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Abstract

I identify circumstances in which an agent wants to make a costly but unobservable irreversible investment that affects the subsequent noisy economic environment. In equilibrium, rivals may eventually infer that the agent is strong even though it initially appeared weak, so long as enough "strength" is seen subsequently.

Comparative statics reveal that the higher is the rivals' opportunity cost, the more likely is the agent not to make the costly investment in equilibrium. In addition, as the amount of noise decreases, the probability that the agent invests increases and the probability that a challenge occurs decreases.

JEL classifications: 026, 611.

Keywords: Chain-store paradox, Discrimination, Entry deterrence, Investment, Noise, Reputation, Signalling, Takeovers

1. Introduction

The levels of investment made by agents often affect the behavior of those with whom they interact. When these investments are not observable publicly[1], agents may form beliefs about them that are based on the history that they observe, since this history may depend on past irreversible investment. I identify circumstances in which an agent wants to invest in a costly but unobservable irreversible characteristic that affects the subsequent economic environment. In the setting that I analyze, the agent may be periodically challenged by rivals who are unsure of the level of investment made by the agent. The presence of noise makes it impossible for rivals to infer the investment exactly from the history of observed outcomes. However, the history of observed outcomes enables rivals to make an educated guess about the agent's likelihood of having made the investment. If the investment has not been made, the agent may be worse off if challenged by rivals. If the investment has been made it is not worthwhile for rivals to challenge the agent.

There are many interpretations of the model. For example, one can interpret the model as one of takeovers in a climate in which price wars to initiate takeovers are illegal. In this case, the "agent" is a conglomerate. The "rivals" are firms that the conglomerate wishes to take over. The conglomerate can invest by buying a "bogus independent" (a subsidiary that is acquired privately (Burns [1986])). The "rivals" challenge the firm by not cooperating in a takeover. The model can be used to analyze the record of wins and losses that are consistent with a conglomerate that has a reputation for being strong.

Alternatively, the agent might be an established stockbroker whose rivals are other stockbrokers. In this case, a rival might challenge the agent by using a new investment strategy. The model can then be used to analyze the array of characteristics in which stockbrokers might invest.

Another interpretation of the model is as one of human capital investment. The "agent" is an individual who invests in human capital. (Even if the quantity of education is observable the effort expended may not be.) The "rivals" are interpreted as competitors who decide whether to challenge the individual's position. Suppose that the "agent" is a member of a group that is discriminated against. In this case, the model can be used to address the question of how investment in education depends on the level of discrimination directed against the agent. It may also be used to predict something about

[1]For an analysis of publicly observable investments, see Klein and Leffler [1981]. In Klein and Leffler [1981] an agent invests in a costly, enduring, publicly observable characteristic. Thus, the investment is a credible signal of the investor's future behavior and so determines the behavior of others. A high level of advertising is an example of such a public investment.

the distribution of skills of gainfully employed "minority" individuals relative to those of the gainfully employed majority.

The formal model can also be placed in the context of entry deterrence in related markets for heterogeneous goods. The "agent" is the incumbent. The "rivals" are the prospective entrants. The irreversible unobservable investment can be interpreted as research and development into a diversified portfolio of projects. The entrants "challenge" the incumbent by introducing currently unobservable substitute goods in related markets.

The outline of the paper is as follows. The details of the model are given in the next section. The equilibria are analyzed in section 3. The comparative statics are given in section 4. I conclude by interpreting the predictions of the model within the multiple frameworks offered above.

2. The Model

I give a verbal description of the model and then discuss the details and my choice of simplifying assumptions. An agent is engaged in a number of different related activities. Initially, the agent chooses whether or not to invest in a diversified portfolio of characteristics; this investment is unobservable and takes place before any challenges occur. In each period the agent may be challenged by a rival who competes with the agent in a characteristic that is initially unobservable to the agent. The characteristic in which the rival competes is made known to the agent only at the time that a challenge takes place. The set of strategies available to the agent when challenged depends both on whether or not the agent has invested in the initial period and on chance, as detailed below. In particular, whether a winning strategy is available when a challenge occurs also depends on the level of investment as well as on chance. (A winning strategy is one that enables the agent to render the challenge ineffective. Thus, in the face of a winning strategy played by the agent, the rival's chosen characteristic fizzles when used to compete with some characteristic or set of characteristics introduced by the agent in response to the challenge.) In particular, even an agent who has not invested may, with luck, have a winning strategy when challenged. For example, the wealth of knowledge accumulated by the agent may sometimes be enough to rout a challenge. Whether the investment has been made or not, the agent always prefers to play a winning strategy if one is available. Thus, when it exists, a winning strategy is a strictly dominant strategy. The rivals cannot observe whether the investment has been made but can observe the number of challenges and the number of wins. If a rival challenges the agent, the probability that the agent wins depends on whether the investment has been made. Because of the presence of noise, the rivals cannot infer the level of investment by noting the response to a particular challenge. Once a challenge occurs, all remaining rivals observe whether the outcome was successful or not (from the agent's point of view). Thus, information about the agent is revealed only through the history of outcomes of challenges by rivals. The details are as follows.

There are $N+1 \geq 3$ players. Player 0 is the agent; the remaining N players are the potential rivals, each of whom may introduce a "new" characteristic that is currently unobserved by the agent. The game is played over a sequence of $N+1$ consecutive periods. In period 0, player 0 chooses whether or not to invest in acquiring a set of characteristics; this investment is unobservable and costs **C**. The random result of the investment is a feasible set of strategies that the agent may use to respond to challenges by prospective rivals. Player j $(= 1, \ldots, N)$ can challenge agent 0 in period j by competing in a "new" characteristic. If player 0 has made the investment then it is referred to as "strong"; the probability that it has a winning strategy when challenged in any period is $1-\sigma$, $0 \leq \sigma \leq .5$. If player 0 has not made the investment then it is "weak"; the probability that it has a winning strategy when challenged in any period is[2] σ. Thus, if the agent made the investment, then with probability $1-\sigma$ the rival experiences a loss after challenging the agent; if not, then with probability $1-\sigma$ the rival wins. The rivals can observe only whether previous rivals won or lost. Thus, rival j knows the following statistic in period j: the number of previous successes of the agent and the number of previous failures. Information about the agent is revealed (though not perfectly) if a challenge takes place. No information about the agent is revealed if no challenge takes place.

At the beginning of each period $j = 1, \ldots, N$, player j must decide between IN and OUT. This decision is immediately made known to all the agents. The decision IN (OUT) is a choice to challenge (not challenge) the agent in period j by competing in some "new" characteristic. If player j chooses OUT then no information is revealed about player 0 in that period. If player j chooses IN and so challenges player 0 then player 0 is successful or not as discussed above. The extensive form of the game (without payoffs) in the case that $N = 2$ is illustrated in Figure 1. If player 0 is strong, then the payoffs to the agent and the rival respectively in that period are given in the top table in Figure 2 below. I refer to the top table as S. If not (if player 0 is weak), then they are given in the bottom table. I refer to the bottom table as W.

[2] The assumption that the probability of success when the agent is strong is equal to the probability of failure when weak is made for technical ease. (See the appendix.) The qualitative results do not depend on it. See the footnotes in Section 3.1 for implications of relaxing this assumption.

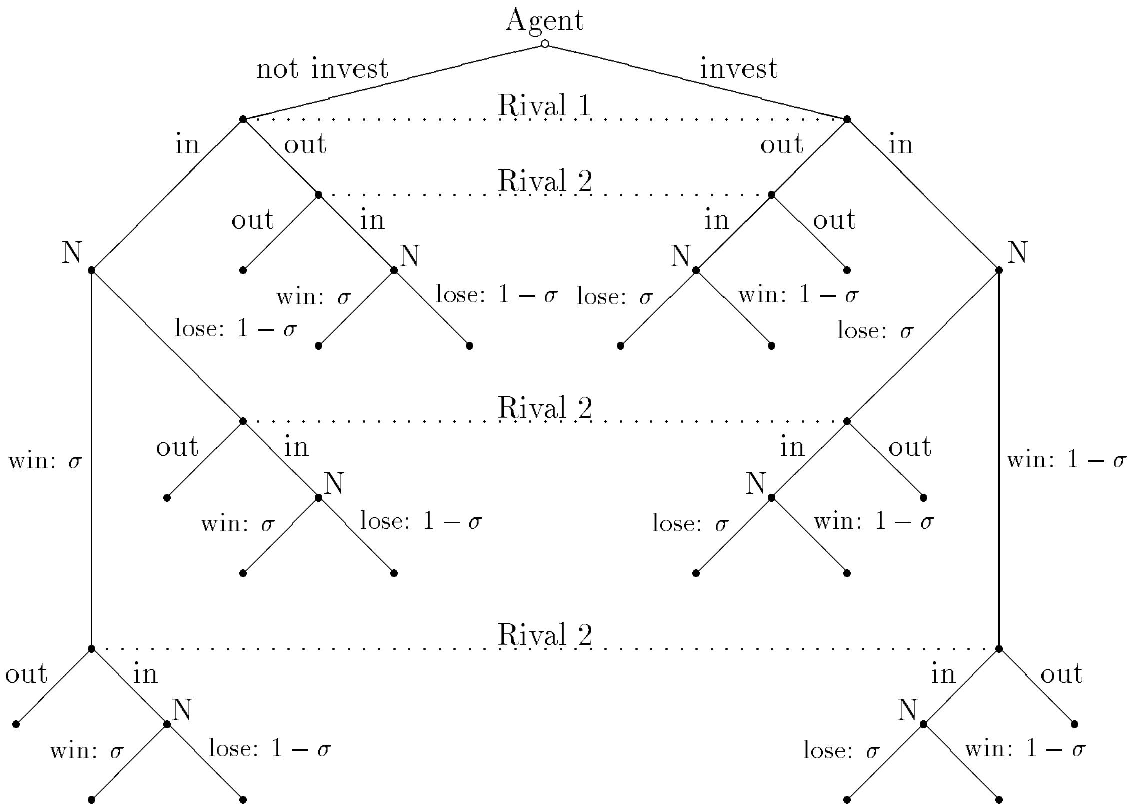

Figure 1. The extensive game for $N = 2$. Player N is nature.

I assume that:

(i) $\mathbf{M} > \mathbf{S} > \mathbf{U}$: the agent prefers that rivals do not challenge it. If challenged, the agent is better off if it wins in that period[3].

(ii) $\mathbf{B} > \mathbf{O} > \mathbf{A}$: the rival prefers to challenge only if the agent loses

(iii) $(\mathbf{O} - \mathbf{A})/(\mathbf{B} - \mathbf{A})$, $(\mathbf{B} - \mathbf{O})/(\mathbf{B} - \mathbf{A}) > \sigma$: the rival is (not) deterred from choosing IN if the agent invests with probability one (zero).

(iv) $\mathbf{O} > (\mathbf{A} + \mathbf{B})/2$: if the rival chooses OUT then the rival prefers, ex post, that the agent did invest. Note that (iii) and (iv) imply that $(\mathbf{O}-\mathbf{A})/(\mathbf{B}-\mathbf{A}) > (\mathbf{B}-\mathbf{O})/(\mathbf{B}-\mathbf{A}) > \sigma$ and that $.5 > (\mathbf{B}-\mathbf{O})/(\mathbf{B}-\mathbf{A})$.

(v) $N\mathbf{m} > N\mathbf{M} - \mathbf{C}$: the investment is too costly to make if no rival ever chooses IN (recall that $\mathbf{C}$ is the cost of investment).

[3]In this case, the agent prefers to commit to be strong if this is costless.

			Rival	
			IN	OUT
Player 0 is strong:	Nature	$1-\sigma$	$(\mathbf{S}, \mathbf{A})$	$(\mathbf{M}, \mathbf{O})$
		σ	$(\mathbf{U}, \mathbf{B})$	$(\mathbf{M}, \mathbf{O})$

			Rival	
			IN	OUT
Player 0 is weak:	Nature	σ	$(\mathbf{S}, \mathbf{A})$	$(\mathbf{m}, \mathbf{O})$
		$1-\sigma$	$(\mathbf{U}, \mathbf{B})$	$(\mathbf{m}, \mathbf{O})$

Figure 2. The payoff tables S and W.

(vi) $N\mathbf{U} < N\mathbf{S} - \mathbf{C}$: if all rivals choose IN and $\sigma = 0$, the agent prefers to invest so that it is strong enough to win each challenge rather than remain weak and lose in each period.

Player 0's payoff is the undiscounted sum of the N period payoffs minus the cost $(\mathbf{C})$ of investment (when investment takes place). The payoff of each rival is given by the appropriate cell in the matrix S or W.

I now discuss some of my simplifying assumptions. I have assumed that the investment is unobservable, while in reality, it may be observed with some uncertainty. I make the polar assumption for simplicity. The agent invests in a diverse portfolio so as to have an advantage in the event that a rival chooses to compete along some dimension that is initially unobserved by the agent. The full details of an agent's investment program are not publicly observable. It may be very difficult for rivals to observe directly how much effort the agent has made in preparing for the possibility of a challenge.

I assume that rivals keep track only of the agent's wins and losses and not on the way in which the win or loss occurred. This is a direct result of my assumption that the presence or absence of the investment and the level of noise determine a distribution over the outcomes when a rival chooses IN. This assumption has implications for the way that rivals update their beliefs about the agent's investment decision. In particular, the presence or absence of a winning strategy in the agent's strategy space is not a clear indicator of whether the agent made the investment in period 0. The presence of a winning strategy could be the result of factors outside the model.[4] In reality, the optimal response could be quite complicated and not perfectly observable

[4] For example, shocks in demand or other economic factors may affect the outcome.

to the rivals.[5] The agent has many ways of responding to a challenge.[6] Or, as is often the case, it may not be clear whether any particular action is a best response to a challenge or not. Nonetheless, it may be clear whether or not the rival won.[7] The rival can observe and keep track of whether it won or lost. If the agent made the investment then with probability $1 - \sigma$ it will be successful if a challenge occurs; if not, then with probability $1 - \sigma$ it will lose if a challenge occurs. Thus, information about the agent is revealed (though not perfectly) if a challenge takes place.

Now I ascertain the fit between the abstract model and each interpretation discussed in the introduction. Consider the interpretation of takeovers. If the conglomerate makes the investment then with high probability it can use the "bogus independent" to initiate price wars that decrease the price of the firms that it wants to take over. If the investment is not made then with low probability a price war still occurs, due to random events in the market. Since price wars are illegal the conglomerate cannot announce directly that it has purchased the "bogus independent".

Consider the "agent" as a stockbroker who invests in acquiring information. If the information is about stocks, then the high variance in realized stock returns is equivalent to a high value of σ. If, instead, the information acquired is client based then, while the distribution of individual thresholds of risk bearing shift over time, an individual's standing relative to a cohort remains stable over time. Thus σ is low.

Consider the "agent" as an individual who invests in human capital. Suppose that the "agent" is a member of a group that is discriminated against. The more intense the discrimination, the more noisy the environment and the more difficult it is for distinguished individuals to signal their high skills to a prospective employer.

Lastly, consider the "agent" as an incumbent who invests in research and development so as to be able to rout an entrant who introduces a "new" substitute product. The set of strategies available to the incumbent when challenged in this way depends not only on whether the research is undertaken but also

[5]In a world of imperfect information, it may not even be clear to the protagonist what is the optimal one period response. There is often a variety of forms a response can take.

[6]Within the framework of entry deterrence for example, in addition to innovations and prices, one may include coupon issuing and advertising as legitimate forms of predatory responses. See Hilke and Nelson [1987] and Schmalensee [1979] for some case studies of responses to entry in the coffee and lemon juice markets.

[7]For example, within the framework of entry deterrence, if the incumbent is a major firm among small firms in an industry and is threatened by entry from another potential major firm, then it is possible for both the incumbent and the entrant to gain at the expense of the smaller firms. See Schmalensee [1979].

on whether the research is successful in meeting the particular challenge posed by the rivals. Once a challenge has been made by one rival, future rivals must update their beliefs about the ability of the incumbent to withstand future challenges. Was the outcome of the challenge largely due to good or bad luck? Or was the outcome influenced more by the incumbent's investment in research and development?

As the details of the formal model are closely related to those of models on entry deterrence and on research and development, I now make a comparison between the assumptions of the present model and those in the related literature.

The literature on research and development has dealt mostly with a race toward a specific innovation of a particular technology, the first firm to innovate being the winner. (For a survey see Reinganum [1989].) In contrast, under the "entry deterrence" interpretation, I consider research and development into a range of products that may or may not foreclose opportunities of entry into related markets. I assume that there are many paths of research that may lead a firm to produce a specific end product. In addition, research can enable a firm to produce a close substitute that improves upon an innovated product introduced by a competitor. Thus, the first firm to innovate is not necessarily a winner; the winner of the race is declared only after all innovations are accounted for. I assume also that different rivals may introduce different heterogeneous products. To respond successfully to a challenge the incumbent must be able to produce a good that can compete successfully with the product introduced by the challenger. The product initially produced by the incumbent has many characteristics; the incumbent may invest in research and development in a multi-dimensional product characteristic space. The end result of this research and development is a feasible set, possibly empty, of substitute products that differ in their characteristics. If this set is large then the incumbent has more chance of being able to effectively compete with a rival that introduces a new product. I assume that there are advantages to incumbency: possibly better access to capital markets, economies of scope, experience, or accumulated knowledge. Under this assumption, as stated in Reinganum [1989], even when the innovations are drastic it is reasonable to conjecture that the incumbent may invest in research and development. Thus under the present interpretation, the research and development is undertaken as a means of enabling the agent to respond to the possible introduction of currently unknown substitute goods. When it is not too noisy and the investment is made the chances are that a winning response to the introduction of a substitute good is available to the agent, whereas, if the investment is not undertaken, a winning strategy is not likely to be available to the agent when needed.

I assume that it is not in the incumbent's interest to patent or introduce any new products when not challenged by a rival. A challenger that knows exactly the gaps in the incumbent's product line is able to introduce a product that has no successful competitor in the incumbent's feasible set.

The related literature on entry deterrence is based on the seminal paper by Selten [1978]. In the classic formulation of reputation building in a model of entry deterrence, Kreps and Wilson [1982] and Milgrom and Roberts [1982] endow an agent with an unobservable characteristic and focus on the observable actions of the agent in each period. The set of feasible actions is independent of the unobservable characteristic. In addition, the outcomes in each period depend only on the actions and not on the unobservable characteristic. Schotter [1983] is the first to ask about the endogenization of the exogenous uncertainty in Kreps and Wilson [1982]. In Pitchik [1993], an incumbent first chooses whether to invest in an unobservable characteristic for each of several markets and then chooses whether or not to fight when entry occurs. Though the set of feasible actions is independent of the unobservable characteristic, the outcomes in each period depend only on the actions and on the unobservable characteristic. In each of the aforementioned papers on entry deterrence, the optimal one stage action (share when entry occurs) is not optimal in the multi-stage game. In contrast, I highlight a one-time irreversible, unobservable investment decision, common to all markets, that takes place at the beginning of the game and that determines a unique set of actions available to the incumbent when challenged in any of the its markets. The optimal choice of action in any stage is also optimal in the multi-stage game. The set of available actions in turn largely determines the observable outcomes of any challenges made by rivals. Beliefs of the rivals depend on the history of observable outcomes and the amount of noise.

Another strand of the entry deterrence literature emanates from Dixit [1980]. In this literature an observable investment is made solely as a signal that deters future entry. Bagwell [1994] shows how even a tiny amount of measurability error (with respect to the entrant's ability to see the signal) can lead to the result that the incumbent no longer has an advantage in Dixit's framework. In contrast, I consider whether investment in an unobservable characteristic is worthwhile when its signalling power is noisy and occurs only after at least one entry has taken place.

In sections 4 and 5 I discuss other comparisons with the literature.

3. Equilibrium

The solution concept I use is sequential equilibrium. The agent does not invest with probability one in equilibrium: if it invests then each rival chooses OUT, so that by assumption (v) the agent is better off not making the investment as its acquisition is never in doubt. Thus, in equilibrium, the agent makes the investment with probability $p \in [0, 1)$.

3.1 The Signalling Equilibrium

Since the distribution of types is endogenous my model is not strictly a signalling model. However, if there is not too much noise, then there is an equilibrium in which the rivals use the history of outcomes as a signal of whether the agent is strong or weak. I refer to this equilibrium as a "signalling" equilibrium. I restrict attention to "signalling" equilibria in which the probability that a rival challenges the agent depends only on the posterior probability that the agent is strong, and not on the history of wins and losses that resulted in the posterior. The reason I do so is that the probability that the agent is strong, not the history of wins and losses per se, determines the rivals' preferences over IN and OUT. Let $E_{p,C}$ be the set of "signalling equilibria" in which the agent invests with probability p and the expected number of challenges is C. For each "signalling equilibrium" in $E_{p,C}$ in which a rival's probability of challenge depends on the history of wins and losses there is an equilibrium in $E_{p,C}$ in which each rival's probability depends only on the rival's belief that the agent is strong. It is the expected number of challenges that is instrumental in calculating the agent's expected payoff[8].

I prove in Theorem 1 that there exists a continuous real-valued function $\hat{\sigma}$ such that the "signalling" equilibrium exists when $0 \leq \sigma \leq \hat{\sigma}(\mathbf{C})$. The "signalling" equilibrium has the following structure. The agent randomizes between making the investment and not doing so. If it chooses to be strong, then, by assumption, its best response to IN wins with probability $1 - \sigma$; if it chooses to remain weak, then its best response wins with probability σ. Each rival's behavior depends on the history. If no challenges have yet occurred or if some challenges have occurred and the number of successful responses equals the number of unsuccessful ones then rival j randomizes between challenging and not challenging. If some challenges have occurred in the past and the

[8]Except for its dependence on proofs in the Appendix, the proof of theorem 1 does not depend directly on the fact that the rivals' probabilities depend only on the rival's belief that the agent is strong. However, one can adapt the proofs in the appendix so as to account for probabilities that depend on the history of wins and losses.

number of successful responses is greater than that of unsuccessful ones then rival j chooses OUT. If some challenges have occurred in the past, and the number of unsuccessful responses is greater than that of successful ones then rival j chooses IN.

Let p be the probability that the agent committed to be strong. Let $p_j(p, k, s)$ (where k denotes the number of previous challenges and s the number of successful responses to a choice of IN) denote agent j's posterior probability that the agent committed to be strong. The posterior probability is then

$$p_j(p, k, s) = \frac{p(1-\sigma)^s \sigma^{k-s}}{p(1-\sigma)^s \sigma^{k-s} + (1-p)(1-\sigma)^{k-s} \sigma^s}$$

In particular, if $s = k - s$ then the posterior probability equals the original prior. Thus, if p is chosen so that the rivals are initially indifferent between IN or OUT then they are again so if the number of successful responses equals the number of unsuccessful responses.[9] In addition, if $k = 1$ and $\sigma < .5$ then the posterior probability of commitment is greater than the prior if $s = 1$ and is less if $s = 0$. (To see this, simply note that the posterior probability equals p if $\sigma = .5$ and is decreasing in σ if $s = k = 1$ but is increasing in σ if $s = 0$, $k = 1$.)

Let q_j, $j = 1, \ldots, N$, be the probability that agent j chooses IN when no challenges have yet taken place. If the agent is strong then one can compute the expected number, $Ex(\mathbf{s}, \sigma)$, of challenges given q_j, $j = 1, \ldots, N$. One can also compute the expected number, $E(\mathbf{s}, S, \sigma)$, of rivals who lose after challenging the agent. Analogously, replacing the first component by $\mathbf{w}$ in Ex and E gives us the equivalent expected numbers in the case that the agent is weak. Note that $Ex(\mathbf{s}, \sigma) = Ex(\mathbf{w}, 1-\sigma)$ and $E(\mathbf{s}, S, \sigma)) = E(\mathbf{w}, S, 1-\sigma)$. (This is again due to the fact that σ is both the probability of failure of a committed agent and the probability of success of an uncommitted agent.) Thus, I can now represent the expected payoff[10] of the agent. Let Π_w (respectively Π_s) denote the payoff of the agent when it is weak (respectively strong) and the rivals behave as specified above:

$$\Pi_w(q_1, \ldots, q_N) = \mathbf{S}E(\mathbf{w}, S, \sigma) + \mathbf{U}(Ex(\mathbf{w}, \sigma) - E(\mathbf{w}, S, \sigma)) + \mathbf{m}(N - Ex(\mathbf{w}, \sigma))$$
$$\Pi_s(q_1, \ldots, q_N) = \mathbf{S}E(\mathbf{s}, S, \sigma) + \mathbf{U}(Ex(\mathbf{s}, \sigma) - E(\mathbf{s}, S, \sigma)) + \mathbf{M}(N - Ex(\mathbf{s}, \sigma)) - \mathbf{C}$$

[9] If the probability of success of a strong agent is $1-\sigma_1$ and that of a weak one is σ_2, then the rivals' indifference is restored when $s/(k-s) = \ln[(1-\sigma_2)/\sigma_1]/\ln[(1-\sigma_1)/\sigma_2]$. Thus, if $.5 > \sigma_2 > \sigma_1$, the number of successes must be greater than the number of failures to restore indifference.

[10] See the proof of proposition 1 in the appendix for an explicit derivation of the payoff functions.

Theorem 1 *A continuous function* $\hat{\sigma} : [0, \infty) \to [0, (\mathbf{B} - \mathbf{O})/(\mathbf{B} - \mathbf{A})]$ *exists such that the "signalling" equilibrium exists if and only if* $\sigma \in [0, \hat{\sigma}(\mathbf{C})]$. *The function* $\hat{\sigma}$ *decreases from* $(\mathbf{B} - \mathbf{O})/(\mathbf{B} - \mathbf{A})$ *to* 0 *as* $\mathbf{C}$ *increases from* $\max\{0, N\mathbf{M} - N\mathbf{m}\}$ *to* $\mathbf{S} + (N - 1)\mathbf{M} - N\mathbf{U}$. *In this equilibrium*

(i) rivals are indifferent between challenging or not and choose each with positive probability so long as the number of successes equals the number of failures

(ii) if a challenge occurs then subsequent rivals choose IN only if the number of failures is greater than the number of successes

(iii) the agent chooses to be strong with probability

$$p = \frac{\mathbf{B}(1 - \sigma) + \mathbf{A}\sigma - \mathbf{O}}{(\mathbf{B} - \mathbf{A})(1 - 2\sigma)}$$

Proof. **The rival**: If the rivals are initially indifferent between IN and OUT then the evolution of the posterior probability implies that the optimal behavior of the rivals is as stated in (i). Each rival is indifferent (and thus is willing to randomize) between IN and OUT if no challenges have yet taken place only if the agent commits with probability

$$p = \frac{\mathbf{B}(1 - \sigma) + \mathbf{A}\sigma - \mathbf{O}}{(\mathbf{B} - \mathbf{A})(1 - 2\sigma)} \equiv r(\sigma)$$

I refer to $r(\sigma)$ as the reservation probability of commitment required to keep a rival indifferent. (For simplicity I do not incorporate the dependency of r on $\mathbf{O}$, $\mathbf{A}$, and $\mathbf{B}$ in the notation.) Assumptions (ii), (iii) and (iv) ensure that $0 < r(\sigma) < 1$, so that it is possible to set $p = r(\sigma)$ as required. Thus, in the "signalling" equilibrium, the agent must choose to be strong with probability $r(\sigma)$. If so, then each rival's behavior is optimal. Thus, assumptions (i) through (iv) ensure that each rival is behaving optimally.

The agent: In order that the agent is willing to randomize we need $\Pi_s(q_1, \ldots, q_N) = \Pi_w(q_1, \ldots, q_N)$. It is clear that Π_w shifts up while Π_s shifts down as σ increases. (See the appendix (proposition 2) for a proof. Intuitively, the more successful the response to a choice of IN the higher the payoff.)

If $q_j = 0$ for all $j = 1, \ldots, N$, then the expected number of rivals who choose IN is 0, independent of σ. In this case, assumption (v) guarantees that the agent strictly prefers to be weak than to be strong. If the agent strictly prefers to be strong for some q, then, by continuity, there exists q such that $\Pi_s(q) = \Pi_w(q)$. Now suppose $\sigma = 0$. Assumptions (i) and (vi) guarantee that the agent prefers to be strong if $q_j = 1$ for all $j = 1, \ldots, N$. Continuity then implies that, when $\sigma = 0$, there exists q^* such that $\Pi_s(q^*) = \Pi_w(q^*)$. Thus the "signalling" equilibrium exists when $\sigma = 0$. However, if $\sigma = .5$, then $Ex(\mathbf{s}, .5) = Ex(\mathbf{w}, .5)$ and $E(\mathbf{s}, S, .5)) = E(\mathbf{w}, S, .5)$, as noted above. But then,

$$\Pi_w(q_1, \ldots, q_N) - \Pi_s(q_1, \ldots, q_N) = \mathbf{m}(N - Ex(\mathbf{s}, .5)) - [\mathbf{M}(N - Ex(\mathbf{s}, .5)) - \mathbf{C}].$$

Assumption (v) again guarantees that the agent prefers to remain weak. Thus, if $\sigma = .5$, the agent prefers to be weak for all q. I summarize the facts as follows. When $\sigma = 0$, there exists q such that $\Pi_w(q) = \Pi_s(q)$. When $\sigma = .5$, $\Pi_w(q) > \Pi_s(q)$ for all q. As σ increases Π_w shifts up and Π_s shifts down. Thus, there exists a continuous real-valued function[11] $\hat{\beta}$ such that Π_w and Π_s intersect if and only if[12] $0 \leq \sigma \leq \hat{\beta}(\mathbf{C})$. Thus, assumptions (i), (v) and (vi) ensure the existence of such a function. However, by assumptions (iii) and (iv), $\sigma < (\mathbf{B} - \mathbf{O})/(\mathbf{B} - \mathbf{A})$ so that $\hat{\sigma}(\mathbf{C}) = \min\{(\mathbf{B} - \mathbf{O})/(\mathbf{B} - \mathbf{A}), \hat{\beta}(\mathbf{C})\}$.

See the appendix (proposition 3) for a proof of the propositions about the monotonicity and limits of $\hat{\beta}$ (and by inference, $\hat{\sigma}$). It is clear that $\hat{\beta}$ decreases in $\mathbf{C}$. Intuitively, the more costly is the characteristic the more successful must it be in order to be acquired in equilibrium. It is clear that[13] $\hat{\beta}(0) = .5$. (By symmetry, $\Pi_s = \Pi_w$ if $\sigma = .5$ and $\mathbf{C} = 0$.) It is also clear that[14] $\hat{\beta}(\mathbf{S} + (N-1)\mathbf{M} - N\mathbf{U}) = 0$. (By previous arguments, there exists a "signalling" equilibrium for all $\mathbf{C}$ when $\sigma = 0$.) □

Corollary 1 *When the "signalling" equilibrium exists, it is sequential.*

Proof. The above theorem implies that the "signalling" equilibrium is sequentially rational. It remains to show that it is consistent. We need to find a sequence of assessments (in which the strategies are completely mixed and the beliefs are derived using Bayes' rule) whose limit is the "signalling" equilibrium. Let the sequence of assessments be such that the strategies of the agent

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[11] Since $\Pi_w - \Pi_s$ is a continuous function, the zero set is compact. Thus, the maximum exists. Let $\hat{\beta}(\mathbf{C})$ be the maximum of all σ such that Π_w and Π_s intersect for some q. The fact that Π_w shifts up and Π_s shifts down when σ increases ensures that Π_w and Π_s intersect for all $\sigma \leq \hat{\beta}(\mathbf{C})$. Continuity of the profit functions in $\mathbf{C}$ ensures continuity of $\hat{\beta}$ in $\mathbf{C}$.

[12] If the probability of success is $1 - \sigma_1$ in the presence of investment and σ_2 in its absence, then there exists continuous real-valued functions $\hat{\sigma}_1$, $\hat{\sigma}_2$ such that intersection occurs if and only if $0 \leq \sigma_1 \leq \hat{\sigma}_1(\mathbf{C})$ and $0 \leq \sigma_2 \leq \hat{\sigma}_2(\mathbf{C})$.

[13] Recall that assumption (v) restricts $\mathbf{C}$ to be greater than $N\mathbf{M} - N\mathbf{m}$ so that $\mathbf{C}$ tends to 0 only if $N\mathbf{M} - N\mathbf{m} \leq 0$.

[14] Assumptions (i) and (vi) imply that $\mathbf{C} \leq \mathbf{S} + (N-1)\mathbf{M} - N\mathbf{U}$.

and rivals are as in statements (i) and (iii) in the above theorem. When the number of wins is greater than the number of losses, let the next rival enter with probability $1-\epsilon^i$ for the ith element of the sequence, where ϵ^i converges to zero. When the number of wins is less than the number of losses, let the next rival enter with probability δ^i for the ith element of the sequence, where δ^i converges to zero. Let the beliefs of the rivals be derived from the above strategies using Bayes' rule. The limit of this sequence is the "signalling" equilibrium given above. Thus, a "signalling" equilibrium is a sequential equilibrium. □

Essentially, $1-\sigma$ has to be high enough relative to $\mathbf{C}$ so that the rivals can interpret a successful response as deriving from commitment rather than noise and so that the agent finds it profitable to commit if the first agent chooses IN with probability one. In addition, the probability that an agent chooses IN has to satisfy two conditions. Firstly, it has to be high enough so that the acquisition of strength is challenged. The value of acquired strength through investment lies in its signalling ability. If this ability is never challenged then it is not worth acquiring. Secondly, the probability that a agent chooses IN has to be low enough relative to the payoffs so that it is not unattractive for the agent to remain weak. There is some value in remaining weak if the agent's ability is not challenged enough.

3.2 The All-Challenge Equilibrium

Along the all-challenge equilibrium path each rival chooses IN and the agent remains weak. If this is an equilibrium path, it must be that the agent's best response is to remain weak if all rivals choose IN. Otherwise, the agent chooses to invest. Essentially, σ must be big enough that a strong agent does not respond successfully to a choice of IN with high probability.

Theorem 2 *Let* $\sigma^*(\mathbf{C}) = [N(\mathbf{S}-\mathbf{U}) - C]/[2N(\mathbf{S}-\mathbf{U})] > 0$.

(i) *The all-challenge equilibrium exists if and only if* $\sigma \in [\sigma^*(\mathbf{C}), .5]$

(ii) *When it exists the all-challenge equilibrium is a sequential equilibrium*

(iii) $\sigma^*(0) = 0.5, \sigma^*(N(\mathbf{S}-\mathbf{U})) = 0$ *and* σ^* *is decreasing*

Proof. **Property (i)**: If $\sigma > 0$ then the rivals can interpret any successes as due to noise when the prior is that the agent is weak with probability one. Thus, if $\sigma > 0$ then the all-challenge equilibrium exists if and only if the agent prefers to remain weak in the event that all agents enter independent of the history of successes and failures. In this case the agent's payoff from remaining weak is $N[\sigma\mathbf{S} + (1-\sigma)\mathbf{U}]$ while that from acquiring stength is $N[(1-\sigma)\mathbf{S}+\sigma\mathbf{U}]-\mathbf{C}$. The former is increasing and the latter is decreasing in σ by assumption (i). At $\sigma = 0$, the latter is greater by assumption (vi). Thus the all-challenge equilibrium exists if and only if[15] $\sigma \geq [N(\mathbf{S}-\mathbf{U})-\mathbf{C}]/[2N(\mathbf{S}-\mathbf{U})]$. Consider the all-challenge assessment consisting of the strategies above and the beliefs of the rivals that the agent did not commit. The above implies that this assessment is sequentially rational.

Property (ii): To show that this assessment is consistent consider the sequence of assessments in which, in the ith element, the agent commits with probability $1-\epsilon^i$, the rivals enter with probability $1-\delta^i$ and the beliefs of the rival are derived at each information set using Bayes' rule. The limit of this sequence of assessments is the all-challenge assessment. Thus, the all-challenge outcome is a sequential equilibrium outcome when it exists.

Property (iii) follows immediately from the definition. □

It is straightforward to show that if it is noisy enough only the all-challenge equilibrium exists; if it is not, only the "signalling" equilibrium exists.

Corollary 2 *Let $\hat{\sigma}$ and σ^* be as in Theorems 1 and 2.*

(i) $\hat{\sigma}(\mathbf{C}) \geq \sigma^*(\mathbf{C})$

(ii) If $\sigma \in [0, \sigma^*(\mathbf{C}))$ then only the "signalling" equilibrium exists; if $\sigma \in (\hat{\sigma}(\mathbf{C}), .5]$ then only the all-challenge equilibrium exists. The "signalling" equilibrium and the all-challenge equilibrium simultaneously exist if and only if $\sigma \in [\sigma^*(\mathbf{C}), \hat{\sigma}(\mathbf{C})]$.

Proof. **Property (i)** is proved as follows. Suppose that $\sigma = \sigma^*(\mathbf{C}) = [N(\mathbf{S}-\mathbf{U})-\mathbf{C}]/[2N(\mathbf{S}-\mathbf{U})] > 0$. In this case, by Theorem 1 and its proof, the result follows if $\Pi_s(q) > \Pi_w(q)$ for some $q = (q_1, \ldots, q_N)$. By Theorem 2, when $\sigma = \sigma^*(\mathbf{C})$, the agent is indifferent between investing and not investing if $q_i = 1$ for all $i = 1, \ldots, N$ and the rivals choose IN independent of the history. Thus, the agent must strictly prefer to invest in the array of characteristics than not to invest if $q_i = 1$ for all $i = 1, \ldots, N$ and the rivals acquiesce with higher probability when investment has taken place. That is, $\Pi_s(1, \ldots, 1) > \Pi_w(1, \ldots, 1)$. **Property (ii)** follows immediately from Property (i) and Theorems 1 and 2. □

Investment is undertaken as a means of enabling the agent to respond to the challenges of rivals. When it is not too noisy, and the investment is made,

[15]If the probability of success is $1-\sigma_1$ in the presence of investment and σ_2 in its absence, then the condition is $\sigma_1 + \sigma_2 \geq [N(\mathbf{S}-\mathbf{U})-\mathbf{C}]/[2N(\mathbf{S}-\mathbf{U})]$.

the chances are that a winning response to a challenge is available to the agent, whereas, if the investment is not undertaken, a winning strategy is not likely to be available to the agent when needed. Thus, when it is not too noisy, investment is made with positive probability in equilibrium.

In contrast, when there is too much noise, investment is not undertaken in equilibrium. One reason is that, the more noise, the less likely is investment to result in a winning response to a challenge. Another reason is that, the more noise, the weaker is the signalling power of investment. Even when the agent responds with a winning strategy, the presence of too much noise compels the remaining rivals to attribute the outcome to luck and relentlessly enter.

4. Comparative Statics

I first consider the effect of a change in the rivals' opportunity cost ($\mathbf{O}$) of choosing IN on the "signalling" equilibrium probability that the agent is strong. This probability is equal to the rivals' reservation probability that the agent is strong, $r(\sigma) = [\mathbf{B}(1-\sigma) + \mathbf{A}\sigma - \mathbf{O}]/[(\mathbf{B}-\mathbf{A})(1-2\sigma)]$. Thus, as $\mathbf{O}$ increases, the probability that the agent is strong in equilibrium decreases. Hence the equilibrium probability that the agent is strong decreases with the rivals' reservation probability. Intuitively, if the rivals' opportunity cost (of choosing IN) increases, then it is easier for the agent to keep them out.[16]

I now analyze the effect of a change in the agent's incentives on the equilibrium probability of choosing IN in the case that $q_i = q$ for all $i = 1, \ldots, N$. The equilibrium condition is $\Pi_w(q_1, \ldots, q_N) = \Pi_s(q_1, \ldots, q_N)$ or

$$\begin{aligned}
&\mathbf{S}E(\mathbf{w}, S, \sigma) + \mathbf{U}(E(\mathbf{w}, \sigma) - E(\mathbf{w}, S, \sigma)) + \mathbf{m}(N - E(\mathbf{w}, \sigma)) = \\
&\qquad \mathbf{S}E(\mathbf{s}, S, \sigma) + \mathbf{U}(E(\mathbf{s}, \sigma) - E(\mathbf{s}, S, \sigma)) + \mathbf{M}(N - E(\mathbf{s}, \sigma)) - \mathbf{C} \qquad (1)
\end{aligned}$$

Consider the consequences of investment becoming relatively more attractive for the agent. If either $\mathbf{S}$ or $\mathbf{M}$ increases, or $\mathbf{C}$ decreases, I claim that the equilibrium values of the challenge probabilities decrease. To see this consider the coefficient of the parameter in question. For example, consider the parameter $\mathbf{S}$. The coefficient of $\mathbf{S}$ in Π_s is larger than the coefficient of $\mathbf{S}$ in Π_w. Though the expected number of rivals is smaller if the agent commits than if not, the expected number of rivals who challenge and lose is larger if the agent has committed. (Formally, this is because $\sigma < .5$ and $E(\mathbf{w}, S, \sigma) = E(\mathbf{s}, S, 1-\sigma)$ so that the power to which $1-\sigma$ is raised in Π_s (Π_w) must be larger (smaller) than the power to which σ is raised in Π_s (Π_w).

[16] This contrasts with the result in Milgrom and Roberts [1982]. In Milgrom and Roberts, the probability increases if the outside profit opportunities of the antagonists increase.

See the appendix (proposition 1) for a formal derivation of Π_s and Π_w.) Thus, Π_s shifts up by more than Π_w if $\mathbf{S}$ increases. Recall that $\Pi_s < \Pi_w$ at $q_i = 0$. It is clear that Π_s and Π_w are decreasing in q_i, $i = 1, \ldots, N$. (This is because the payoffs that are generated when a rival chooses IN are higher if the agent's response is successful. If the agent's response is unsuccessful then a choice of IN encourages others to choose IN. See the appendix (proposition 1) for a formal proof.) Hence each is decreasing in q. Thus, the new equilibrium value of q must be less than the old equilibrium value. Thus, q decreases as $\mathbf{S}$ increases. Analogous arguments produce the same result if $\mathbf{M}$ increases or $\mathbf{C}$ decreases. Thus, in a symmetric equilibrium, the effect of increasing the short-run attractiveness of being strong is to decrease the (conditional) challenge probability of each rival and thereby decrease the expected number of challenges in equilibrium. If strength becomes relatively more valuable to the agent, then it is harder for the rivals to challenge.

Similarly, if $\mathbf{U}$ or $\mathbf{m}$ increases, then it can be shown that the solution q must increase. Intuitively, if weakness increases in value relative to strength, then it is easier for the rivals to challenge.

Lastly, consider the consequences of decreasing σ. The derivative with respect to σ of $r(\sigma)$ is $(\mathbf{B} + \mathbf{A} - 2\mathbf{O})/[(\mathbf{B} - \mathbf{A})(1 - 2\sigma)^2]$ which is negative by assumption (iv). Recall that Π_w shifts down and Π_s shifts up if σ decreases. This implies that the probability of a challenge must decrease in the case that $q_i = q$ for all $i = 1, \ldots, N$. Thus, if σ decreases then the probability that the agent makes the investment increases and the probability that a challenge occurs decreases. Intuitively, the stronger is the investment the more it is acquired and the less it is challenged.

5. Conclusion

I provide conclusions of the model under the interpretations set forth in the introduction. I begin with the interpretation of the model as one of investment in R&D to deter entry by firms who introduce "new" products.

As in Dixit [1980] and Klein and Leffler [1981] the possibility that an agent in my model makes an investment has an effect on the behavior of other agents. The observable nature of the investment in Dixit [1980] and Klein and Leffler [1981] implies that the effect of investment is immediate. If instead the investment is unobservable, as I have assumed, the effect can be delayed. Even if noise is absent, information regarding the strength of the agent is delayed until some agent challenges the agent for the first time. In the presence of noise, a weak agent can masquerade as a strong agent if luck is initially on its side. Alternatively, a strong agent can recover from a hint of weakness so long as enough strength is subsequently seen.

Whether an agent makes a costly investment in my model depends on the amount of noise inherent in any signal sent by the agent. If there is too much noise (i.e. if $\sigma \in (\hat{\sigma}(\mathbf{C}), .5])$ then there is no incentive for the agent to invest. Rivals interpret any successes as due to noise and relentlessly challenge. No reputation for strength can be earned. However, if the conditional probability of success is great enough relative to the cost of investment (i.e. if $\sigma \in [0, \sigma^*(\mathbf{C}))$, the agent invests with positive probability in equilibrium. In the latter case, the "signalling" equilibrium path is quite rich in terms of reputation building. Initially, the agent can be either weak in all markets or strong in all markets. An agent who chooses to be weak is free riding on the presumed presence of the strong agent. An agent who chooses to be strong hopes to send the appropriate signals of strength if challenged. The first challenge can occur in any period, including the first. (In Kreps and Wilson [1982] a challenge can take place in equilibrium only in the end game. In Milgrom and Roberts [1982] a challenge may occur in the first period, due to the presence of rivals who enter independent of the consequences. In the present model a challenge may occur early on because it is likely enough that a weak agent is present and because a weak agent is one who most probably does not have the resources to successfully respond to a challenge.) So long as enough successful responses to choices of IN are seen, a reputation for being strong is earned and remaining rivals are discouraged from choosing IN. If enough unsuccessful responses are seen, a reputation for being weak is earned and further challenges are encouraged. However, an agent with an initial mixed record of failures and successes can still earn a reputation for being strong with a subsequent string of successes. (As stated earlier, the driving force behind this result is the assumption that the agent's initial choice leads to a random distribution over the space of possible strategies.)

In Kreps and Wilson (1982), the lack of noise coupled with the eternal presence of a unique winning strategy compels rivals to conclude that one occurrence of weakness is proof that an agent is weak. However, the presence of noise in the present model, coupled with the endogenous random presence of alternative winning strategies, allows the agent to recover from the odd share. In the absence of noise entrants conclude with certainty from one failure that the incumbent is weak.

Consider the interpretation of takeovers. If there is not too much noise (i.e. if $\sigma \in [0, \sigma^*(\mathbf{C}))$) then, in equilibrium, an "agent" with an initially mixed record of successes and failures can still earn a reputation for being strong if a suitably long string of successes occurs, an outcome that (unlike those in Kreps and Wilson [1982], Milgrom and Roberts [1982] and Pitchik [1993]) is consistent with the stylized facts in Burns [1986].

The model predicts that private investments are not made when the probability of success is too low relative to the cost of acquiring the characteristic (i.e. the signalling equilibrium does not exist when $\sigma \in (\hat{\sigma}(\mathbf{C}), .5])$. This implies something about the array of characteristics in which an agent invests. For example, in the financial sector, the high variance in realized stock returns makes it difficult for even sophisticated clients to verify the claimed ability of stockbrokers. The burgeoning of "No Frills" stockbrokers is consistent with the idea that it is difficult to acquire a superior reputation for choosing stocks, as is the fact that the large brokerage houses that exist are not used by large agents for investment advice. Brokerage houses compete at the level of service. Reputations for superior performance are not built in equilibrium.

Recall that another interpretation of the model is that of human capital investment; under this interpretation the model explains why some groups invest more in education than others. Suppose that the "agent" is a member of a group that is discriminated against. The more intense the discrimination, the more noisy the environment and the more difficult it is for distinguished individuals to signal their high skills to a prospective employer. (That is, when $\sigma \in (\hat{\sigma}(\mathbf{C}), .5]$ the signalling equilibrium does not exist. In this case, the unique equilibrium is the all-challenge equilibrium in which investment does not take place.) Thus, it may not be worthwhile for such an individual to make the initial investment. (This confirms a result obtained in Lundberg and Startz (1983).) If we assume that the cost of investment is inversely correlated with the skill of the individual then the model also predicts something about the distribution of skills of individuals who are gainfully employed in the marketplace. Suppose there is more than one well-defined group in society and that some groups are discriminated against while others are not. Suppose that the distribution of the costs of investment in human capital is independent of the group. In this case, one implication of the model is that as we look across groups, employed individuals from a more discriminated group will be more skilled at their jobs than employed individuals from a less discriminated group. (For a fixed level of σ a signalling equilibrium exists so long as $\mathbf{C}$ is small enough. In addition, when a signalling equilibrium exists, the probability of a challenge decreases as $\mathbf{C}$ decreases.)

6. Appendix

Proposition 1 *Both Π_s and Π_w decrease in q_i.*

Proof. Monotonicity follows by induction on N. Let $\Pi_s^N(q_1,\ldots,q_N) = F_1^N(q_1,\ldots,q_N) - \mathbf{C}$, where $F_i^n(q_1,\ldots,q_N)$ is the revenue to a commited agent if there are n rivals and it takes i successes to deter further challenges. Similarly, let $\Pi_w^N(q_1,\ldots,q_N) = G_1^N(q_1,\ldots,q_N)$, where $G_i^n(q_1,\ldots,q_N)$ is the revenue to a weak agent if there are n rivals and it takes i successes to deter further challenges. Since the probability of a successful response to a challenge is $1-\sigma$ (σ) if the agent is strong (weak), $G_i^n(q_1,\ldots,q_N)$ is obtained from $F_i^n(q_1,\ldots,q_N)$ by replacing σ with $1-\sigma$ and $\mathbf{M}$ with $\mathbf{m}$. The revenue $F_i^n(q_1,\ldots,q_N)$ is defined recursively as follows.

$$\begin{aligned}
F_1^1(q_1) &= q_1[(1-\sigma)\mathbf{S} + \sigma\mathbf{U}] + (1-q_1)\mathbf{M} \\
F_1^2(q_1,q_2) &= q_1[(1-\sigma)(\mathbf{S}+\mathbf{M}) + \sigma(\mathbf{U} + F_1^1(1))] + (1-q_1)[\mathbf{M} + F_1^1(q_2)] \\
F_2^2(q_1,q_2) &= q_1[(1-\sigma)(\mathbf{S}+F_1^1(1)) + \sigma(\mathbf{U} + F_1^1(1))] + (1-q_1)[\mathbf{M} + F_1^1(q_2)].
\end{aligned}$$

In general,

$$\begin{aligned}
F_1^{n+1}(q_1,q_2,\ldots,q_N) = q_1[(1-\sigma)(\mathbf{S}+n\mathbf{M}) + \\
\sigma(\mathbf{U} + F_2^n(1,q_3,\ldots,q_N))] + (1-q_1)[\mathbf{M} + F_1^n(q_2,\ldots,q_N)]
\end{aligned}$$

and

$$\begin{aligned}
F_i^{n+1}(q_1,q_2,\ldots,q_N) = q_1[(1-\sigma)(\mathbf{S}+F_{i-1}^n(q_2,\ldots,q_N)) + \\
\sigma(\mathbf{U} + F_{i+1}^n(q_2,\ldots,q_N))] + (1-q_1)[\mathbf{M} + F_{i-1}^n(q_2,\ldots,q_N)] \text{ for } i>1,
\end{aligned}$$

where $F_i^n(q_1,q_2,\ldots,q_N) = F_n^n(q_1,q_2,\ldots,q_n)$ if $i>n$.

I now proceed with the inductive proof. F_1^n is decreasing in q_1 since $\mathbf{M} > \mathbf{S} > \mathbf{U}$ by assumption (i). F_1^1 is decreasing in q_2 by the inductive step. F_1^2 is decreasing in q_1 since

- $\mathbf{M} > (1-\sigma)\mathbf{M} + \sigma F_1^1(1)$ by the inductive step, and
- $F_1^1(q_2) > (1-\sigma)\mathbf{S} + \sigma\mathbf{U}$ by the inductive step.

Similarly F_2^2 is decreasing in q_2 by the inductive step. In addition, F_2^2 is decreasing in q_1 since

- $M > (1-\sigma)F_1^1(1) + \sigma F_1^1(1)$ by the inductive step, and
- $F_1^1(q_2) > (1-\sigma)\mathbf{S} + \sigma\mathbf{U}$ by the inductive step.

Lastly, $F_1^2 > F_2^2$ since $F_1^1(1) < \mathbf{M}$ by the inductive step.

Now assume that F_i^n is decreasing in i and that F_{i-1}^n and F_{i+1}^n are each decreasing in q_j for $n = 1, \ldots, N$, $i = 1, , \ldots, n$, $j = 1, \ldots, n$. Then, F_1^{n+1} and F_{i+1}^n are each decreasing in q_2, q_3, $\ldots$,q_n by the inductive step. Let D be the derivative operator with respect to q_1. Then

$$\begin{aligned} DF_1^{n+1}(q_1, \ldots, q_{n+1}) = [(1-\sigma)(\mathbf{S} + n\mathbf{M}) + \sigma(\mathbf{U} + F_2^n(1, q_3, \ldots, q_n))] - \\ [\mathbf{M} + F_1^n(q_2, \ldots, q_n)] < 0 \end{aligned}$$

since (1) $F_1^n(q_2, \ldots, q_n) > (1-\sigma)(\mathbf{S} + (n-1)\mathbf{M}) + \sigma F_2^n(1, q_3, \ldots, q_n))$ by the inductive step, and (2) $\mathbf{M} > (1-\sigma)\mathbf{M} + \sigma\mathbf{U}$. Also

$$\begin{aligned} DF_i^{n+1}(q_1, \ldots, q_{n+1}) = [(1-\sigma)(\mathbf{S} + F_{i-1}^n(q_2, \ldots, q_n)) + \\ \sigma(\mathbf{U} + F_{i+1}^n(q_2, \ldots, q_n))] - [\mathbf{M} + F_{i1}^n(q_2, \ldots, q_n)] < 0 \end{aligned}$$

since

- $F_{i-1}^n > F_{i+1}^n$ by the inductive step and
- $\mathbf{M} > (1-\sigma)\mathbf{M} + \sigma\mathbf{U}$.

Lastly, F_i^{n+1} is decreasing in i by the inductive step.

Thus, for each N, $F_1^N(q_1, \ldots, q_N)) - \mathbf{C}$ is decreasing in q_i for each $i = 1, \ldots, N$ as required to show. An analogous inductive proof shows that for each N, $G_1^N(q_1, \ldots, q_N)$ is also decreasing in q_i for each $i = 1, \ldots, N$. □

Proposition 2 Π_s *shifts down and* Π_w *shifts up as* σ *increases*

Proof. The proof is again by induction on N. Let $\Pi_s^N(q_1, , \ldots, q_N) = F_1^N(q_1, \ldots, q_N) - \mathbf{C}$ where $F_i^n(q_1, \ldots, q_N)$ is defined in the proof of proposition 1. Recall that F_i^n is decreasing in i and that $F_i^n(q)$ is less than or equal to $n\mathbf{M}$ for all n, for all $i = 1, \ldots, n$. I can now proceed with the inductive proof. F_1^1 is decreasing in σ since $\mathbf{S} > \mathbf{U}$ by assumption (i). Now assume that F_{i-1}^n and F_{i+1}^n are each decreasing in σ for $i = 1, , \ldots, n$. Let D be the derivative operator with respect to σ. Then

$$\begin{aligned} DF_1^{n+1}(q) = q_1\{\mathbf{U} - \mathbf{S} + F_2^n(1, q_3, \ldots, q_n) - n\mathbf{M} + \\ \sigma DF_2^n(1, q_3, \ldots, q_n)\} + (1-q_1)DF_1^n(q_2, \ldots, q_n) < 0 \end{aligned}$$

(since $\mathbf{U} < \mathbf{S}$, $F_2^n < n\mathbf{M}$, and $Df_i^n < 0$ by the inductive step) and for $i > 1$

$$\begin{aligned} Df_i^{n+1}(\sigma) = q_1\{\mathbf{U} - \mathbf{S} + F_{i+1}^n(1, q_3, \ldots, q_n) - F_{i-1}^n(1, q_3, \ldots, q_n) + \\ (1-\sigma)DF_{i-1}^n(q_2, \ldots, q_n) + \sigma DF_{i+1}^n(q_2, \ldots, q_n)\} + \\ (1-q_1)DF_{i-1}^n(q_2, \ldots, q_n) < 0 \end{aligned}$$

(since $\mathbf{S} < \mathbf{U}$ by assumption (i), $F^n_{i+1} < F^n_{i-1}$ and both DF^n_{i-1} and DF^n_{i+1} are less than zero by the inductive step). Thus, F^n_1 is decreasing in σ for all N as required to show. An analogous inductive proof shows that G^n_1 is increasing in σ. $\square$

Proposition 3 *The function $\hat{\beta}$ decreases in $\mathbf{C}$. As $\mathbf{C}$ increases from 0 to $\mathbf{S} + (N-1)\mathbf{M} - N\mathbf{U}$, $\hat{\beta}$ decreases from .5 to 0.*

Proof. Recall that at $q = 0$, $\Pi_w > \Pi_s$ for all σ. From proposition 2, Π_s shifts down and Π_w shifts up as σ increases. By definition, $\hat{\beta}(\mathbf{C})$ is the maximum of all those σ such that Π_w and Π_s intersect for some q. If $\mathbf{C}$ decreases then Π_s shifts up. Thus, $\hat{\beta}$ must increase if $\mathbf{C}$ decreases (i.e. $\hat{\beta}$ decreases in $\mathbf{C}$). Recall that $\hat{\beta}(0) = .5$, as shown in the proof of Theorem 1. When $\mathbf{C} = \mathbf{S} + (N-1)\mathbf{M} - N\mathbf{U}$, $\Pi_w = \Pi_s$ at $q_j = 1$ if $\sigma = 0$. Thus, if $q_i = 1$, $\Pi_s = \Pi_w$ if $\sigma = 0$. Hence, $\hat{\beta}(\mathbf{C}) = 0$ in this case. $\square$

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