Approximate cores of games and economies with clubs.[*]

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Abstract

We introduce the framework of parameterized collections of games and provide three nonemptiness of approximate core theorems for arbitrary games with and without sidepayments. The parameters bound (a) the number of approximate types of players and the size of the approximation and (b) the size of nearly effective groups of players and their distance from exact effectiveness. The theorems are based on a new notion of partition-balanced profiles and approximately partition-balanced profiles. The results are then applied to a new model of an economy with clubs. In contrast to the extant literature, our approach allows both widespread externalities and uniform results.

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1 Introduction.

It is well understood that, except in highly idealized situations, cores of games may be empty and competitive equilibrium may not exist. For example, within the context of an exchange economy, the conditions required for existence of equilibrium typically include convexity, implying infinite divisibility of commodities, and also nonsatiation. Even these two conditions may well not be satisfied; goods are usually sold in prespecified units and there are some commodities that many individuals prefer not to consume. In the context of economies with coalition structures, such as economies with clubs and/or local public goods, the added difficulties of endogenous group formation compound the problems; even if all conditions for existence of equilibrium and nonemptiness of the core are satisfied by the sub-economies consisting of the membership of each possible club, the core of the total economy may be empty. One possible approach to this problem is to restrict attention to models where equilibria exist, for example, economies with continuums of agents. But a model with a continuum of agents can only be an approximation to a finite economy. Another approach is to consider solution concepts for which existence is more robust, for example, approximate equilibria and cores. It seems reasonable to suppose that there are typically frictions that prevent attainment of an exact competitive equilibrium. At any time, most markets may have some unsatisfied demand or supply and most purchases might be made at prices that are only close to equilibrium prices. It also seems reasonable to suppose that there are typically costs of forming coalitions. These sorts of observations motivate the study of existence of approximate equilibria and nonemptiness of approximate cores, initiated for exchange economies by Shapley and Shubik (1966).

In this paper, we introduce the notion of parameterized collections of games and show that, under apparently mild conditions, approximate cores of all sufficiently large games without side payments are nonempty. A collection of games is parameterized by (a) the number of approximate types of players and the goodness of the approximation and (b) the size of nearly effective groups of players and their distance from exact effectiveness. All games described by the same parameters are members of the same collection. The conditions required on a parameterized collection of games to ensure nonemptiness of approximate cores are merely that most players have many close substitutes, per capita payoffs are bounded (per capita boundedness), and all or almost all gains to collective activities can be realized by groups bounded in size (small group effectiveness). Per capita boundedness simply rules out arbitrarily large average payoff. The final condition, small group effectiveness, may appear to be restrictive, but, in fact, in the context of a "pregame," per capita boundedness and small group effectiveness are equivalent (Wooders 1994b).[1]

As an application of our work, we develop a new model of an economy with clubs and obtain analogues of our non-emptiness results for games. Our model allows

[1] See also Wooders (1991,1992) for related results for games with and without side payments.

utilities from forming a club to be a®ected by the size and composition of the economy containing the club. For example, there may be widespread externalities.

To position our model and results in the literature, recall that Shapley and Shubik showed that large exchange economies with quasi-linear preferences have nonempty approximate cores. Under the assumption of per capita boundedness { ¯niteness of the supremum of average payo® { Wooders (1980,1983) demonstrated nonemptiness of approximate cores of large games derived from pregames. These result extend those of Shapley and Shubik to general games with and without side payments. Since then, there have been a number of advances in this literature, including Shubik and Wooders (1983), Kaneko and Wooders (1982), and Wooders and Zame (1984). The prior literature on approximate cores of large games all uses the framework of a pregame. A pregame consists of a compact metric space of player types, possibly ¯nite, and a worth function ascribing a payo® possibilities set to every possible group of players. The worth function depends continuously on the types of players in a coalition. The pregame framework treats collections of games that can all be described by a single worth function. This has hidden consequences; for example, as we will illustrate, the equivalence between small group e®ectiveness and per capita boundedness noted above depends on the structure of a pregame. Moreover, in general the payo® to a coalition cannot depend on the total player set of the game in which it is embedded; widespread externalities are ruled out.[2]

To illustrate how parameterized collections can treat a broader class of situations than pregames, consider, for example, a sequence of economies where the nth economy has n identical players. Due to widespread negative externalities, in the nth economy each agent can realize a payo® of 1 + 1=n. Also suppose, for simplicity, that in the nth economy, a coalition containing m · n can realize the total payo® of m(1 + 1=n) { within each economy there are no gains to coalition formation. (It is easy to modify the example to allow such gains.) The pregame framework rules out such sequences of games. In contrast, parameterized collections of games incorporate games with widespread externalities and our results apply. The framework of parameterized collections incorporates the prior models and uses less restrictive conditions than in the prior literature.[3] This example also illustrates that our club-theoretic results cannot be obtained in the pregame context.

In the remainder of this introduction, we ¯rst discuss our game-theoretic framework and results in more detail and then discuss economies with clubs. Related literature is discussed in the body of the paper.

[2] An exception is Wooders (1983), which allows positive externalities.

[3] In spirit, the pregame framework is similar to the economic frameworks of Kannai (1970) and Hildenbrand (1974), for example, while our approach is more in the spirit of the economic models of Anderson (1978) and Manelli (1991a,b).

1.1 The game-theoretic model and results.

We provide three theorems showing non-emptiness of approximate cores of arbitrary games. Given the speci¯cation of an approximate core { the particular approximate core notion and the parameters describing the closeness of the approximation { we obtain a lower bound ´ on the number of players so that any game in the class of games described by the speci¯ed parameters with at least ´ players has a non-empty approximate core. While our three theorems each use di®erent notions of approximate cores, both the notions of approximate cores and the theorems build on each other. Our framework encompasses games derived from pregames with or without side payments and our results encompass, as special cases, a number of non-emptiness of approximate core results in the literature. In the concluding section of the current paper we remark on other applications of the notion of parametrized collections of games.

Our ¯rst result, for the "-remainder core, requires a ¯nite number T of types of players and a bound B on e®ective group sizes. Roughly, a payo® is in the "-remainder core if it is in the core of a subgame containing all but a fraction " of the players. The result provides a lower bound, depending on T; B; and ", on the number of players required to ensure nonemptiness of the "-remainder core for all games with T types and bound B on e®ective group sizes: An important aspect of this result, like the result of Kaneko and Wooders (1982), is that the conclusion is independent of the payo® sets for the games. The result is eminently applicable to models with bounded coalition sizes, such as marriage and matching games (cf., Kelso and Crawford (1982) or Roth and Sotomayor (1992)).

The "-core of a game is the set of feasible payo®s that cannot be improved upon by any coalition of players by at least " for each member of the coalition. A payo® is in the weak "-remainder core if it induces a payo® in the "-core of a subgame containing all but a fraction " of the players. Our non-emptiness theorem for the weak "-remainder core requires only that groups bounded in size are e®ective for the realization of almost all gains to cooperation. Instead of the assumption of a ¯nite number of types, to show non-emptiness of the weak "-remainder core we require only that there be a partition of the set of players into a ¯nite number of approximate types. Such an assumption would be satis¯ed by games derived from a pregame with a compact metric space of player types, for example.

Under two additional restrictions on the class of games, we obtain a similar nonemptiness result for "-cores. The restrictions are that: (a) per capita payo®s are bounded; and (b) the games are strongly comprehensive (that is, the boundaries of the total payo® set are bounded away from being \°at"). A corollary relaxes assumption (b).

1.2 Economies with clubs.

There are now numerous papers in the literature studying cores and equilibria of economies with local public goods, where a feasible state of the economy includes a partition of the set of agents into disjoint jurisdictions or clubs for the purposes of collective consumption of public goods within each club or jurisdiction.[4] There have been far fewer works on economies where an agent can belong to multiple clubs { two of the few are Buchanan (1965) and Shubik and Wooders (1982). In this paper we develop a model of an economy with clubs where: (a) an agent may belong to multiple clubs { indeed, as many clubs as there are groups containing that agent; (b) all agents may di®er from each other; (c) each club may provide a unique bundle of goods and/or services, including private goods, public goods subject to exclusion, and conviviality; and (d) the payo® to a group of players may depend on the economy in which it is embedded { widespread externalities are permitted.

A club is a group of people who collectively consume and/or produce a bundle of goods and/or services for the members of the club. Often clubs have been treated as synonymous with coalitions of agents providing congestable and excludable public goods for their members. We observe, however, that clubs engage in a variety of activities. These activities may or may not require input of private goods. The goods provided by the club may include the enjoyment of the company of the other club members. In clubs of intellectuals, the exchange of ideas may be the aspect of the club that brings enjoyment to its members. Clubs may provide only private goods; for example, many academic departments have co®ee clubs. Other clubs o®er some goods and/or services to the general public. Some sorts of clubs o®er private goods and/or services to their members in addition to public goods. There is frequently no requirement that members of the same club consume the same bundles of goods. Thus, in this paper for each club we assume that there is an abstract set of feasible club activities.[5]

It may be the case that some sorts of clubs are ruled out for legal, technical, or social reasons. For example, a marriage may be viewed as a club, and polyandrous marriages may be illegal. Thus, for each coalition of agents in the economy there is an admissible club structure of that coalition. Admissible club structures are required to satisfy certain natural properties. In addition our model is required to satisfy the conditions that: (a) average utilities are bounded independently of the size of the economy; and (b) as the economy grows large, there is a limit to increasing returns to club size.

Although the conditions on our model are remarkably non-restrictive, by appli-

[4] Some early papers include, for example, Wooders (1978) and Greenberg and Weber (1986). See Conley and Wooders (1998) and Konishi, Le Breton, and Weber (1998) for more recent references.

[5] The notion of public projects, introduced in Mas-Colell (1980) and extended to local public projects in Manning (1992), is related. Our formulation here, however, is more abstract and accomodates private goods clubs as well as clubs for the provision of public goods.

cation of our game-theoretic results we are able to show several forms of the result that approximate cores of large economies – with sufficiently many players – are non-empty. Our result applies simultaneously to all games in a parameterized collection.

1.3 Organization of the paper.

The paper is organized as follows. The next section introduces the basic definitions, including the notion of parametrized collections of games. Section 3 presents our three theorems on non-emptiness of approximate cores in the order presented above. Section 4 consists of our club model and results. Section 5 presents the mathematical foundation and provides mathematical examples illustrating the conditions of the theorems. Section 6 concludes the body of the paper. The final section is an appendix containing the proofs.

2 Definitions.

2.1 Cooperative games: description and notation.

Let $N = \{1, \ldots, n\}$ denote a set of players. A non-empty subset of N is called a coalition. For any coalition S let $\mathbb{R}^S$ denote the $|S|$-dimensional Euclidean space with coordinates indexed by elements of S. For $x \in \mathbb{R}^N$; x_S will denote its restriction to $\mathbb{R}^S$. To order vectors in $\mathbb{R}^S$ we use the symbols $\gg$; $>$ and $\geq$ with their usual interpretations. The non-negative orthant of $\mathbb{R}^S$ is denoted by $\mathbb{R}^S_+$ and the strictly positive orthant by $\mathbb{R}^S_{++}$. We denote by 1_S the vector of ones in $\mathbb{R}^S$, that is, $1_S = (1, \ldots, 1) \in \mathbb{R}^S$. Each coalition S has a feasible set of payoffs or utilities denoted by $V_S \subset \mathbb{R}^S$. By agreement, $V_\emptyset = \{0\}$ and $V_{\{i\}}$ is non-empty, closed and bounded from above for any i. In addition, we will assume that

$$\max \left\{ x : x \in V_{\{i\}} \right\} = 0 \text{ for any } i \in N;$$

this is by no means restrictive since it can always be achieved by a normalization.

It is convenient to describe the feasible utilities of a coalition as a subset of $\mathbb{R}^N$. For each coalition S let $V(S)$, called the payoff set for S, be defined by

$$V(S) := \left\{ x \in \mathbb{R}^N : x_S \in V_S \text{ and } x_a = 0 \text{ for } a \notin S \right\} :$$

A game without side payments (called also an NTU game or simply a game) is a pair $(N; V)$ where the correspondence $V : 2^N \to \mathbb{R}^N$ is such that $V(S) \subset \left\{ x \in \mathbb{R}^N : x_a = 0 \text{ for } a \notin S \right\}$ for any $S \subset N$ and satisfies the following properties :

(2.1) $V(S)$ is non-empty and closed for all $S \subset N$.

(2.2) $V(S) \cap \mathbb{R}^N_+$ is bounded for all $S \subset N$, in the sense that there is a real number $K > 0$ such that if $x \in V(S) \cap \mathbb{R}^N_+$, then $x_i \leq K$ for all $i \in S$.

(2.3) $V(S_1 \bigcup S_2) \supseteq V(S_1) + V(S_2)$ for any disjoint $S_1, S_2 \subset N$ (superadditivity).

We next introduce the uniform version of strong comprehensiveness assumed for our third approximate core result. Roughly, this notion dictates that payoff sets are both comprehensive and uniformly bounded away from having level segments in their boundaries. Consider a set $W \subset \mathbb{R}^S$. We say that W is comprehensive if $x \in W$ and $y \leq x$ implies $y \in W$. The set W is strongly comprehensive if it is comprehensive, and whenever $x \in W$, $y \in W$, and $x < y$ there exists $z \in W$ such that $x << z$.[6] Given (i) $x \in \mathbb{R}^S$, (ii) $i, j \in S$, (iii) $0 \leq q \leq 1$ and (iv) $\varepsilon \geq 0$, define a vector $x^q_{i,j}(\varepsilon) \in \mathbb{R}^S$, where

$$\begin{aligned} (x^q_{i,j}(\varepsilon))_i &= x_i - \varepsilon, \\ (x^q_{i,j}(\varepsilon))_j &= x_j + q\varepsilon, \text{ and} \\ (x^q_{i,j}(\varepsilon))_k &= x_k \text{ for } k \in S \setminus \{i, j\}. \end{aligned}$$

The set W is q-comprehensive if W is comprehensive and if, for any $x \in W$, it holds that $(x^q_{i,j}(\varepsilon)) \in W$ for any $i, j \in S$ and any $\varepsilon \geq 0$.[7] This condition for $q > 0$ uniformly bounds the slopes of the Pareto frontier of payoff sets away from zero. Note that for $q = 0$, 0-comprehensiveness is simply comprehensiveness. Also note that if a game is q-comprehensive for some $q > 0$ then the game is q'-comprehensive for all q' with $0 \leq q' \leq q$.

Let $V_S \subset \mathbb{R}^S$ be a payoff set for $S \subset N$. Given q, $0 \leq q \leq 1$, let $W^q_S \subset \mathbb{R}^S$ be the smallest q-comprehensive set that includes the set V_S.[8] For $V(S) \subset \mathbb{R}^N$ let us define the set $c_q(V(S))$ in the following way:

$$c_q(V(S)) := \left\{ x \in \mathbb{R}^N : x_S \in W^q_S \text{ and } x_a = 0 \text{ for } a \notin S \right\}.$$

Notice that for the relevant components – those assigned to the members of S – the set $c_q(V(S))$ is q-comprehensive, but not for other components. With some abuse of the terminology, we will call this set the q-comprehensive cover of $V(S)$. When $q > 0$ we can think of a game as having some degree of "side-paymentness" or as

[6] Informally, if one person can be made better off (while all the others remain at least as well off), then all persons can be made better off. This property has also been called "nonleveledness."

[7] The notion of q-comprehensiveness can be found in Kaneko and Wooders (1996), For the purposes of the current paper, q-comprehensiveness can be relaxed outside the individually rational payoff sets.

[8] Notice that there exist q-comprehensive sets that contain V_S, specifically $\mathbb{R}^S$. The set W^q_S is the intersection of all q-comprehensive sets containing V_S.

allowing transfers between players, but not necessarily at a one-to-one rate. This is an eminently reasonable assumption for games derived from economic models.

A game with side payments (also called a TU game) is a game $(N;V)$ with 1-comprehensive payoff sets, that is $V(S) = c_1(V(S))$ for any $S \subset N$. This implies that for any $S \subset N$ there exists a real number $v(S) \geq 0$ such that $V_S = \left\{ x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S) \right\}$. The numbers $v(S)$ for $S \subset N$ determine a function v mapping the subsets of N to $\mathbb{R}_+$. Then the TU game is represented as the pair $(N;v)$.

2.2 Parameterized collections of games.

To introduce the notion of parameterized collections of games we will need the concept of Hausdorff distance. For every two non-empty subsets E and F of a metric space $(M;d)$, define the Hausdorff distance between E and F (with respect to the metric d on M), denoted by $\mathrm{dist}(E;F)$, as

$$\mathrm{dist}(E;F) := \inf\{\varepsilon \in (0;\infty) : E \subset B_\varepsilon(F) \text{ and } F \subset B_\varepsilon(E)\};$$

where $B_\varepsilon(E) := \{x \in M : d(x;E) \leq \varepsilon\}$ denotes an ε-neighborhood of E.

Since payoff sets are unbounded below, we will use a modification of the concept of the Hausdorff distance so that the distance between two payoff sets is the distance between the intersection of the sets and a subset of Euclidean space. Let m^* be a fixed positive real number. Let M^* be a subset of Euclidean space $\mathbb{R}^N$ defined by $M^* := \left\{ x \in \mathbb{R}^N : x_a \geq -m^* \text{ for any } a \in N \right\}$. For every two non-empty subsets E and F of Euclidean space $\mathbb{R}^N$ let $H_\infty[E;F]$ denote the Hausdorff distance between $E \cap M^*$ and $F \cap M^*$ with respect to the metric $\|x - y\|_\infty := \max_i |x_i - y_i|$ on Euclidean space $\mathbb{R}^N$.

The concepts defined below lead to the definition of parameterized collections of games. To motivate the concepts, each is related to analogous concepts in the pregame framework.

<u>$\delta-$substitute partitions:</u> In our approach we approximate games with many players, all of whom may be distinct, by games with finite sets of player types. Observe that for a compact metric space of player types, given any real number $\delta > 0$ there is a partition (not necessarily unique) of the space of player types into a finite number of subsets, each containing players who are "δ-similar" to each other. Parameterized collections of games do not restrict to a compact metric space of player types, but do employ the idea of a finite number of approximate types.

Let $(N;V)$ be a game and let $\delta \geq 0$ be a non-negative real number. A δ-substitute partition is a partition of the player set N into subsets with the property that any two players in the same subset are "within δ" of being substitutes for each other.

Formally, given a set $W \subset \mathbb{R}^N$ and a permutation τ of N, let $\sigma_\tau(W)$ denote the set formed from W by permuting the values of the coordinates according to the associated permutation τ. Given a partition $\{N[t] : t = 1, \ldots, T\}$ of N, a permutation τ of N is type-preserving if, for any $i \in N$, $\tau(i)$ belongs to the same element of the partition $\{N[t]\}$ as i. A δ-substitute partition of N is a partition $\{N[t] : t = 1, \ldots, T\}$ of N with the property that, for any type-preserving permutation τ and any coalition S,

$$H_\infty\left[V(S), \sigma_\tau^{-1}(V(\tau(S)))\right] \leq \delta.$$

Note that in general a δ-substitute partition of N is not uniquely determined. Moreover, two games may have the same partitions but have no other relationship to each other (in contrast to games derived from a pregame).

<u>(δ,T)- type games.</u> The notion of a (δ,T)-type game is an extension of the notion of a game with a finite number of types to a game with approximate types.

Let δ be a non-negative real number and let T be a positive integer. A game (N, V) is a (δ, T)-type game if there is a T-member δ-substitute partition $\{N[t] : t = 1, \ldots, T\}$ of N. The set $N[t]$ is interpreted as an approximate type. Players in the same element of a δ-substitute partition are δ-substitutes. When $\delta = 0$, they are exact substitutes.

<u>profiles.</u> Another notion that arises in the study of large games is that of the profile of a player set, a vector listing the number of players of each type in a game. This notion is also employed in the definition of a parameterized collection of games, but profiles are defined relative to partitions of player sets into approximate types.

Let $\delta \geq 0$ be a non-negative real number, let (N, V) be a game and let $\{N[t] : t = 1, \ldots, T\}$ be a partition of N into δ-substitutes. A profile relative to $\{N[t]\}$ is a vector of non-negative integers $f \in \mathbb{Z}_+^T$ and a subprofile s of a profile f is a profile satisfying the condition that $s \leq f$. Given $S \subset N$ the profile of S is a profile, say $s \in \mathbb{Z}_+^T$, where $s_t = |S \cap N[t]|$. A profile describes a group of players in terms of the numbers of players of each approximate type in the group. Let $\|f\|$ denote the number of players in a group described by f, that is, $\|f\| = \sum f_t$.

<u>β-effective B-bounded groups:</u> In all studies of approximate cores of large games, some conditions are required to limit gains to collective activities, such as boundedness of marginal contributions to coalitions, as in Wooders and Zame (1984,1989) or the less restrictive conditions of per capita boundedness and/or small group effectiveness, as in Wooders (1980,1983,1994a,b), for example. Small groups are effective if all or almost all gains to collective activities can be realized by groups bounded in size of membership. The following notion formulates the idea of small effective groups in the context of parameterized collections of games.

Informally, groups of players containing no more than B members are β-effective if, by restricting coalitions to having fewer than B members, the loss to each player is no more than β: This is a form of small group effectiveness for arbitrary games. Let $(N;V)$ be a game. Let $\beta \geq 0$ be a given non-negative real number and let B be a given positive integer. For each group $S \subset N$; define a corresponding set $V(S;B) \subset \mathbb{R}^N$ in the following way:

$$V(S;B) := \bigcup \left[\sum_k V(S^k) : \left\{ S^k \right\} \text{ is a partition of } S, \ \left| S^k \right| \leq B \right].$$

The set $V(S;B)$ is the payoff set of the coalition S when groups are restricted to have no more than B members. Note that, by superadditivity, $V(S;B) \subset V(S)$ for any $S \subset N$ and, by construction, $V(S;B) = V(S)$ for $|S| \leq B$. We might think of $c_q(V(S;B))$ as the payoff set to the coalition S when groups are restricted to have no more than B members and transfers are allowed between groups in the partition. If the game $(N;V)$ has q-comprehensive payoff sets then $c_q(V(S;B)) \subset V(S)$ for any $S \subset N$: The game $(N;V)$ with q-comprehensive payoff sets has β-effective B-bounded groups if for every group $S \subset N$

$$H_\infty \left[V(S); c_q(V(S;B)) \right] \leq \beta.$$

When $\beta = 0$, 0-effective B-bounded groups are called strictly effective B-bounded groups.

parameterized collections of games $G^q((\delta;T);(\beta;B))$. With the above definitions in hand, we can now define parameterized collections of games.

Let T and B be positive integers and let q be a real number, $0 \leq q \leq 1$. Let $G^q((\delta;T);(\beta;B))$ be the collection of all $(\delta;T)$-type games that are superadditive, have q-comprehensive payoff sets, and have β-effective B-bounded groups.

Less formally, given non-negative real numbers q; β and δ; and positive integers T and B; a game $(N;V)$ belongs to the class $G^q((\delta;T);(\beta;B))$ if:

(a) the payoff sets satisfy q-comprehensiveness;

(b) there is a partition of the total player set into T sets where each element of the partition contains players who are δ-substitutes for each other; and

(c) almost all gains to collective activities (with a maximum possible loss of β for each player) can be realized by partitions of the total player sets into groups containing fewer than B members.

Our results hold for all parameters δ and β that are sufficiently small, that is, $2(\delta + \beta) < m^*$; where m^* is a positive real number used in the definition of the Hausdorff distance. (Since m^* can be chosen to be arbitrarily large, this requirement is nonrestrictive.)

3 Non-emptiness of approximate cores of games.

Recall the definition of the core.

<u>the core.</u> Let $(N; V)$ be a game. A payoff x is undominated if, for all $S \subset N$ and $y \in V(S)$, it is not the case that $y_S >> x_S$. The payoff x is feasible if $x \in V(N)$. The core of a game $(N; V)$ consists of all feasible and undominated payoffs.

3.1 The ε-remainder core.

The concept of the ε-remainder core is based on the idea that all requirements of the core should at least be satisfied for almost all players with the remainder of players representing a small fraction of "unemployed" or "underemployed" players. This approximate core notion can be viewed as a stepping stone to other notions of approximate cores. There are game-theoretic situations, however, in which the notion of the ε-remainder core may naturally arise – for example, the demand games of Selten (1981).

<u>the ε-remainder core.</u> Let $(N; V)$ be a game. A payoff x belongs to the ε-remainder core if, for some group $S \subset N$, $\frac{|N|-|S|}{|N|} \leq \varepsilon$ and x_S belongs to the core of the subgame $(S; V)$.

Note that the following theorem requires no restrictions on the degree of comprehensiveness – the usual notion of comprehensiveness suffices.

Theorem 1. Non-emptiness of the ε-remainder core. Let T and B be positive integers. For any $\varepsilon > 0$, there exists an integer $\eta_1(\varepsilon; T; B)$ such that if

(a) $(N; V) \in G^q((0; T); (0; B))$ and

(b) $|N| \geq \eta_1(\varepsilon; T; B)$

then the ε-remainder core of $(N; V)$ is non-empty.

The assumptions of Theorem 1 provide a strong conclusion, which is a stepping stone to our more broadly applicable results. The Theorem requires a fixed number T of exact player types and strictly effective small groups of size less than or equal to B. Under these assumptions, the theorem states that for any $\varepsilon > 0$ there exists a lower bound $\eta_1(\varepsilon; T; B)$ on the number of the players such that all games satisfying the assumptions with more than $\eta_1(\varepsilon; T; B)$ players have non-empty ε-remainder cores. Since the bound depends only on ε, T, and B, the bound is uniform across all the games characterized by the parameters; there is no restriction to replica games. Our result extends the result of Kaneko and Wooders (1982) from replication sequences to arbitrary large games. As in Kaneko and Wooders (1982) the result is independent of the characteristic function of the games; the same bound holds for all games in the collection parameterized by T and B. In Section 5 we provide an example illustrating application of the result.

3.2 The weak ε-remainder core.

For a less restrictive definition of the approximate core we can treat a significantly more general class of games than those of Theorem 1, in particular, we can allow approximate types ($\delta > 0$) and almost effective groups ($\beta > 0$). For example, the class of models covered by our next Theorem includes replica models of economies with private goods as in Debreu and Scarf (1963) and models of local public good economies satisfying per capita boundedness, as in Wooders (1988).

<u>the weak ε-remainder core.</u> Let $(N; V)$ be a game. A payoff x belongs to the $(\varepsilon_1; \varepsilon_2)$-weak core[9] if there is some $S \subset N$, $\frac{|N| - |S|}{|N|} \leq \varepsilon_1$ such that

(a) x_S is feasible in the subgame $(S; V)$, and

(b) x_S is ε_2-undominated in the subgame $(S; V)$.[10]

Since one possibility is that $\varepsilon_1 = \varepsilon_2 = \varepsilon$, we typically refer to this notion as the weak ε-remainder core.

The following result extends the nonemptiness results of Wooders (1980,1983,1992), Shubik and Wooders (1983), and Wooders and Zame (1984,1989) from pregames to parameterized collections of games. For the same values of the parameters T and B the bound on the sizes of games in the following theorem can be chosen to equal the bound in the preceding theorem. Note that there are no restrictions on the value of q – strong comprehensiveness is not required.

Theorem 2. Non-emptiness of the weak $(\varepsilon; (\delta + \beta))$-remainder core. Let T and B be positive integers. For any $\varepsilon > 0$ there exists an integer $\eta_1(\varepsilon; T; B)$ such that if

(a) $(N; V) \in G^q((\delta; T); (\beta; B))$ and

(b) $|N| \geq \eta_1(\varepsilon; T; B)$

then the weak $(\varepsilon; (\delta + \beta))$-remainder core of $(N; V)$ is non-empty.

Observe that by definition the $(\varepsilon; 0)$-weak core coincides with the ε-remainder core. Therefore, Theorem 2 is a strict generalization of Theorem 1 (Theorem 1 is a subcase for $\delta = \beta = 0$). But both Theorem 1 and Theorem 2 are based on the idea that some small proportion of the players can be ignored. An example in Section 5 illustrates this point.

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[9] The weak ε-remainder core might be called the ε_1-remainder ε_2-core but this is awkward.

[10] That is, $x_S + 1_S \varepsilon_2$ is undominated in the game $(S; V)$.

3.3 The ε-core.

<u>the ε-core.</u> Let $(N;V)$ be a game. A payoff x belongs to the ε-core if
(a) x is feasible, and (b) x is ε-undominated.[11]

Note that when $\varepsilon = 0$, the ε-core coincides with the core.

Our third Theorem provides conditions for the non-emptiness of the ε-core of large games. The proof is based on the idea of compensating the "remainder" players from the previous theorems, as in Wooders (1980,1983,1992) and Wooders and Zame (1984,1989). This compensation is possible under q-comprehensiveness (with $q > 0$) and one more condition, typically called per capita boundedness.

<u>per capita boundedness.</u> Let C be a positive real number. A game $(N;V)$ has a per capita payoff bound of C if, for all coalitions $S \subseteq N$,

$$\sum_{a \in S} x_a \leq C|S| \text{ for any } x \in V(S).$$

Theorem 3. Non-emptiness of the $(\varepsilon + \delta + \beta)$-core. Let T and B be positive integers. Let C and q be positive real numbers, $q > 0$. Then for each $\varepsilon > 0$ there exists an integer $\eta_2(\varepsilon; T; B; C; q)$ such that if:
(a) $(N;V) \in G^q((\delta; T); (\beta; B))$,
(b) $(N;V)$ has per capita payoff bound C, and
(c) $|N| \geq \eta_2(\varepsilon; T; B; C; q)$
then the $(\varepsilon + \delta + \beta)$-core of $(N;V)$ is non-empty.

In Section 5, we present an example of a game with a compact metric space of player attributes and show, through examples, the indispensability of the conditions in Theorem 3. The following Corollary shows that Theorem 3 can be applied to obtain non-emptiness of approximate cores of games that are "close" to q-comprehensiveness games (with $q > 0$).[12] The proof of this result is left to the reader.

Corollary. Non-emptiness with near q-comprehensiveness. Let $(N;W)$ be a game. Suppose that for some $q > 0$ there is a game $(N;V) \in G^q((\delta; T); (\beta; B))$ and positive real numbers $\varepsilon > 0$ and $\gamma > 0$ such that:
(a) $(N;V)$ has per capita payoff bound C,
(b) $|N| \geq \eta_2(\varepsilon; T; B; C; q)$ and
(c) $H_1[W(S); V(S)] < \frac{\gamma}{2}$ for all $S \subseteq N$:
Then the $(\varepsilon + \delta + \beta + \gamma)$-core of $(N;W)$ is non-empty.

[11] That is, $x + \varepsilon 1_N$ is undominated.
[12] Any comprehensive payoff set can be approximated arbitrarily closely by a q-comprehensive payoff set, for q small (Wooders 1983, Appendix).

3.4 Remarks.

Remark 1. The intuition behind our results. Our results are all based on one fact: with a finite number of types of players and bounded effective group sizes, large games have non-empty approximate cores. This is a property of profiles rather than a property of games. To illustrate this point, consider, for example, a collection of games where all players are identical and two-player coalitions are effective. Independent of further specification of the characteristic function, all games in the collection with an even number of players have nonempty cores. Therefore, there is a subgame with a nonempty core containing at least all but one of the players – there is at most one player left-over. Thus, any game with at least n players will have a non-empty $\frac{1}{n}$-remainder cores. These sorts of observations hold generally and lead to Theorem 1. Theorem 2 follows by approximation techniques. Then, using q-comprehensiveness and per capita boundedness, left-over players can be compensated and Theorem 3 follows.

Remark 2. q-comprehensiveness or convexity? It is possible to obtain a result similar to Theorem 3 using convexity of payoff sets and “thickness” instead of q-comprehensiveness (see Kovalenkov and Wooders 1997a). Strong comprehensiveness, however, can be naturally satisfied by games derived from economies. Moreover, “1-strongly comprehensive games” are games with side payments, so we can incorporate this important special case. Furthermore, in models of economies with local public goods or with clubs, convexity may be difficult to satisfy. Although examples show that none of the assumptions can be omitted, our Corollary relaxes q-comprehensiveness.

Remark 3. Explicit bounds. It may be possible to compute the bounds on the size of the total player sets given in Theorems 1, 2, and 3 in terms of the parameters describing the games. A simple bound is obtained in Kovalenkov and Wooders (1997b), although under somewhat different assumptions. Also, the proofs of that paper, relative to those of this paper, are quite complex.

Remark 4. Absolute or relative sizes? It is possible to obtain similar results with bounds on relative sizes of effective coalitions. In a finite game with a given number of players, assumptions on absolute sizes and on relative sizes of effective coalitions are equivalent. We have chosen to develop our results using bounds on absolute sizes of near-effective coalitions since this seems to reflect typical economic and social situations. Examples include: marriage and matching models (see Kelso and Crawford (1982) and Roth and Sotomayor (1990)); models of economies with shared goods and crowding (see Conley and Wooders (1998) for a survey); and private goods exchange economies (see Mas-Colell (1979) and Kaneko and Wooders (1989)). In fact, assumptions on proportions of economic agents typically occur only when there is a continuum of players, cf. Ostroy (1984).

Remark 5. Limiting gains to coalition formation. In the pregame framework several different conditions limiting returns to coalition formation have been used. For situations with a fixed distribution of a finite number of player types, Wooders (1980,1983) and Shubik and Wooders (1983) require per capita boundedness. To treat compact metric spaces of player types, Wooders and Zame (1984,1989) require boundedness of marginal contributions to coalitions while Wooders (1992,1994a,b) requires the less restrictive condition of small group effectiveness. As noted in the introduction, in the context of games derived from pregames, small group effectiveness and per capita boundedness are equivalent. In Section 5, we show that in the broader framework of parameterized collections of games both β-effective B-bounded groups and per capita boundedness are required.

4 Economies with clubs.

We define admissible club structures in terms of natural properties and take as given the set of all admissible club structures for each coalition of agents. Generalizing Mas-Colell's (1980) notion of public projects to club activities, there is no necessary linear structure on the set of club activities. Indeed, our results could be obtained even without any linear structure on the space of private commodities. We remark that it would be possible to separate crowding types of agents (those observable characteristics that affect the utilities of others, or, in other words, their external characteristics) from taste types, as in Conley and Wooders (1996,1997), and have agents' roles as club members depend on their crowding types. In these papers, however, the separation of crowding type and taste type has an important role; the authors show that prices for public goods – or club membership prices – need only depend on observable characteristics of agents and not on their preferences. The current paper treats only the core so the separation of taste and crowding type would have no essential role and therefore is not made.

agents. There are T "types" of agents. Let $m = (m_1, \dots, m_T)$ be a given profile, called the population profile. The set of agents is given by

$$N_m = \{(t,q) : q = 1, \dots, m_t \text{ and } t = 1, \dots, T\};$$

and (t,q) is called the qth agent of type t. It will later be required that all agents of the same type may play the same role in club structures. For example, in a traditional marriage model, all females could have the role of "wife". Define $N_m[t] := \{(t,q) : q = 1, \dots, m_t\}$. For our first Proposition members of $N_m[t]$ will be exact substitutes for each other and for our next two Propositions, approximate substitutes.

commodities. The economy has L private goods. A vector of private goods is denoted by $y = (y_1, \dots, y_\ell, \dots, y_L) \in \mathbb{R}^L_+$.

clubs. A club is a subset of agents. For each $S \subset N_m$, a club structure of S, denoted by $\mathcal{S}$, is a set of clubs whose union coincides with S. The non-empty set of admissible club structures for S is denoted by $\mathcal{C}(S)$. These sets are required to satisfy the following two properties:

1. If S and S' are disjoint subsets of agents and $\mathcal{S}$ and $\mathcal{S}'$ are club structures of S and S' respectively, then $\{C : C \in \mathcal{S} \bigcup \mathcal{S}'\}$ is a club structure of $S \bigcup S'$ (unions of admissible club structures of disjoint coalitions are club structure of the unions of the coalitions).

2. Let S and S' be subsets of agents with the same profiles, let $\mathcal{S}$ be a club structure of S and let φ be a type-preserving 1-1 mapping from S onto S' (that is, if $(t,q) \in S$ then $\varphi((t,q)) = (t,q')$ for some $q' = 1,\ldots,m_t$). Then
$$\mathcal{S}' = \{C \subset S' : \varphi^{-1}(C) \in \mathcal{S}\}$$
is a club structure of S' (admissible club structures depend only on profiles, that is, all agents of the same type have the same roles in clubs).

Note that our assumptions ensure that the partition of any set S into singletons is an admissible club structure. The first assumption is necessary to ensure that the game derived from the economy is superadditive. It corresponds to economic situations where one option open to a group is to form smaller groups. The second assumption corresponds to the idea that the opportunities open to a group depend on the profile of the group.

club activities. For each club C there is a set of club activities $A(C)$. An element α of $A(C)$ requires input $x(C;\alpha) \in \mathbb{R}^L$ of private goods. For any two clubs C and C' with the same profile we require that if $\alpha \in A(C)$, then $\alpha \in A(C')$ and $x(C;\alpha) = x(C';\alpha)$. For 1-agent clubs $\{(t,q)\}$, we assume that there is an activity α_0 with $x(\{(t,q)\};\alpha_0) = 0$, that is, there is an activity requiring no use of inputs.

preferences and endowments. Only private goods are endowed. Let $\omega^{tq} \in \mathbb{R}^L_+$ be the endowment of the $(t,q)^{th}$ participant of private goods.

Given $S \subset N_m$, $(t,q) \in S$, and a club structure $\mathcal{S}$ of S, the consumption set of the $(t,q)^{th}$ agent (relative to $\mathcal{S}$) is given by
$$Z^{tq}(\mathcal{S}) := X^{tq}(S) \times \prod_{C \in \mathcal{S}} A(C);$$
where $X^{tq}(S) \subset \mathbb{R}^L$ is the private goods consumption set relative to S, assumed to be closed. Thus, the entire consumption set of the $(t,q)^{th}$ agent is given by
$$Z^{tq} := \bigcup_{S \subset N_m : (t,q) \in S} \; \bigcup_{\mathcal{S} \in \mathcal{C}(S)} Z^{tq}(\mathcal{S}).$$

We assume that the $(t;q)^{th}$ agent can subsist in isolation. That is

$$(\omega^{tq};\alpha_0) \in Z^{tq}(\{(t;q)\}):$$

It is also assumed that for each $(t;q)$; each $S \subset N_m; (t;q) \in S$, and each club structure $\mathcal{S}$ of S, the preferences of the $(t;q)^{th}$ agent are represented by a continuous utility function $u^{tq}(\cdot;\mathcal{S})$ defined on $Z^{tq}(\mathcal{S})$.

states of the economy. Let S be a non-empty subset of N_m and let $\mathcal{S}$ be a club structure of S. A feasible state of the economy S relative to $\mathcal{S}$, or simply a state for $\mathcal{S}$, is a pair $(y^S;\alpha^{\mathcal{S}})$ where:

(a) $y^S = \{y^{tq}\}_{(t;q)\in S}$ with $y^{tq} \in X^{tq}(\mathcal{S})$ for $(t;q) \in S$;

(b) $\alpha^{\mathcal{S}} = \{\alpha^C\}_{C\in\mathcal{S}}$ with $\alpha^C \in A(C)$ for $C \in \mathcal{S}$; and

(c) the allocation of private goods is feasible, that is,

$$\sum_{C\in\mathcal{S}} x(C;\alpha^C) + \sum_{(t;q)\in S} y^{tq} = \sum_{(t;q)\in S} \omega^{tq}:$$

feasible payoffs. A payoff $U = (\hat{u}^{tq})_{(t;q)\in N_m}$ is feasible for a coalition S if $\bar{u}^{tq} = 0$ for all $(t;q) \in N_m \backslash S$ and there is club structure $\mathcal{S}$ of S and a feasible state of the economy for S relative to $\mathcal{S}; (y^S;\alpha^{\mathcal{S}})$; such that $\bar{u}^{tq} = u^{tq}(y^{tq};\alpha^{\mathcal{S}};\mathcal{S})$ for each $(t;q) \in S$.

the game induced by the economy. For each coalition $S \subset N_m$; define

$$V(S) = \{(\mathbf{u}^{tq})_{(t;q)\in N_m} : \text{there is a payoff } (\hat{u}^{tq})_{(t;q)\in N_m} \text{ that is feasible for } S \text{ and } \mathbf{u}^{tq} \le \bar{u}^{tq} \text{ for all } (t;q) \in S\}:$$

It is immediate that the player set N_m and function V determine a game $(N_m;V)$ with comprehensive payoff sets.

ε-domination. Let $\mathcal{N}_m$ be a club structure of the total agent set N_m and let $(y^{N_m};\alpha^{\mathcal{N}_m})$ be a feasible state of the economy N_m relative to $\mathcal{N}_m$. A coalition S can ε-dominate the state $(y^{N_m};\alpha^{\mathcal{N}_m})$ if there is a club structure $\mathcal{S} = \{S_1;\ldots;S_K\}$ of S and a feasible state of the economy $(y'^S;\alpha'^{\mathcal{S}})$ such that for all consumers $(t;q) \in S$ it holds that

$$u^{tq}(y'^{tq};\alpha'^{\mathcal{S}};\mathcal{S}) > u^{tq}(y^{tq};\alpha^{\mathcal{S}};\mathcal{N}_m) + \varepsilon:$$

the core of the economy and ε-cores. The state $(y^{N_m};\alpha^{\mathcal{N}_m})$ is in the core of the economy if it cannot be improved upon by any coalition S. Notions of the ε-remainder core of the economy, the weak ε-remainder core of the economy, and the ε-core are defined in the obvious way. It is clear that if $(y^{N_m};\alpha^{\mathcal{N}_m})$ is a state of the economy in a core of the economy — any one of the approximate cores that we've defined or the core itself — then the utility vector induced by that state is in the corresponding core of the induced game. Similarly, if $(\hat{u}^{tq})_{(t;q)\in N_m}$ is in a core of the game then there is a state in the core of the economy $(y^{N_m};\alpha^{\mathcal{N}_m})$ such that the utility vector induced by that state is $(\hat{u}^{tq})_{(t;q)\in N_m}$.

4.1 Non-emptiness of approximate cores.

To obtain our results we require few restrictions on the economy. Our first Proposition requires exact player types and strictly effective small groups.

(A.0) For each t and all $q, q' \in \{1, ..., m_t\}$, $u^{tq}(\cdot) = u^{tq'}(\cdot)$ and $\omega^{tq} = \omega^{tq'}$. In addition, in the game induced by the economy the players (t, q) and (t, q') are exact substitutes. (All agents of the same type are identical in terms of their endowments, preferences and crowding types – their effects on others.)

(A.1) There is a bound B such that for any population profile m, any coalition $S \subset N_m$, and any club structure $\mathcal{S}$ of S, if $U = (\overline{u}^{tq} : (t, q) \in S)$ is a feasible payoff for the club structure $\mathcal{S}$ then there is a partition of S into coalitions, say $\{S^1, ..., S^K\}$ and club structures of these coalitions, $\{\mathcal{S}^1, ..., \mathcal{S}^K\}$ such that for each k $|S^k| \leq B$ and $U^k := (\overline{u}^{tq} : (t, q) \in S^k)$ is a feasible payoff for $\mathcal{S}^k$.

Our approach requires that the set of individually rational and feasible outcomes is compact. It is possible to introduce conditions on the primitives of the economy as, for example, Debreu's (1962) condition of positive semi-independence, but for the purposes of this application, we will simply assume compactness.

(A.2) For each subset of agents $S \subset N_m$ the set $V(S) \cap \mathbb{R}^N_+$ is compact.

The following result is an immediate application of Theorem 1.

Proposition 1. Non-emptiness of the ε-remainder core. Assume (A.0)-(A.2) hold. Given $\varepsilon > 0$, there exists an integer $\eta_1(\varepsilon, T, B)$ such that if m, the profile of the economy, satisfies the property that $\|m\| \geq \eta_1(\varepsilon, T, B)$ then the ε-remainder core of the economy is non-empty.

Proposition 1 is most natural if there is only one private good or if private goods are indivisible so that all gains from trade in private goods can be realized by trade within coalitions of bounded sizes. If we require only non-emptiness of the weak ε-remainder core we can weaken the restrictions on the economy – players of the same type need only be approximate substitutes and small groups need only be nearly effective. For brevity, these assumptions will not be made on the primitives of the economy. Thus, instead of (A.0) and (A.1), for the following two Propositions, we will assume (A.0′) and (A.1′):

(A.0′) For some $\delta \geq 0$ the players in the set $N_m[t] = \{(t, q) : q = 1, ..., m_t\}$ are δ-substitutes for each other in the game induced by the economy.

(A.1′) There is an $\beta \geq 0$ and an integer B so that the game derived from the economy has β-effective B-bounded groups.

Then the following result, for the weak ε-reminder core, follows from Theorem 2.

Proposition 2. Non-emptiness of the weak $(\varepsilon, (\delta+\beta))$-remainder core. Assume (A.0$'$), (A.1$'$) and (A.2) hold. Given $\varepsilon > 0$, there exists an integer $\eta_1(\varepsilon, T, B)$ such that if m, the profile of the economy, satisfies the property that $\|m\| \geq \eta_1(\varepsilon, T, B)$ then the weak $(\varepsilon, (\delta+\beta))$-remainder core of the economy is non-empty.

For our next result, we require that each agent always owns some commodity which other agents value. Specifically, we assume that the L^{th} commodity is a "quasi-money" with which everyone is endowed and for which everyone has a separable preference. In the following, let y^{tq}_{-L}, and $X^{tq}_{-L}(S)$ denote the restriction of y^{tq} and $X^{tq}(S)$ respectively to their first $(L-1)$ coordinates. We also assume that, for some real number $q^* \in (0, 1]$, the marginal utility of the L^{th} commodity, for all sufficiently large amounts of the commodity, is greater than or equal to q^*:

(A.3) (q^*-comprehensiveness): For good L and all participants $(t, q) \in N_m$ there is a positive real number ω such that $\omega^{tq}_L \geq \omega > 0$ (everybody is endowed with the L^{th} good). Moreover, for any state of the economy (y^S, α^S) we have :

(a) $X^{tq}(S) = X^{tq}_{-L}(S) \times \mathbb{R}_+$ (the consumption set is separable and the projection of the L^{th} coordinate is $\mathbb{R}_+$),

(b) $u^{tq}(y^{tq}, \alpha^S, S) = u^{tq}_{-L}(y^{tq}_{-L}, \alpha^S, S) + u^{tq}_L(y^{tq}_L, \alpha^S, S)$ for some functions $u^{tq}_{-L}(\cdot, \cdot)$ and $u^{tq}_L(\cdot, \cdot)$ (utility is separable),

(c) for a real number q^*, $0 < q^* \leq 1$, for all players (t, q) the marginal utility of the $(t, q)^{th}$player for the L^{th} good on the range $(\frac{\omega}{2}, \infty)$ is between q^* and 1.

Assumption (A.3) ensures that the remainder players can be "paid off," (at the rate q^*) so that they cannot profitably joint improving coalitions. Alternatively, it could simply be assumed that utility functions are linear in one commodity. (That is, $u^{tq}(x^{tq}, \alpha^S, S) = u^{tq}_{-L}(x^{tq}_{-L}, \alpha^S, S) + x^{tq}_L$.) This implies q^*-comprehensiveness of the game derived from the economy. Then the next result follows from Theorem 3.

Proposition 3. Non-emptiness of the $(\varepsilon + \delta + \beta)$-core. Assume that (A.0$'$), (A.1$'$), (A.2) and (A.3) hold. Given $\varepsilon > 0$, there exists an integer $\eta_2(\varepsilon, T, B, C, q^*, \omega)$ such that if m, the profile of the economy, satisfies $\|m\| \geq \eta_2(\varepsilon, T, B, C, q^*, \omega)$ then the $(\varepsilon + \delta + \beta)$-core of the economy is non-empty.

4.2 Further applications.

The class of economies defined above is very broad. The results can be applied to extend results already in the literature on economies with coalition structures, such as those with local public goods (called club economies by some authors), cf., Shubik and Wooders (1982,1997).

For example, there are a number of papers showing core-equilibrium equivalence in finite economies with local public goods and one private good and satisfying strict effectiveness of small groups, cf., Conley and Wooders (1998) and references therein. In these economies, from the results of Wooders (1983) and Shubik and Wooders (1983), existence of approximate equilibrium where an exceptional set of agents is ignored is immediate. (Just take the largest subgame having a non-empty core and consider the equilibria for that subeconomy; ignore the remainder of the consumers.) Our results allow the immediate extension of these results to results for all sufficiently large economies – no restriction to replication sequences is required.

5 Mathematical foundations.

5.1 Partition-balanced profiles.

This section formalizes some key ideas about profiles that underlie the non-emptiness of approximate cores of large games. Throughout this section, let the number of types of players be fixed at T. Thus, every profile f has T components and $f \in \mathbb{R}^T$. Our key definitions follow.

<u>B–partition-balanced profiles.</u> A profile f is B-partition-balanced if any game $(N; V) \in G^q((0;T);(0;B))$ where the profile of N is f (that is, $|N[i]| = f_i$ for any $i = 1; \ldots; T$) has a nonempty core.

<u>replicas of a profile.</u> Given a profile f and a positive integer r; the profile rf is called the r^{th} replica of f:

The Lemma below is a very important step. It states that, for any profile f; there is a replica of that profile that is B-partition-balanced. The smallest such replication number is called the depth of the profile. Note that the depth of a profile depends on the profile.

Lemma 1. (Kaneko and Wooders, 1982, Theorem 3.2)[13] **The balancing effect of replication.** Let B be a positive integer and let f be any profile. Then there is an integer $m(f;B)$; the depth of f, such that, for any positive integer k, the profile $km(f;B)f$ is B-partition-balanced.

[13] This type of argument also appears in Wooders (1980,1983) and other papers on approximate cores. Kaneko and Wooders (1982) highlight the fact that the argument does not depend on knowledge of the structure of payoffs. For a recent discussion and an interesting application of this sort of result to dynamic matching processes, see Myerson (1991).

Proof: For $q = 0$ this result is simply Theorem 3.2 in Kaneko and Wooders (1982), based on Wooders (1983, Lemma 5) and the technique used there to show nonemptiness of approximate cores. (The Kaneko-Wooders collection ¼ of all basic coalitions consists in our case of all groups bounded in size by B.) For $q > 0$ only a minor change in Kaneko- Wooders' proof is needed: If transfers between groups are possible and x_S belongs to the interior of some payo® set $V_S = c_q \sum_k V_{S^k} : S^k$ is a partition of $S; \bar{S}^{k-} \cdot B$, then it does not mean (as it is in the case when $q = 0$) that x_{S^k} belongs to the interior of all S^k: But importantly x_{S^k} belongs to the interior of at least one S^k; which is enough for the argument in the Kaneko-Wooders proof. (Informally, if some transfers are required between groups to support the point x_S there must be nonpositive net transfers to at least one S^k and this coalition S^k can improve.)]

Alternatively, the result stated above can be obtained from Wooders (1983, Lemmas 2,5,6,7 and Theorem 3). ■

The following concept of "-B-partition-balanced pro¯les completes our construction.

<u>"-B-partition-balanced pro¯les.</u> Given a positive integer B and a non-negative real number "; 0 · " · 1; a pro¯le f is "-B-partition-balanced if there is a subpro¯le f^0 of f such that $\frac{kf^0k}{kfk}$ ¸ 1 ¡ " and f^0 is B{partition-balanced.

The next result is key: given " > 0 and B; any su±ciently large pro¯le is "-B-partition-balanced. Note that this result is uniform across all large pro¯les.

Proposition. The balancing e®ect of large numbers. Given a positive integer B and a positive real number "; 0 < " · 1; there is a positive integer k("; B) such that any pro¯le f with kfk ¸ k("; B) is "-B-partition-balanced.

The idea of the proof: Lemma 1 provides a way to replicate a pro¯le that ensures "-B-partition-balancedness of the resulting replica. The manner of replication depends on the initially given pro¯le. Using Lemma 1, however, for any given " we can construct a number of \small" pro¯les that, when appropriately replicated, create B-partition-balanced replicas that \"-approximate" all su±ciently large pro¯les. The proof is presented in Appendix. ■

5.2 Mathematical examples.

In the following example we enlarge the player set of a given game so that the number of players of each type is arbitrary and illustrate the application of Theorem 1.

Example 1. Let $(N; V)$ be a game satisfying comprehensiveness. Suppose $jNj = T$: We construct a collection of games with T player types and strictly e®ective group sizes bounded by $B = T$: The games are indexed by $m \ 2 \ Z^T_+$: Given a vector m define $N_m[t] = \{(t,q) : q = 1,\ldots,m_t\}$ and define $N_m = \bigcup N_m[t]$. Next, define a characteristic function V_m in the following way. Let S be any coalition in N_m containing no more than one player from each set $N_m[t]$ and let S' be a subset of N with the same profile. Formally, let τ be a type-consistent 1-to-1 correspondence between S and S'. Define a payoff set V_{mS} for a coalition S as follows:

$$V_{mS} := V_{\tau(S)}$$

That is, any coalition S of players in N_m with the same profile as some coalition S' of players in N has the same payoff possibilities as S'. The function V_m is extended to the remaining coalitions in N_m by superadditivity. Specifically, for any $S \subset N_m$,

$$V_m(S) := \bigcup_{P(S)} \sum_{S' \in P(S)} V_m(S')$$

where $P(S)$ is a partition of S with the property that $|S' \cap N[t]| \leq 1$ for all members S' of the partition. The game (N_m, V_m) satisfies the condition on the class of games of Theorem 1.

Theorem 1 implies that given $\epsilon > 0$ there is a size of game $\eta_1(\epsilon, T, B)$ such that for all possible choices of m, if $\|m\| = |N_m| \geq \eta_1(\epsilon, T, B)$ then the game (N_m, V_m) has a non-empty ϵ-remainder core. But the theorem implies more; the bound $\eta_1(\epsilon, T, B)$ is independent of the initial characteristic function V. To clarify this remark, let (N, V') be another game with the same player set as (N, V) but there is no necessary relationship between V and V'. Then for all possible choices of m, if $\|m\| = |N_m| \geq \eta_1(\epsilon, T, B)$ then the game (N_m, V'_m) has a non-empty ϵ-remainder core where V'_m is defined from V' just as V_m was defined from V.

The following example continues Example 1 and illustrates the application of Theorem 2.

Example 2. Let $\mu > 0$ be real number and let B and T be positive integers. Consider the collection of games (N_m, V_m) defined in Example 1. Given a profile $m \in Z^T_+$ with $\|m\| \geq \eta_1(\epsilon, T, B)$ consider a superadditive game (N_m, W_m) satisfying the following properties:

$$V_m(S) \subset W_m(S) \subset V_m(S) + \mathbf{1}_{N_m}\mu \text{ for all } S \subset N_m.$$

For the game (N_m, W_m) it may be that none of the players are exact substitutes for each other and it may be that there are increasing returns to group size. The games (N_m, W_m), however, are members of the class $G^0((\mu, T), (\mu, B))$ and our Theorem applies. Given $\epsilon > 0$ the bound $\eta_1(\epsilon, T, B)$, depending only on ϵ, T, and B, has the property that if the game N_m has more than $\eta_1(\epsilon, T, B)$ players the $(\varepsilon, 2\mu)$-weak core of the game is non-empty. The result of the Theorem applies uniformly to all games (N_m, V_m) derived from a game (N, V) that has T types of players and strictly effective groups bounded in size by B.

Besides replica games, or indeed, any game with a fixed number of player types and a bound on near effective group sizes, our third theorem can accommodate games derived from pregames with a compact metric space of player types. The following example for games with side payments illustrates how our result can apply to such situations. For brevity, our example is somewhat informal.

Example 3. Consider the collection of games $G^q((\delta, T), (\beta, B))$ where $q = 1$, $T = 8$, $\delta = 1/4$, $B = 4$, $\beta = 0$ and where the games all have a per capita bound of $C = 2$. We illustrate how our results cover games derived from pregames with a compact metric space of player types.

Suppose a pregame has two sorts of players, firms and workers.[14] The set of possible types of workers is given by the points in the interval $[0, 1)$ and the set of possible types of firms is given by the points in the interval $[1, 2]$.

To derive a game from the information given above, let N be any finite player set and let ξ be an attribute function, that is, a function from N into $[0, 2]$. If $\xi(i) \in [0, 1)$ then i is a worker and if $\xi(i) \in [1, 2]$ then i is a firm. Firms can profitably hire up to three workers and the payoff to a firm i and a set of workers $W(i) \subset N$, containing no more than 3 members, is given by $v(\{i\} \bigcup W(i)) = \xi(i) + \sum_{j \in W(i)} \xi(j)$. Workers and firms can earn positive payoff only by cooperating so $v(\{i\}) = 0$ for all $i \in N$. For any coalition $S \subset N$ define $v(S)$ as the maximum payoff the group S could realize by splitting into coalitions containing either workers only, or 1 firm and no more than 3 workers. This completes the specification of the game.

We leave it to the reader to verify that every game derived from the pregame is a member of the class $G^1((\frac{1}{4}, 8), (0, 4))$ and has a per capita bound of 2. Our theorem states that given $\varepsilon > 0$, if $|N| \geq \eta_2(\varepsilon, 8, 4, 2, 1)$ then the game (N, v) has a non-empty $(\varepsilon + \frac{1}{4})$-core. In fact, the Theorem states this conclusion for an arbitrary game (N, V) described by the same parameter values, $T = 8$, $B = 4$, $\delta + \beta = \frac{1}{4}$, $C = 2$ and $q = 1$.

The following three examples illustrate that none of the assumptions of Theorem 3 can be omitted. The first two examples show that within the framework of parametrized collections of games, the equivalence of small group effectiveness and per capita boundedness that occurs when there are many players of each type, shown

[14] We refer the reader to Wooders and Zame (1984) or Wooders (1992) for a definition of a pregame with a compact metric space of player types.

in Wooders (1994b) for games with side payments, no longer holds; both small group effectiveness and per capita boundedness are required. Example 5 also shows that the non-emptiness of approximate cores of games derived from a pregame satisfying small group effectiveness and with a compact metric space of player types, shown in Wooders (1992), does not hold for arbitrary games.

Example 4. Small group effectiveness. Consider a sequence of games $(N^m; v^m)_{m=1}^{\infty}$ with side payments and where the m^{th} game has $3m$ players. Suppose that any coalition S consisting of at least $2m$ players can get up to $2m$ units of payoff to divide among its members, that is, $v^m(S) = 2m$. Assume that if $|S| < 2m$, then $v^m(S) = 0$. Observe that each game has one exact player type and a per capita bound of 1. That is, $q = 1, T = 1, C = 1$, and $\delta = 0$. However, the $\frac{1}{7}$-core of the game is empty for arbitrarily large values of m.

For any feasible payoff there are m players that get in total no more than $\frac{2m}{3m}m = \frac{2}{3}m$. There are another m players that get in total no more than $\frac{2m}{2m}m = m$. These $2m$ players can form a coalition and receive $2m$ in total. This coalition can improve upon the given payoff for each of its members by $\frac{1}{6}$, since $(2m - \frac{5}{3}m)\frac{1}{2m} = \frac{1}{6}$.

Example 5.[15] The per capita bound. Consider a sequence of games with side payments $(N^m; v^m)_{m=1}^{\infty}$ where the m^{th} game has $2m + 1$ players. Assume that any player alone can get only 0 units or less, that is $v^m(\{i\}) = 0$ for all $i \in N$. Also assume that any two-player coalition can get up to $2m$ units of payoff to divide; $v^m(S) = 2m$ if $|S| = 2$. An arbitrary coalition can gain only what it can obtain in partitions where no member of the partition contains more than two players. The games $(N^m; v^m)_{m=1}^{\infty}$ are members of the collection of games with one exact player type and strictly effective small groups of two. That is, $q = 1$, $T = 1$, $B = 2$, and $\delta = \beta = 0$. However, the $\frac{1}{7}$-core of the game is empty for arbitrarily large values of m.

To see this, observe that for any feasible payoff there is a player whose payoff is no more than $\frac{2m^2}{2m+1}$. There is the another player whose payoff must be no more than $\frac{2m^2}{2m} = m$. These two player may form a coalition and realize $2m$. Thus they gain $m - \frac{2m^2}{2m+1} = \frac{m}{2m+1} \geq \frac{m}{3m} = \frac{1}{3}$. Obviously, together this two-player coalition can improve upon the given payoff by $\frac{1}{6}$ for each member of the coalition.

The final example motivates the requirement of some transferability of payoff, and, in this paper, the condition of Theorem 3 that q is greater than zero.

Example 6. The positivity of q. Consider a sequence of games without side payments $(N^m; V^m)_{m=1}^{\infty}$ where the m^{th} game has $2m + 1$ players. Suppose that

[15] A similar example in Wooders and Zame (1984).

any player alone can earn only 0 units or less. Suppose that any two-player coalition can distribute a total payo® of 2 units in any agreed-upon way, while there is no transferability of payo® between coalitions. Suppose only one- and two-player coalitions are e®ective. Then the game is described by the following parameters: $q = 0; T = 1; B = 2; \pm = {}^{-} = 0$: Moreover, the game has per capita bound $C = 1$: Thus the game satis¯es strict small group e®ectiveness and per capita boundedness. However, the $\frac{1}{3}$-core of the game is empty for arbitrarily large values of m: (At any feasible payo®, at least one player gets 0 units and some other player no more than 1 unit. These two players can form a coalition and gain $\frac{1}{2}$ each.)

6 Conclusions.

Except in certain idealized situations, cores of games are typically empty. This has the consequences that important classes of economies typically have empty cores and a competitive equilibrium does not exist. Examples include economies with indivisibilities and other nonconvexities, economies with public goods subject to crowding, and production economies with non-constant returns to scale. The standard justi¯cation for convexity, assumed in Arrow-Debreu-McKenzie models of exchange economies, is that the economies are \large," rendering nonconvexities negligible { the convexifying e®ect of large numbers. Similarly results on non-emptiness of approximate cores rely on large numbers of players and the balancing e®ect of large numbers. An important aspect of our results in this paper is that they are for arbitrary games and the bounds depend on the parameters describing the games; the compact metric space of player types assumed in previous work is a special case. Moreover, our approach allows both widespread externalities and uniform results.

It appears that the framework of parametrized collections of games and our approach will have a number of uses. In ongoing research this framework is used to demonstrate further market-like properties of arbitrary games[16]: approximate cores are nearly symmetric { treat similar players similarly; arbitrary games are approximately market games; and arbitrary games satisfy a \law of scarcity," dictating that an increase in the abundance of players of a given type does not increase the core payo®s to members of that type. In addition, some initial results have been obtained on convergence of cores and approximate cores. A particularly promising direction appears to be the application of ideas of lottery equilibrium in games of Garratt and Qin (1994) to parameterized collections of games. Another possible application is to games with asymmetric information, as in Allen (1994), for example, and Forges

[16] See Shapley and Shubik (1966,1969) for seminal results of this nature and Wooders (1994a,b) for more recent references to related results in the context of pregames and economies with clubs/local public goods.

(1998).[17]

7 Appendix.

A technical lemma is required. Denote by $\|\cdot\|$ the sum-metric in R^T, that is, for $x, y \in R^T$, $\|x - y\| := \sum_i |x_i - y_i|$. For any positive integer η, define

$$\Delta_{[\eta]} := \left\{ x \in \Delta_+ : \eta x \in Z_+^T \right\}.$$

Lemma 2. For each $\varepsilon > 0$ there exists a positive integer $\eta(\varepsilon)$ such that for any $f \in R^T_+$ there is a vector $g \in R^T_+$ satisfying $g \le f$, $\|f\| - \|g\| = \|f - g\| \le \varepsilon \|f\|$ and $\frac{g}{\|g\|} \in \Delta_{[\eta(\varepsilon)]}$.

Proof of Lemma 2: Let us first prove that for $\eta(\varepsilon) > \frac{T}{\varepsilon}$ we have that $(1+\varepsilon)\Delta_+ \subset \Delta_{[\eta(\varepsilon)]} + \varepsilon \Delta_+$. Consider any $a = (a_1, \dots, a_T) \in (1+\varepsilon)\Delta_+$. (That is $\sum_{i=1}^T a_i = 1 + \varepsilon$ and $a_i \ge 0$ for each $i = 1, \dots, T$.) Let us define $I^k \in R^T$ such that $I^k_l = 1$ for $k = l$ and 0 otherwise. Notice that $I^k \in \Delta_{[\eta(\varepsilon)]}$ for any k. If there exist j such that $a_j \ge 1$, then $(a - I^j) \in \varepsilon \Delta_+$ and thus $a = I^j + (a - I^j) \in \Delta_{[\eta(\varepsilon)]} + \varepsilon \Delta_+$. If $a_j < 1$ for any j, then let us consider $a_j = \frac{l_j}{\eta(\varepsilon)} + r_j$, where l_j is an integer, $l_j < \eta(\varepsilon)$, and $0 \le r_j < \frac{1}{\eta(\varepsilon)}$. Then

$$a = (a_1, \dots, a_T) \in \left(\frac{l_1}{\eta(\varepsilon)}, \dots, \frac{l_T}{\eta(\varepsilon)}\right) + \frac{T}{\eta(\varepsilon)}\Delta_+ \subset \Delta_{[\eta(\varepsilon)]} + \frac{T}{\eta(\varepsilon)}\Delta_+ \subset \Delta_{[\eta(\varepsilon)]} + \varepsilon \Delta_+.$$

Now given a profile f, observe that $\frac{f}{\|f\|} \in \Delta_+ \subset (\Delta_{[\eta(\varepsilon)]} + \varepsilon \Delta_+)\frac{1}{1+\varepsilon}$. Therefore there exists $h \in \frac{1}{1+\varepsilon}\Delta_{[\eta(\varepsilon)]}$ such that $\frac{f}{\|f\|} \in (\{h\} + \frac{\varepsilon}{1+\varepsilon}\Delta_+)$. Now, define $g := h\|f\|$. Then $g \le f$ and, by construction, $\frac{g}{\|g\|} = (1+\varepsilon)h \in \Delta_{[\eta(\varepsilon)]}$. Moreover $\frac{\|f\| - \|g\|}{\|f\|} = 1 - \frac{1}{1+\varepsilon} = \frac{\varepsilon}{1+\varepsilon} < \varepsilon$. ■

Proof of Proposition: Given a positive integer η, we first define an integer that will play an important role in the proof. Arbitrarily select $x \in \Delta_{[\eta]}$ and define $y(x) := \eta x \in Z^T_+$. Since $y(x)$ is a profile, by Lemma 1 there is an integer $m(y(x); B)$ such that for any integer k the profile $km(y(x); B)y(x)$ is B-partition-balanced. There exists such an integer $m(y(x); B)$ for each $x \in \Delta_{[\eta]}$. Since $\Delta_{[\eta]}$ contains only a finite number of points, there is a finite integer $M(\eta; B)$ such that $\frac{M(\eta;B)}{m(y(x);B)}$ is an integer for any $x \in \Delta_{[\eta]}$.

[17] We are grateful to Francoise Forges for pointing out this possible application.

By Lemma 2, given $\frac{\varepsilon}{2} > 0$, there exists a positive integer $\eta' := \eta(\frac{\varepsilon}{2})$, such that for any $f \in \mathbb{R}^T_+$ there exists a vector $g \in \mathbb{R}^T_+$ satisfying

$$
\begin{aligned}
&g \le f, \\
&\|f\| - \|g\| = \|f - g\| \le \tfrac{\varepsilon}{2}\|f\| \text{ and} \\
&\tfrac{g}{\|g\|} \in \Delta_{[\eta']}.
\end{aligned}
$$

Arbitrarily select $f \in \mathbb{R}^T_+$ and let $g \in \mathbb{R}^T_+$ be a vector satisfying the above conditions. Define $y^* := \eta' \frac{g}{\|g\|}$. Since $\frac{g}{\|g\|} \in \Delta_{[\eta']}$, it holds that $y^* \in \mathbb{Z}^T_+$. Therefore y^* is a profile. Moreover, by the choice of $M(\eta'; B)$, the $kM(\eta'; B)^{th}$-replica of the profile y^* is B-partition-balanced for any integer k.

Observe that there is an integer k^0, possibly equal to zero, such tha

$$k^0 M(\eta'; B) y^* \le g < (k^0 + 1) M(\eta'; B) y^*.$$

Define

$$f' := k^0 M(\eta'; B) y^* = k^0 M(\eta'; B)\eta' \frac{g}{\|g\|}.$$

Obviously, f' is a profile ($f' \in \mathbb{Z}^T_+$) and $f' \le g \le f$. Suppose that $k^0 > 0$. Then

$$\frac{f'}{\|f'\|} = \frac{g}{\|g\|} \text{ and } \|g\| - \|f'\| = \|g - f'\| \le M(\eta'; B)\eta'.$$

Moreover, the profile f' is B-partition-balanced since it is a replica of the profile y^*.

Now, define $k(\beta; B) := M(\eta'; B)\eta' \frac{2}{\varepsilon}$. If $\|f\| \ge k(\beta; B)$, then

$$
\begin{aligned}
&k^0 > 0, \; f' \le g \le f, \\
&\|f - g\| \le \tfrac{\varepsilon}{2}\|f\|; \text{ and} \\
&\|g - f'\| \le M(\eta'; B)\eta' \le \tfrac{\varepsilon}{2}\|f\|.
\end{aligned}
$$

Therefore $\|f\| - \|f'\| = \|f - f'\| \le \varepsilon\|f\|$. Thus f' is a subprofile of f, $\frac{\|f'\|}{\|f\|} \ge 1 - \varepsilon$, and f' is B-partition-balanced. ■

Proof of Theorem 1: Fix the number of types T and consider the bound $k(\varepsilon; B)$ from Proposition. Let $\eta_1(\varepsilon; B; T) := k(\varepsilon; B)$. Let $(N; V)$ be a game with $|N| \ge k(\varepsilon; B)$. Denote the profile of N by f. By Proposition, f is ε-B-partition-balanced. That is, there is a B-partition-balanced subprofile f' of f such that $\frac{\|f'\|}{\|f\|} \ge 1 - \varepsilon$. Now select some $S \subset N$ such that $|S[i]| = f'_i$ for any $i = 1, \ldots, T$. Then $\frac{|N| - |S|}{|N|} \le \varepsilon$ by choice of S and the subgame $(S; V)$ has a non-empty core. Thus the ε-remainder core of $(N; V)$ is non-empty. ■

Proof of Theorem 2: For any $S \subset N$ define $V'(S) := \bigcap \sigma_\tau^{-1}(V(\tau(S)))$, where the intersection is taken over all type-preserving permutations τ of the player set

N. Then $(N;V') \in G^q((0;T);(\beta;B))$. Moreover, from the definition of $V'(S)$ it follows that $V'(S) \subset V(S)$: (Informally, taking the intersection over all type-preserving permutations makes all players of each approximate type no more productive than the least productive members of that type.) From the definition of δ-substitutes, it follows that $H_1[V'(S);V(S)] \le \delta$ for any $S \subset N$.

Now for any $S \subset N$, define $V^q(S) := c_q(V'(S;B))$. Then $(N;V^q) \in G^q((0;T);(0;B))$. Moreover, $V^q(S) \subset V'(S) \subset V(S)$ and $H_1[V^q(S);V(S)] \le H_1[V^q(S);V'(S)] + H_1[V'(S);V(S)] \le \beta + \delta$.

By Theorem 1, if $|N| \ge \eta_1(\varepsilon;T;B)$ then the ε-remainder core of the game $(N;V^q)$ is non-empty. That is, there exists $S \subset N$ such that $\frac{|N|-|S|}{|N|} \le \varepsilon$ and such that $(S;V)$ has a non-empty core. Let x be a payoff in the core of the game $(S;V^q)$. Since $V^q(S) \subset V(S)$, the payoff x is feasible and $(\beta+\delta)$-undominated for the game $(S;V)$. Thus, the weak $(\varepsilon;(\beta+\delta))$-remainder core of $(N;V)$ is non-empty. ■

Proof of Theorem 3: As in the proof of Theorem 2 first construct the game $(N;V^q) \in G^q((0;T);(0;B))$. As noted in the proof of Theorem 2, $V^q(S) \subset V(S)$ and $H_1[V^q(S);V(S)] \le \beta + \delta$ for any $S \subset N$. In addition, the game $(N;V^q)$ has a per capita bound of C. We required that $2(\beta+\delta) < m^*$. Assume first that $2\varepsilon \le m^*$: Thus $(\varepsilon + \beta + \delta) < m^*$.

Applying Theorem 1 for $\varepsilon' := \frac{q}{BC}\varepsilon$ to the game $(N;V^q)$ we found that for $|N| \ge \eta_1(\varepsilon';B;T)$ there is some subset of players $S \subset N$ with $\frac{|N|-|S|}{|N|} \le \varepsilon'$ such that the game $(S;V^q)$ has a non-empty core. Let x be a payoff in the core of $(S;V^q)$. We now construct a payoff $y \in R^N$ for the game $(N;V)$: For $a \in S$, define $y_a := x_a - \varepsilon$ and for $a \notin S$, define $y_a := BC - \varepsilon$. Observe that y is in the ε-core of the game $(S;V^q)$.

We next need to show that $y \in V^q(N)$. Since $\frac{|N|-|S|}{|N|} \le \varepsilon' = \frac{q}{BC}\varepsilon$, it holds that

$$q\varepsilon|N| \ge BC(|N|-|S|):$$

Since $q \le 1$, it follows that

$$q\varepsilon|S| \ge (BC-\varepsilon)(|N|-|S|):$$

Informally, this means that we can take ε away from each player in S, transfer this amount to the players in $N\setminus S$ at the rate q; and increase the payoff to each player in $N\setminus S$ to $BC - \varepsilon$: Therefore since $x \in V^q(S)$; by superadditivity and by q-comprehensiveness of payoff sets it holds that $y \in V^q(N)$.

We now prove that the payoff y is ε-undominated in the game $(N;V^q)$. The strategy of the proof is to show that if y is ε-dominated in the game $(N;V^q)$ then it can be ε-dominated by some coalition (to be called) $A \subset S$: We thus obtain a contradiction. The proof proceeds through two steps.

The first step is to construct the coalition A. Suppose that y is ε-dominated in the game $(N;V^q)$ by some coalition W: Specifically, suppose there exists a payoff vector

z such that

$$z \in V^q(W) = c_q(V^q(W;B)) \text{ and}$$

$$z_W >> y_W + \mathbf{1}_W \varepsilon.$$

Since $z \in V^q(W)$ there exists some partition $\left\{W^k\right\}$ of W, $\left|W^k\right| \leq B$ and some payoff $z' \in \sum_k V^q(W^k)$ such that z can be obtained from z' by making "transfers" at the rate q between agents in W. Let

$$A := \bigcup \left\{ W^k : W^k \subset S \right\} \text{ and let } A^L := \bigcup \left\{ W^k : W^k \setminus S \neq \emptyset \right\};$$

that is, A consists of those members of subsets in $\{W^k\}$ that are contained in S and A^L consists of those members of subsets of $\{W^k\}$ that contain at least one player from $N \setminus S$.

The second step is to show that the set A is non-empty and can ε-dominate the payoff y. Since y is in the ε-core of the subgame $(S; V^q)$ it is clear that the coalition W must contain at least one member of $N \setminus S$; therefore the set A^L must be non-empty. Observe that for any $W^k \subset A^L$ and $x' \in V^q(W^k)$, it holds that $\sum_{a \in S^k} x'_a \leq BC$. There exists, however, $a \in W^k \setminus S$ such that $z_a >> y_a + \varepsilon = BC$. Thus, z can be feasible in $V^q(W)$ only by some transfers from the players in the set A to the players in the set A^L. This implies that the set A is non-empty. Moreover the coalition A is not a net beneficiary of transfers needed to support the payoff z. This implies that there is a payoff $z'' \in V^q(A)$ such that for all players $a \in A$,

$$z''_a \geq z_a > y_a + \varepsilon.$$

Since $A \subset S$, this is a contradiction to the construction of y as a payoff in the ε-core of the game $(S; V^q)$. We conclude that y is ε-undominated in the game $(N; V^q)$.

Since the payoff y is ε-undominated in the game $(N; V^q)$, for $|N| \geq \eta_1(\frac{q}{BC}\varepsilon; B; T)$ the payoff y is in the ε-core of the game $(N; V^q)$. This implies that y is feasible and $(\varepsilon + \beta + \delta)$-undominated in the initial game $(N; V)$, providing that $|N| \geq \eta_1(\frac{q}{BC}\varepsilon; B; T)$. Let $\eta_2(\varepsilon; B; T; C; q) := \eta_1(\frac{q}{BC}\varepsilon; B; T)$. Thus, we proved that for $|N| \geq \eta_2(\varepsilon; B; T; C; q)$ the $(\varepsilon + \beta + \delta)$-core of the game $(N; V)$ is non-empty.

For $\varepsilon > \frac{m^*}{2}$ let us define $\eta_2(\varepsilon; B; T; C; q) := \eta_2(\frac{m^*}{2}; B; T; C; q)$. Then for $|N| \geq \eta_2(\varepsilon; B; T; C; q)$ again the $(\varepsilon + \beta + \delta)$-core of the game $(N; V)$ is non-empty. ■

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