit of money. However, the retained bond can purchase exactly the same amount of goods as does a unit of money. Thus, it does not matter to the household how much of the matured bonds to be redeemed.

Different redemption fractions lead to different levels of total "moneyness" in the economy, $(m + b)$. A higher redemption fraction reduces the bond-money ratio and the total moneyness in the goods market. Since real output is the same for all r, the higher redemption fraction reduces the price level and increases the values of the assets. Also, as the bond-money ratio falls, households shift some members from holding bonds to holding money; i.e., n increases. This shift is necessary for maintaining perfect substitutability between the two assets, by ensuring that money and bonds yield the same services in the goods market.

It is remarkable that (matured) nominal bonds circulate in the goods market in equilibrium, given that the model has imposed a few restrictions that seem to reduce the desirability of bonds relative to money. For example, the government does not accept matured bonds as payments for newly issued bonds and it refuses to redeem bonds that have passed their maturity. In the end, however, the fiat nature of bonds makes them equally acceptable in the exchange as the other fiat object – money.

The indeterminacy of equilibria in the current model differs from that in a conventional monetary model where the real money balance appears in the utility function. There, the price level is indeterminate when the utility function depends on the real money balance in particular ways. Here, the real money balance does not appear in the utility function. Instead, the substitution between money and other money-like assets in the goods market is the key to the indeterminacy.

Notice that the continuum of equilibria requires $\gamma > 1$. If $\gamma \leq 1$ and the redemption fraction is less than one, then the amount of matured bonds will increase over time relative to the money stock, making the bond-money ratio unbounded. However, this is an artifact of the assumption $b > 0$. If the government can lend as well as borrow, the continuum of equilibria can exist for $\gamma < 1$ as well.[10]

3.3. Effects of Open Market Operations

Since there are a continuum of equilibria, it is difficult to pin down the effects of monetary policies, such as open market operations. To illustrate, let us examine a tightening open market operation, modelled as an increase in z. Let $r < 1$ and $\gamma > 1$.

Assume first that the redemption fraction does not respond to the increase in z. Then, the tightening operation increases the bond-money ratio and *increases* the price level. To explain this seemingly paradoxical result, note that new bonds are sold for money at a discount. For each unit of money that the government extracts from the economy by issuing bonds, more than one unit of bonds is injected into the economy. Once matured, these bonds circulate as perfect substitutes for money, thus increasing the sum of money and bonds.

The households absorb the increased quantity of bonds by allocating more buyers to transact with matured bonds, thus maintaining the perfect substitutability between the two assets in the goods market. The bond price and the nominal interest rate do not change. This is because the households fully anticipate the increase in z, and so they increase the amount of money allocated

[10] Alternatively, if the government is willing to redeem bonds that have passed the maturity, then the continuum of equilibria can exist for $\gamma < 1$ (as well as $\gamma > 1$), provided $1 > r > \max\{0, 1 - \gamma\}$.

to the bonds market to completely offset the increase in the new issues.[11] Real consumption and output do not respond to the increase in bond issuing, either.

The effect on the price level can be quite different if the redemption fraction responds to the tightening operation, a possibility that can occur given the continuum of equilibria. For example, if the redemption fraction increases sufficiently with the tightening operation, then the bond-money ratio and the price level can fall, rather than rise, with the increase in z. Thus, whether the tightening open market operation will increase or decrease the price level depends on the direction and magnitude in which the redemption fraction responds to the operation.[12] Real consumption and output, however, are still invariant with respect to the operation.

The analysis so far has maintained the assumption that the government always adjusts monetary transfers to keep the money growth rate unchanged. It may be interesting to examine an alternative policy which sets monetary transfers as a constant fraction, l, of the money stock. Under this alternative policy, open market operations affect the money growth rate, and hence affect real output and the nominal interest rate. To see this, denote the money growth rate as $\gamma_{+1} = m_{+1}/m$. The price of newly issued bonds is $S = \beta/\gamma_{+1}$. The equations (3.3), (3.4) and (3.5) all hold after replacing γ by γ_{+1}. In contrast, the result (3.6) no longer holds; instead, the law of motion of money now implies:

$$\gamma_{+1} - 1 - l = z\left(r - \frac{\beta}{\gamma_{+1}}\right).$$

This equation solves for γ_{+1} for given (l, z, r). If $l < -(1-\beta)z$ and $r > \beta - l/z$, then $\gamma_{+1} > 1$. In this case, an increase in z increases γ_{+1}. The quantity of goods traded in each match and the level of real consumption both fall.

4. Government in the Goods Market

Now I introduce the government in the goods market. Assume that government goods are perfect substitutes for private goods. This assumption is not critical, as discussed later.

4.1. Government Traders and Private Households' Decisions

The government has a measure N_g of buyers and a measure σ_g of sellers. Each government buyer holds M_g units of money. The total measure of buyers in the economy is $(1-\sigma+N_g)$ and the total measure of sellers $(\sigma+\sigma_g)$. Extend the matching technology as follows. A buyer (private or government) gets a trade match with a private seller with probability $\alpha\sigma$, and with a government

[11] If z is stochastic and the households must allocate money between the two markets before z is realized, then the nominal interest rate may change.

[12] One can push the above argument further by introducing sunspots into the economy that serve as a device to coordinate the households' decisions on the redemption ratio. Then, it is possible that the price level can remain invariant with respect to open market operations. Such invariant prices, of course, have nothing to do with any cost associated with adjusting nominal prices.

seller with probability $\alpha\sigma_g$. Similarly, a seller gets a trade match in a period with a private money holder with probability αn, with a (private) bond holder with probability $\alpha(1-\sigma-n)$, and with a government buyer with probability αN_g.

An important assumption is that the government sells goods only for money and buys goods only with money. More specifically, government agents trade in the following reasonable but exogenous ways:[13]

(i) A government buyer. A government buyer carries only money. In a trade match with a private seller, a government buyer spends all his money, M_g/N_g, to buy goods at the price which a private money holder pays to a private seller. Denote this price by P^m, which is the price averaged over all trades between private agents who use money as payments, not including the trades that use bonds as payments.[14]

(ii) A government seller. A government seller accepts only money for trade. In a trade match with a private money holder, a government seller sells goods only at the average price P^m, but leaves the quantity of goods for the private money holder to decide. Denote this quantity of goods by q^g.

Notice that private and government agents both take the price P^m as given. Also, I abstract from the trades between two government agents, since such trades are inconsequential to private households' behavior.

A household's consumption now is:

$$c = \alpha \left\{ n \left[\sigma q^m + \sigma_g q^g \right] + \sigma \left(1 - \sigma - n \right) q^b \right\}, \tag{4.1}$$

which includes goods purchased from the government. The household also has a net receipt of money from trades with the government, $\alpha(\sigma M_g - \sigma_g n P^m q^g)$. Incorporating this additional term, the law of motion of the household's money holdings becomes:

$$m_{+1} = m + (r - S)d + \alpha\sigma \left(NX^m - nx^m \right) + \alpha \left(\sigma M_g - \sigma_g n P^m q^g \right) + L. \tag{4.2}$$

The law of motion of the household's bond holdings is still (2.9).

The household's additional choice is q^g, the quantity of goods to ask from a government seller. The corresponding constraint in such a trade is:

$$P^m q^g \leq am/n. \tag{4.3}$$

[13]Another way to model government agents' trading behavior is to specify the government's objective function. I do not take this approach because it is not clear what is a reasonable objective function for the government.

[14]The maximum amount of goods that a government buyer can obtain from a private seller is $M_g/\left(N_g P^m\right)$. I assume that a private seller is always willing to trade this quantity, i.e., $\omega^m M_g/N_g \geq \psi\left(M_g/\left(N_g P^m\right)\right)$.

Let λ^g be the Lagrangian multiplier of this constraint and multiply λ^g by the number of trades between a household's buyers and government sellers, $\alpha\sigma_g n$. The optimal choice of q^g satisfies:

$$u'(c) = (\omega^m + \lambda^g)\, P^m. \tag{4.4}$$

The quantities of goods and assets traded in matches between two private agents still satisfy (2.11) and (2.12). The amount of purchases of new bonds satisfies (2.15) and the redemption fraction satisfies (2.13). Moreover, the optimal choice of n still obeys (2.16).[15]

There are two revisions in the household's optimal conditions. First, when choosing the division of money between the two markets, a household takes into account the service generated by money in trades with government sellers, $\alpha\sigma_g\lambda^g$. So, the condition for a becomes

$$\alpha(\sigma\lambda^m + \sigma_g\lambda^g) = \rho \text{ if } a \in (0,1). \tag{4.5}$$

Second, for the same reason, I revise the envelope condition for money as follows:

$$\omega^m_{-1}/\beta = \omega^m + (1-a)\rho + a\alpha\left(\sigma\lambda^m + \sigma_g\lambda^g\right). \tag{4.6}$$

With these revisions, I can adapt the previous definition of a symmetric equilibrium to the current economy, where γ refers to the growth rate of money held by the private sector. Symmetry requires an additional condition:

$$P^m = p^m \equiv x^m/q^m. \tag{4.7}$$

Again, I focus on equilibria with $0 < a < 1$, $n > 0$, $0 < r \leq 1$ and $\gamma > \beta$. Then, the nominal interest rate is:

$$\frac{1}{S} = \frac{\gamma}{\beta} = 1 + \alpha\left(\sigma\frac{\lambda^m}{\omega^m} + \sigma_g\frac{\lambda^g}{\omega^m}\right). \tag{4.8}$$

The net nominal interest rate is positive for all $\gamma > \beta$.

Notice that

$$P^m = x^m/q^m = \psi(q^m)/\left(\omega q^m\right) < \psi'(q^m)/\omega^m, \tag{4.9}$$

where the inequality follows from the fact that $\psi(q)/q < \psi'(q)$ for all $q > 0$. Then, (2.12) and (4.4) imply $\lambda^g > \lambda^m \geq 0$. That is, the money constraint between a private buyer and a government seller always binds. This yields:

$$q^g = \frac{am}{nP^m} = q^m\frac{am}{nx^m}. \tag{4.10}$$

If the money constraint also binds in a trade between two private agents, then $q^g = q^m$.

[15] To see this, note that a private money holder's surplus from a trade with a government seller, $u'(c)q^g - (\omega^m + \lambda^g)\, P^m q^g$, is zero under (4.4). Thus, a private buyer gets a positive surplus from trade entirely from matches with private sellers, just as a bond holder does. So, (2.16) continues to hold. Clearly, this result depends on the assumption that a government seller sells goods at a fixed price P^m. In addition to simplifying the algebra, this assumption strengthens the later result that matured bonds do not circulate in the goods market. If a money holder obtained a positive surplus from a trade with a government seller, as well as from a private seller, there would be an additional advantage to holding money relative to bonds.

4.2. Matured Bonds Do Not Circulate in the Goods Market

Matured bonds do not circulate in this economy, as stated below (see Appendix A for a proof):

Proposition 4.1. *For $\gamma > \beta$, there is no stationary equilibrium where $0 < r < 1$.*

Bonds exit from the goods market because of decentralized exchanges and the government's trading policies. Let me explain the result in two steps.

First, for matured bonds to circulate in the goods market, a household must be indifferent between money and matured bonds. This requires that the two assets have the same value, i.e., $\omega^m = \omega^b$. Since each asset derives a value from the services that it generates in the markets, by relaxing the trade constraints, these services must be the same for the two assets. That is, $\alpha\sigma\lambda^b/\omega = \alpha\sigma\lambda^m/\omega + \alpha\sigma_g\lambda^g/\omega$, where ω is the common value of money and matured bonds. Because money can relax the trading constraint in more trades than bonds do, perfect substitutability between the two assets requires $\lambda^b > \lambda^m$. Notice that an additional unit of asset can relax the trading constraint greatly if the buyer in the trade can only buy a small quantity of goods. Thus, $\lambda^b > \lambda^m$ if and only if $q^b < q^m$.

Second, for matured bonds to circulate in the goods market, each household must also allocate a positive measure of members (or time) to trade using matured bonds. For this decision to be optimal, a bond holder must obtain an expected surplus from trade that is equal to or greater than what a money holder obtains. However, a bond holder encounters fewer trades than a money holder does, due to the government's refusal to bond payments. To satisfy the requirement on the expected surplus, a bond holder's surplus in each trade must be at least as high as a money holder's in a similar trade. Since a buyer's surplus increases in the quantity of goods in the trade, the requirement becomes $q^b \geq q^m$.[16] This contradicts the previous requirement $q^b < q^m$. Therefore, matured bonds do not circulate in the goods market.

The government's policy of selling goods for only money is necessary, but <u>not</u> sufficient, for the exit of bonds from the goods market. Decentralized exchanges are also important for the result. By implying an implicit (time) cost of trading, decentralized exchanges force a household to consider the number of trades, as well as the quantities in each trade. The government's refusal to bond payments drives a wedge between the trading costs of using money and matured bonds, and hence makes the two assets imperfect substitutes in the goods market.

To appreciate the importance of decentralized trades, let us consider a Walrasian goods market and argue that the same legal restriction does not prevent bonds from circulating in the goods market. To make the argument more general, let a household's utility function be $u(c, g)$, rather than $u\,(c + g)$, where g is consumption of government goods. So, government goods may be indispensable to the household, rather than being perfect substitutes for private goods. Let the

[16] The inequality need not be strict because the surplus a money holder obtains in a trade with a government seller is zero under the particular description of government agents' trading strategies, as discussed earlier.

price level of private goods be P^m and of government goods P^g, measured in terms of money. Because a household can arbitrage between money and bonds, either directly or indirectly through goods, the price level of private goods in terms of matured bonds must be P^m. Let b' be the amount of matured bonds that the household holds after the arbitrage. Then, the household faces the following trading constraints in the goods market:

$$P^m c + P^g g \leq m + b, \quad P^g g \leq m + b - b'.$$

The second constraint does not bind, unless the household finds it optimal to spend all its money holdings exclusively on government goods. So, there is a range of values for the choice b' that are consistent with equilibrium.[17] The government's insistence on money as the only means of payment for its goods does not drive bonds out of the Walrasian goods market; it merely shifts money from purchases on private goods to government goods.

Two remarks are useful. First, bonds are out of the goods market as long as government sellers in the goods market have a positive measure, no matter how small this measure is. In this sense, the continuum of equilibria with circulating bonds established earlier is not robust. Second, a positive measure of private households can also drive bonds out of the goods market if, for reasons that are not modelled here, they commit to not accepting bonds as a payment.

4.3. Robustness

In this subsection I check the robustness of Proposition 4.1 by examining two variations of the model. The first uses a different assumption on when bonds can be redeemed and the second on how assets can be used in the goods market.

Suppose first that matured bonds can be redeemed in any period at or after maturity. This assumption seems to reduce the cost of carrying bonds to the goods market because, if such bonds fail to trade for goods, they can be redeemed for money. However, Proposition 4.1 continues to hold. For an interior choice of r to be optimal, it must be true that $\omega^m = \omega^b$. Then, the same argument following Proposition 4.1 leads to a contradiction to $r \in (0, 1)$ and $n \in (0, 1 - \sigma)$.

Now return to the assumption that bonds can be redeemed only at maturity, but eliminate the assumption that money and bonds must be used separately in trades. Assume instead that a buyer can carry both money and bonds into each match. In this alternative setup, a household does not need to divide buyers into money holders and bond holders. Each buyer carries $(am + b)/(1 - \sigma)$ units of assets into the goods market. To show that Proposition 4.1 still holds, suppose to the contrary that there is a symmetric equilibrium with $r \in (0, 1)$. Then, $\omega^m = \omega^b$. Denote this common value as ω. I show that this leads to a contradiction.

[17] The household can also make an arbitrage between private and government goods, but this arbitrage merely determines the relative price P^g/P^m, rather than imposing additional requirements on the choice b'.

To proceed, I need to revise the asset constraints in trade. Use superscripts mb to indicate a trade match between two private agents (where the buyer can use both m and b) and a superscript g to indicate a trade match between a private buyer and a government seller. The constraints facing the buyer in these two matches are, respectively,

$$\frac{am+b}{1-\sigma} \geq x^{mb} = \frac{\psi(q^{mb})}{\Omega}, \tag{4.11}$$

$$\frac{am}{1-\sigma} \geq x^{g} = \frac{\psi(q^{g})}{\Omega}. \tag{4.12}$$

Let λ^{mb} be the Lagrangian multiplier of (4.11) and λ^g of (4.12). Similar to (2.12), the optimal choice of q^{mb} satisfies:

$$u'(c) = (\omega + \lambda^{mb})\psi'(q^{mb})/\Omega.$$

The quantity q^g still satisfies (4.4), once the average price is redefined as $P^{mb} = X^{mb}/Q^{mb}$. As before, $\lambda^g > \lambda^{mb}$ and so $\lambda^g > 0$. The envelope conditions for money and matured bonds require that the two assets yield the same service in the market. When $\omega^m = \omega^b$, as supposed, this requirement becomes $\alpha\sigma\lambda^{mb} = \alpha(\sigma\lambda^{mb} + \sigma_g\lambda^g)$, which contradicts $\lambda^g > 0$. Therefore, there is no symmetric equilibrium where matured bonds circulate in the goods market.

5. Stationary Equilibrium

Now that the only stationary equilibrium is one in which bonds do not circulate, it is useful to characterize the equilibrium and examine its properties. This equilibrium has $r = 1$, $n = 1 - \sigma$ and $b = 0$ (since all bonds are redeemed at maturity).

5.1. Characterization

There are two cases of an equilibrium. In the first case, the money constraint in a trade between two private agents does not bind. That is, $\lambda^m = 0$. This case requires $q^g \geq q^m$ and $\psi'(q^m) = u'(c)$ (see (2.12)). Denote the quantity q^m in this case by q_1, c by c_1, and q^g by q_1^g. Then,

$$c_1 = u'^{-1}\left(\psi'(q_1)\right), \quad q_1^g = \frac{1}{\sigma_g}\left[\frac{c_1}{\alpha(1-\sigma)} - \sigma q_1\right].$$

To solve for q_1, use (4.8), (4.4) and (4.9) to derive the following equation:

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma_g\left(\frac{q_1\psi'(q_1)}{\psi(q_1)} - 1\right). \tag{5.1}$$

Recall that $q\psi'(q)/\psi(q)$ is assumed to be non-decreasing. If $q\psi'(q)/\psi(q)$ is strictly increasing for all $q > 0$, then a unique solution for q_1 exists for $\gamma \geq \underline{\gamma}$, where

$$\underline{\gamma} \equiv \beta\left[1 + \alpha\sigma_g(\Psi - 1)\right] \text{ and } \Psi \equiv \lim_{q\to 0} q\psi'(q)/\psi(q). \tag{5.2}$$

This solution satisfies the required conditions $q_1^g \geq q_1$ and $\psi'(q_1) = u'(c_1)$ if and only if $q_1 \leq q_0$, where q_0 is defined by:

$$\psi'(q_0) = u'(\alpha(1-\sigma)(\sigma+\sigma_g)q_0). \tag{5.3}$$

If $q\psi'(q)/\psi(q)$ is constant, then this constant is $\underline{\gamma}$ and the case $\lambda^m = 0$ exists only for $\gamma = \underline{\gamma}$.

The second case has $\lambda^m > 0$, which requires $q^g = q^m$ and $\psi'(q^m) < u'(c)$. Denote the quantity q^m in this case by q_2, c by c_2, and q^g by q_2^g. Then,

$$q_2^g = q_2, \quad c_2 = \alpha(1-\sigma)(\sigma+\sigma_g)q_2.$$

To solve for q_2, use (4.8), (2.12), (4.4) and (4.9) to derive the following equation:

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma\left(\frac{u'(c_2)}{\psi'(q_2)} - 1\right) + \alpha\sigma_g\left(\frac{u'(c_2)q_2}{\psi(q_2)} - 1\right). \tag{5.4}$$

There is a unique solution to this equation for $\gamma > \beta$. However, this solution must satisfy $\psi'(q_2) < u'(c_2)$, which holds if and only if $q_2 < q_0$ where q_0 is defined above. A necessary condition is $\gamma > \underline{\gamma}$.

Note that the right-hand sides of (5.1) and (5.4) are equal to each other when $q = q_0$. Define γ_0 as the level of γ that satisfies both (5.1) and (5.4) at $q = q_0$. Now I can establish the following proposition (see Appendix B for a proof):

Proposition 5.1. *A stationary equilibrium with* $r = 1$ *exists iff* $\gamma \geq \underline{\gamma}$ *and* $0 < z < \gamma/\beta$, *where* $\underline{\gamma}$ *is defined in (5.2). The equilibrium is unique if* $\gamma > \underline{\gamma}$. *When* $[q\psi'(q)/\psi(q)]$ *is a strictly increasing function for all* q, *there exists* $\gamma_0 > \underline{\gamma}$ *such that case* 1 *occurs when* $\underline{\gamma} < \gamma \leq \gamma_0$ *and case* 2 *occurs when* $\gamma > \gamma_0$. *When* $[q\psi'(q)/\psi(q)]$ *is a constant,* $\gamma_0 = \underline{\gamma}$ *and only case* 2 *occurs for* $\gamma > \underline{\gamma}$.

Figure 2 depicts the two cases for a strictly increasing function $[q\psi'(q)/\psi(q)]$. The two horizontal lines are $(\gamma/\beta - 1)$ for two levels of γ. The curve $R1(q)$ depicts the right-hand side of (5.1), and the curve $R2(q)$ of (5.4). The two curves intersect at point E_0, which occurs when $\gamma = \gamma_0$ and gives $q = q_0$. The equilibrium level of q must satisfy $0 < q \leq q_0$ in both cases. When $\gamma \leq \underline{\gamma}$, no such solution exists. When $\underline{\gamma} < \gamma \leq \gamma_0$, the solution is depicted by a point like E_1, which is case 1. When $\gamma > \gamma_0$, the solution is depicted by a point like E_2, which is case 2.[18]

An interesting feature of the equilibrium is that the money constraint in the goods market does not always bind, although the net nominal interest rate is positive. When trading with a government seller, a private buyer faces a binding constraint on money, provided $\gamma > \beta$. However, when trading with a private seller, a private buyer's money constraint binds only when $\gamma > \gamma_0$. This is a consequence of the assumption that a government seller always sells at the price P^m, while a private seller takes into account the increasing marginal cost of production. This difference implies that a private seller sells a smaller quantity of goods than a government seller does, i.e.,

[18]If $[q\psi'(q)/\psi(q)]$ is constant for all $q > 0$, the curve $R1(q)$ becomes a horizontal line going through E_0.

$q^m < q^g$. Since a private buyer has the same amount of money when trading with these two types of sellers, he is more likely to face a binding money constraint when trading with a government seller than with a private seller.

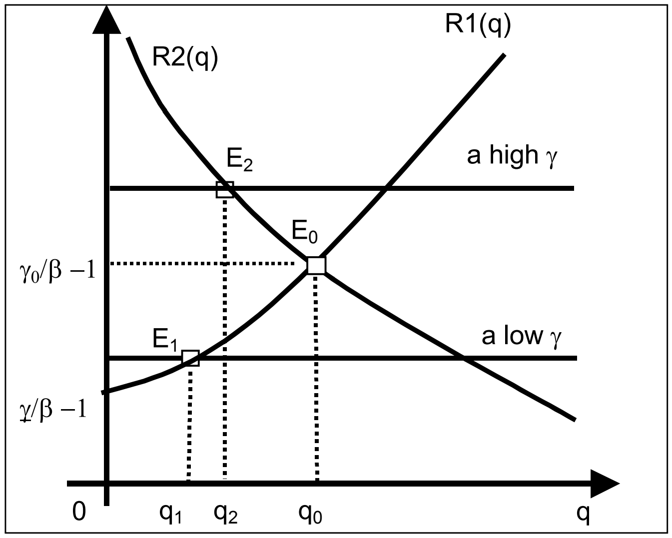

Figure 2.

In both cases of the unique equilibrium, $b = 0$ (since $r = 1$). The fraction of money taken to the goods market is $a = 1 - z\beta/\gamma$. The price level and the value of money are, respectively,

$$P^m = \frac{am}{(1-\sigma)\, q^g}, \qquad \omega^m = \frac{(1-\sigma)\,\psi(q^m)}{am}. \tag{5.5}$$

5.2. Effects of Monetary Policies

I examine the effects of two monetary policies, an increase in the money growth rate and an increase in the amount of new bond issues. Assume that $[q\psi'(q)/\psi(q)]$ is a strictly increasing function, so that cases 1 and 2 both exist for some intervals of the money growth rate. It is useful to distinguish private and public consumption/output, as listed below:

	consumption	output
private	$c = \alpha(1-\sigma)(\sigma q^m + \sigma_g q^g)$;	$y = \alpha\sigma(1-\sigma)\left(q^m + q^g \frac{M_g}{am}\right)$
government	$c_g = \alpha\sigma(1-\sigma)\, q^g \frac{M_g}{am}$;	$y_g = \alpha(1-\sigma)\,\sigma_g q^g$
aggregate	$c + c_g = y + y_g = \alpha(1-\sigma)\left(\sigma q^m + \sigma_g q^g + \sigma q^g \frac{M_g}{am}\right)$.	

Examine first the effects of an increase in the money growth rate γ. An increase in the money growth rate affects q^m in cases 1 and 2 in opposite directions. This can be seen from Figure 2, where the solution for q^m is q_1 for $\gamma < \gamma_0$ (case 1) and q_2 for $\gamma > \gamma_0$ (case 2). When $\gamma < \gamma_0$, the money constraint does not bind in a trade between two private agents, but binds in a trade between a private money holder and a government seller. An increase in γ shifts the purchasing power of money from trades with government sellers to trades between private agents, thus increasing q^m. When $\gamma > \gamma_0$, the money constraint binds in all trade matches. By reducing a buyer's real balance, an increase in γ reduces q^m. Therefore, the amount of goods traded between two private agents increases with γ for $\gamma < \gamma_0$ and decreases with γ for $\gamma > \gamma_0$. At $\gamma = \gamma_0$, the quantity of goods in such trades reaches the maximum.

Despite the above ambiguity, an increase in the money growth rate always reduces the quantity of goods traded between a private buyer and a government seller, q^g. This is because money growth reduces the real money balance and makes the binding money constraint in such a trade tighter. Moreover, the reduction in q^g is more than the possible increase in q^m, if q^m ever increases with money growth. Thus, a higher money growth rate always reduces private consumption. It also reduces government output, government consumption, and aggregate output/consumption (note that a increases in γ).

Private output consists of the goods sold to private agents and to government agents. Because the first part responds to money growth non-monotonically and the second part always decreases with money growth, the overall response of private output to money growth depends on both the level of money growth and the ratio between the two parts of private output. If the ratio of government money holdings to private holdings is low, i.e., if M_g/m is significantly smaller than $a\sigma_g/\sigma$, then most of private output is sold to private agents. In this case, private output responds to money growth in a hump-shaped pattern, although the hump does not necessarily occur at γ_0. If $M_g/m \geq a\sigma_g/\sigma$, private output decreases with money growth for all $\gamma > \underline{\gamma}$.

I now examine a tightening open market operation, modelled as an increase in z with corresponding changes in transfers to maintain the money growth rate at γ.[19] A tightening operation shifts money from the goods market to the bonds market; i.e., it reduces a. Given the money growth rate, however, this shift has no effects on (q^m, q^g). So, private consumption and the government output remain the same as before. The price level falls. Since the government's money holdings do not change, the lower price enables the government to purchase more goods from the private sector, leading to higher private output and aggregate output.

These real effects were absent in the previous model where the government does not participate in the goods market. They are also absent in models where the goods market is Walrasian, e.g., the certainty counterpart of Lucas (1990). There, a fully anticipated increase in open market operations changes only the price level and other nominal variables, provided that monetary transfers keep the money growth rate constant.

Noticing that government consumption increases to respond to the tightening operation in this model, one may argue that a standard cash-in-advance model can also generate real effects

[19] To keep m growing at γ, monetary transfers satisfy:

$$\frac{L}{m} = \gamma - 1 - \left(1 - \frac{\beta}{\gamma}\right) z + \alpha \left(\sigma_g a - \sigma \frac{M_g}{m}\right).$$

of the operation if government spending is assumed to increase with the tightening open market operation. The question, of course, is why government spending necessarily increases to accompany a tightening operation. The current model provides a mechanism that is consistent with the model's assumption that the government uses only money to purchase goods. Moreover, government spending in a standard model does not generate the same effects as in the current model. For example, government spending is likely to crowd out private consumption in a standard model, which does not occur in the current model.

6. Conclusion

In this paper I construct two search models in which fiat money coexists in equilibrium with interest-bearing, default-free nominal bonds. In both models, there is no restriction that exogenously excludes matured bonds from circulating in the goods market. In equilibrium, newly issued bonds are sold at a discount for money and thus they bear positive interest, regardless of whether matured bonds circulate as a medium of exchange. In the first model, matured bonds circulate in the goods market as perfect substitutes for money and there are a continuum of stationary equilibria of this sort. In the second model, matured bonds do not circulate in the goods market and there is a unique stationary equilibrium. I examine the effects of money growth and open market operations in these two economies.

The two models have different predictions on the effects of monetary policies. Take, for example, a permanent increase in the issuing of bonds in the open market, accompanied by a change in the lump-sum monetary transfer to maintain the growth rate of privately held money stock constant. In both models, the open market operation shifts money from the goods market to the bonds market. In the first model, this shift of money between the two market has no effect on real output/consumption or the nominal interest rate. If households do not change the redemption fraction of matured bonds, the tightening operation increases the bond-money ratio and *increases* the total amount of media of exchange in the goods market, thus increasing the price level. If households change the redemption fraction to respond to the open market operation, the response of the price level is different to predict. In the second model, the shift of money between the two markets reduces the price level. Also in contrast to the first model, real output and government consumption increase in response to the tightening operation, while private consumption and government output remain unchanged.

Money growth has different effects in the two models. Consider an increase in the money growth rate, achieved by increases in lump-sum monetary transfers. When bonds circulate as a medium of exchange (i.e., in the first model), higher money growth reduces real output and consumption. When bonds do not circulate as a medium of exchange (i.e., in the second model), real output rises with money growth if the money growth rate is initially low, and falls when the money growth rate is high. However, higher money growth increases the nominal interest rate in both models.

The results in this paper cast doubts on the effects of policies obtained in traditional monetary models that assume money as the unique medium of exchange. Once nominal bonds are allowed to circulate, as in the first model here, the continuum of equilibria presents a difficulty for predicting the effect of open market operations on the price level. To avoid the continuum of equilibria, one can introduce legal restrictions, as is done in the second model in this paper. However, an explicit monetary model with legal restrictions reveal some policy effects that would not be discovered in a heuristic monetary model. For example, an increase in bond issuing leads to higher private output.

Toward the goal of combining the microfoundation of monetary theory with Lucas's (1990) analysis, there is still a distance to go from this paper. In the ongoing research, I am extending the framework in two directions. The first is to allow bonds to have longer maturities. With this extension, the legal restriction in the goods market still succeeds in driving out matured bonds from circulation, but it may not necessarily drive out unmatured bonds. If unmatured bonds circulate in the goods market, they are imperfect substitutes for money. This introduces deeper discounting on newly issued, long-term bonds, and hence generates a non-trivial term structure of interest rates. The second extension is to make the model stochastic so that open market operations can generate the so-called liquidity effect, as in Lucas (1990). Since the operations also have non-trivial real effects like those in the second model here, there is an interesting interaction between these real effects and the liquidity effect.

References

[1] Aiyagari, Rao, Wallace, Neil and Randall Wright, 1996, "Coexistence of money and interest-bearing securities," *Journal of Monetary Economics* 37: 397-419.

[2] Green, Edward and Ruilin Zhou, 1998, "A Rudimentary Model of Search with Divisible Money and Prices," *Journal of Economic Theory* 81, 252-271.

[3] Kiyotaki, Nobuhiro and Randall Wright, 1991, "A Contribution to the Pure Theory of Money," *Journal of Economic Theory* 53, 215-235.

[4] Kiyotaki, Nobuhiro and Randall Wright, 1993, "A Search-Theoretic Approach to Monetary Economics," *American Economic Review* 83, 63-77.

[5] Kocherlakota, Narayana, 2001, "Optimal Co-existence of Money and Nominal Bonds," manuscript, University of Minnesota.

[6] Lagos, Ricardo and Randall Wright, 2001, "A Unified Framework for Monetary Theory and Policy Analysis," Manuscript, University of Pennsylvania.

[7] Li, Yiting, 1998, "Middlemen and Private Information," *Journal of Monetary Economics* 42, 131-159.

[8] Lucas, Robert E., 1990, "Liquidity and Interest Rates," *Journal of Economic Theory* 50, 237-264.

[9] Molico, Miguel, 1997, *The Distribution of Money and Prices in Search Equilibrium*, Ph.D. dissertation, University of Pennsylvania.

[10] Shi, Shouyong, 1995, "Money and Prices: A Model of Search and Bargaining," *Journal of Economic Theory* 67, 467-496.

[11] Shi, Shouyong, 1996, "Credit and Money in a Search Model with Divisible Commodities," *Review of Economic Studies* 63, 627-652.

[12] Shi, Shouyong, 1997, "A Divisible Search Model of Fiat Money," *Econometrica* 65, 75-102.

[13] Shi, Shouyong, 1999, "Search, Inflation and Capital Accumulation," *Journal of Monetary Economics* 44, 81-103.

[14] Shi, Shouyong, 2001, "Liquidity, Bargaining, and Multiple Equilibria in a Search Monetary Model," *Annals of Economics and Finance* 2, 325-351.

[15] Trejos, Alberto and Randall Wright, 1995, "Search, Bargaining, Money and Prices," *Journal of Political Economy* 103, 118-141.

[16] Wallace, Neil, 2001, "Whither Monetary Economics," *International Economic Review* 42, 847-869.

A. Proof of Proposition 4.1

For $0 < r < 1$ to be consistent with an equilibrium, matured bonds must be perfect substitutes for money in the private goods market, i.e., $\omega^b = \omega^m$ (see (2.13)). The constraint (2.4) must bind, i.e., $\lambda^b > 0$; otherwise (2.18) would require $\omega_{-1}^m/(\beta\omega^m) = 1$, which would in turn require $\gamma = \beta$. For $\lambda^b > 0$, it must be true that $n < 1 - \sigma$, because $b/(1 - \sigma - n) = \infty$ otherwise. The choice $n \in (0, 1 - \sigma)$ is optimal iff $q^m = q^b$ (see (2.16)). Denote this common value of q^m and q^b by q^c. Substituting λ^b from (2.12) into (2.18) and using $\omega_{-1}^m/\omega^m = \gamma$, I have:

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma \left(\frac{u'(c)}{\psi'(q^c)} - 1 \right). \tag{A.1}$$

Clearly, $\psi'(q^c) < u'(c)$ for all $\gamma > \beta$. So, (2.12) implies $\lambda^m > 0$, which in turn implies $q^g = q^m = q^c$ by (4.10). Substituting λ^m from (2.12) and λ^g from (4.4) into (4.8), I get:

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma \left(\frac{u'(c)}{\psi'(q^c)} - 1 \right) + \alpha\sigma_g \left(\frac{u'(c)q^c}{\psi(q^c)} - 1 \right). \tag{A.2}$$

Because $\psi(q^c)/q^c < \psi'(q^c) < u'(c)$, (A.1) and (A.2) cannot both hold. Therefore, there does not exist a stationary equilibrium with $0 < r < 1$. **QED**

B. Proof of Proposition 5.1

The condition $0 < z < \gamma/\beta$ is required for $a \in (0, 1)$ in both cases 1 and 2, because $a = 1 - z\beta/\gamma$. When $[q\psi'(q)/\psi(q)]$ is a constant, the constant is $\underline{\gamma}$ and $\gamma_0 = \underline{\gamma}$. Then, case 1 exists only at $\gamma = \underline{\gamma}$. The equilibrium for $\gamma > \underline{\gamma}$ is given by case 2. When $[q\psi'(q)/\psi(q)]$ is strictly increasing for all $q > 0$, the condition $\gamma > \underline{\gamma}$ is required for $q_1 > 0$. For $\gamma \leq \gamma_0$, the solutions to (5.1) and (5.4) satisfy $q_1 \leq q_0 < q_2$, and so case 1 characterizes the equilibrium. For $\gamma > \gamma_0$, the solutions to (5.1) and (5.4) satisfy $q_2 < q_0 < q_1$, and so case 2 characterizes the equilibrium.

To show that these solutions constitute a unique equilibrium, I need to verify that an individual household does not have incentive to deviate to $r < 1$ when all other households choose $r = 1$, $n = 1-\sigma$, and trade according to the described quantities. Consider the following deviation by an individual household. The household keeps an amount, ε, of bonds that matured in the *previous* period and uses them to trade for goods in the current period. All other households continue to use the equilibrium strategies. This means that, if they receive any bond in the trade, they will value it at $\Omega^b < \Omega^m$.

Because trades are decentralized, the deviating household must allocate some buyers to trade with the bonds. Let Δ be the measure of the household's buyers who carry the bonds into the goods market, each carrying an amount ε/Δ. Each of these bond holders has a trade match with a private seller with probability $\alpha\sigma$. Let me compute the benefit and cost of the deviation.

The benefit comes from potential trades with private sellers. When meeting a private seller in a trade match, a bond holder can offer the amount of bonds for q_ε units of goods. The seller will accept the trade, as long as $\Omega^b\varepsilon/\Delta \geq \psi(q_\varepsilon)$. Thus, the bond holder will offer q_ε such that $\Omega^b\varepsilon/\Delta = \psi(q_\varepsilon)$, i.e., $q_\varepsilon = \psi^{-1}\left(\Omega^b\varepsilon/\Delta\right)$. Because each bond holder gets such a trade with probability $\alpha\sigma$, the total number of such trades is $\alpha\sigma\Delta$ and the total utility generated from such trades is $\alpha\sigma\Delta u'(c)q_\varepsilon$. The amount of bonds that the household fails to trade away is $(1-\alpha\sigma)\,\Delta\,(\varepsilon/\Delta) = (1-\alpha\sigma)\,\varepsilon$. Thus, the total benefit of the deviation is

$$\alpha\sigma\Delta u'(c)q_\varepsilon + (1-\alpha\sigma)\,\varepsilon\omega^b.$$

The deviation has two costs. The first cost is the foregone value of money. If the household did not deviate in the previous period, the amount ε of bonds would be redeemed for ε units of money, which would have a value $\varepsilon\omega^m_{-1}/\beta$. The second cost is that allocating Δ buyers to hold bonds takes them away from trading with money. Because each trade by a money holder generates a surplus $u'(c)\left[q^m - \psi(q^m)/\psi'(q^m)\right]$ (recall that the surplus from trading with a government seller is zero), the second cost of the deviation is $\alpha\sigma\Delta$ times this amount. Therefore, the total cost of the deviation is

$$\varepsilon\frac{\omega^m_{-1}}{\beta} + \alpha\sigma\Delta u'(c)\left(q^m - \frac{\psi(q^m)}{\psi'(q^m)}\right).$$

The net gain from the deviation, divided by Δ, is less than the following amount:

$$\alpha\sigma u'(c)q_\varepsilon - \left(\frac{\gamma}{\beta} - 1 + \alpha\sigma\right)\psi(q_\varepsilon) - \alpha\sigma u'(c)\left(q^m - \frac{\psi(q^m)}{\psi'(q^m)}\right),$$

where I have substituted $\omega^b\varepsilon/\Delta = \psi(q_\varepsilon)$, $\omega^b < \omega^m$, and $\omega^m_{-1}/\omega^m = \gamma$. Denote the above expression temporarily as $f(q_\varepsilon)$. For the deviation to be not profitable, it is sufficient that $f(q_\varepsilon) \leq 0$ for all $q_\varepsilon \geq 0$. Since f is concave, it is maximized at q^*_ε which solves:

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma\left(\frac{u'(c)}{\psi'(q^*_\varepsilon)} - 1\right).$$

Using this definition to substitute $(\gamma/\beta - 1)$ in $f(q^*_\varepsilon)$, I can show that $f(q^*_\varepsilon) \leq 0$ iff $q^*_\varepsilon - \psi(q^*_\varepsilon)/\psi'(q^*_\varepsilon) \leq q^m - \psi(q^m)/\psi'(q^m)$. Because $[q - \psi(q)/\psi'(q)]$ is an increasing function of q, $f(q^*_\varepsilon) \leq 0$ iff $q^*_\varepsilon \leq q^m$. In turn, $q^*_\varepsilon \leq q^m$ iff

$$\frac{\gamma}{\beta} - 1 \geq \alpha\sigma\left(\frac{u'(c)}{\psi'(q^m)} - 1\right).$$

This is satisfied in both cases 1 and 2 (see (5.1) and (5.4)). Therefore, the deviation is not profitable. **QED**