ANSWERS TO TEST NUMBER 7

Question 1: (25 points)

a) A point estimate of the difference between the mean log durations is the difference between the sample means,

$$\bar{X}_1 - \bar{X}_2 = .593 - .973 = -.38.$$

[Give yourself 1 point for understanding this.]

The variance of the difference of the sample means is

$$s^2_{\bar{X}_1-\bar{X}_2} = s_c^2\left[\frac{1}{n_1}+\frac{1}{n_2}\right] = s_c^2\left[\frac{1}{13}+\frac{1}{15}\right] = s_c^2(0.0769+0.0667) = 0.1436\, s_c^2.$$

This variance is a pooled estimate

$$s_c^2 = \frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2} = \frac{(12)(.294^2)+(14)(.349^2)}{26} = \frac{1.037+1.705}{26} = .1055$$

which when substituted into the equation for the variance of the difference of the sample means yields

$$s^2_{\bar{X}_1-\bar{X}_2} = 0.1436\, s_c^2 = (.1436)(.1055) = .0151454.$$

The standard deviation of the difference of the sample means thus becomes $s_{\bar{X}_1-\bar{X}_2} = \sqrt{.0151454} = 0.1230668$.

[Give yourself 3 points for understanding how to calculate the standard deviation of $\bar{X}_1 - \bar{X}_2$.]

The critical value of t for a 90% confidence interval with 26 degrees of freedom is $t((1-\alpha/2); 26) = t(.95; 26)$. This can be calculated from XLISPSTAT as follows:

```
> (t-quant .95 26)
1.7056179197376315
```

The upper and lower confidence limits are thus

$$\bar{X}_1 - \bar{X}_2 \pm 1.706\, s_{\bar{X}_1-\bar{X}_2} = -.38 \pm (1.706)(.1230668) = -.38 \pm .2099521$$

yielding the interval [-.5899, -.1700].

[Give yourself 3 points for knowing how to calculate the confidence interval.]

b) To test the null hypothesis that $\mu_1-\mu_2 = 0$ against the alternative hypothesis that $\mu_1 - \mu_2 \neq 0$ we calculate the t-value

$$t = \frac{(\bar{X}_1-\bar{X}_2)-(\mu_1-\mu_2)}{s_{\bar{X}_1-\bar{X}_2}} = \frac{-.38}{.1230668} = -3.087754$$

which is clearly lower than -1.706, the critical value of t for $\alpha = .10$ in a two-tailed test.

[Give yourself 3 points for knowing how correctly test the hypothesis.]

c) We know that

$$\frac{\Sigma(X_i - \bar{X})^2}{\sigma^2} = \chi^2(n-1)$$

and that

$$s^2 = \frac{\Sigma(X_i - \bar{X})^2}{n-1}$$

from whence

$$\Sigma(X_i - \bar{X})^2 = (n-1)\, s^2$$

and

$$\frac{(n-1)\, s^2}{\sigma^2} = \chi^2(n-1).$$

Also, the ratio of two χ^2 distributions, each divided by its degrees of freedom, is distributed as $F(v_1, v_2)$ where v_1 is the degrees of freedom of the χ^2 distribution in the numerator and v_2 is the degrees of freedom of the χ^2 distribution in the denominator.

$$\frac{\chi^2(v_1)/v_1}{\chi^2(v_2)/v_2} = F(v_1, v_2)$$

Thus we can take the ratio of

$$\frac{(n_1-1)\, s_1^2/\sigma_1^2}{n_1-1} \qquad \text{and} \qquad \frac{(n_2-1)\, s_2^2/\sigma_2^2}{n_2-1}$$

to obtain

$$\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} = F(n_1-1, n_2-1)$$

Our null hypothesis is $H_0 : \sigma_1^2 = \sigma_2^2$, so the above becomes

$$\frac{s_1^2}{s_2^2} = F(12, 14)$$

[Give yourself 5 points for understanding the relevant F statistic and how to calculate it.]

The critical values of $F(12, 14)$ for $\alpha = 10$ are

$$F(95; 12, 14) = 2.53 \qquad \text{and} \qquad F(05; 12, 14) = \frac{1}{F(.95; 14, 12)} = \frac{1}{2.64} = .38$$

These critical values can be calculated using Xlispstat as follows:

```
> (F-quant .95 12 14)
2.5342432527485617
> (F-quant .95 14 12)
2.6371235576309124
> (/ 1 2.6371235576309124)
0.3792010416449203
> (F-quant .05 12 14)
0.3792010416449203
```

The rejection region contains the values of the test statisic bigger than 2.53 and smaller than .38.

The test statistic is

$$\frac{s_1^2}{s_2^2} = \frac{.294^2}{.349^2} = \frac{0.0864}{0.1218} = .7094$$

so we cannot reject the null hypothesis that the two variances are the same.

[Give yourself 5 points for understanding how to calculate the critical values and a further 5 points for knowing how to calculate and interpret the test statistic to reach the correct conclusion.]

Question 2: (25 points)

The table we are given can be expanded by summing the rows and columns.

	Doctors	Lawyers	Engineers	Total
Protestant	64	110	152	326
Catholic	60	86	78	224
Jewish	57	21	10	88
Total	181	217	240	638

By dividing the totals by 638 we can obtain the marginal probabilities of 'having a particular religion' and 'having a particular profession'.

	Doctors	Lawyers	Engineers	Total	
Protestant	64	110	152	326	.511
Catholic	60	86	78	224	.351
Jewish	57	21	10	88	.138
Total	181	217	240	638	
	.284	.340	.376		1.00

[Give yourself 5 points for understanding how to calculate the marginal probabilities.]

Given two events A and B, the joint probability of A given B can be written

$$P(A \cap B) = P(A|B)P(B)$$

If A and B are independent, the conditional probability of A given B is the probability of A. The value of A will be independent of the value of B so that

$$P(A|B) = P(A).$$

Hence

$$P(A \cap B) = P(A)\,P(B).$$

Thus, if 'religion' and 'profession' are statistically independent, the probability that a member of the association in question will be both a lawyer and a catholic will equal the marginal probability of being a lawyer times the marginal probability of being a catholic. An estimate of the joint probability distribution of 'religion' and 'profession' for members of the association will thus be given by the following table—the entry in each cell is the product of the marginal probability for the corresponding row and the marginal probability for the corresponding column.

	Doctors	Lawyers	Engineers	Total
Protestant	.145	.174	.192	.511
Catholic	.100	.119	.132	.351
Jewish	.039	.047	.052	.138
Total	.284	.340	.376	1.000

From these estimates of the joint probabilities we can then calculate the number of members we would expect to find in each cell if 'religion' and 'profession' were independent—we simply multiply all the joint probabilities by 638. This yields

	Doctors	Lawyers	Engineers	Total
Protestant	64 [92.5]	110 [111.0]	152 [122.5]	326
Catholic	60 [63.8]	86 [75.9]	78 [84.2]	224
Jewish	57 [24.9]	21 [30]	10 [33.2]	88
Total	181	217	240	638

The numbers we would expect, given independence, are in the square brackets.

[Give yourself 5 points for understanding the above-noted implications of independence and 5 points for knowing how to correctly calculate the numbers that would be in each cell if independence holds.]

The question is whether the degree of divergence between the actual and expected numbers in the cells is greater than could reasonably be accounted for by sampling error. To address it we calculate the statistic

$$\Sigma \frac{(f_i - F_i)^2}{F_i}$$

which will be distributed as $\chi^2[(nr-1)(nc-1)]$, where f_i and F_i are the actual and expected numbers, respectively, in the i-th cell and nc and nr are the number of rows and columns in the distribution of the joint probabilities. This statistic is calculated as follows

i	f_i	F_i	$(f_i - F_i)^2/F_i$l
Protestant/Doctor	64	92.5	8.780
Protestant/Lawyer	110	111	.009
Protestant/Engineer	152	122.5	7.104
Catholic/Doctor	60	63.8	.226
Catholic/Lawyer	86	75.9	1.344
Catholic/Engineer	78	84.2	.456
Jewish/Doctor	57	24.9	41.382
Jewish/Lawyer	21	30	2.700
Jewish/Engineer	10	33.2	16.212
Total			78.213

Since $nr = nc = 2$, $(nr - 1)(nc - 1) = 4$. Using Xlispstat, we calculate the quantiles of the $\chi^2(4)$ distribution beyond which 5% and 1% of the probability weight lies as 9.49 and 13.28, respectively.

```
> (chisq-quant .95 4)
9.487729036781156
> (chisq-quant .99 4)
13.27670413598762
```

We must reject the null hypothesis of independence at both the 5% and 1% levels.

[Give yourself 5 points for knowing how to calculate the χ^2 statistic and an additional 5 points for understanding how to interpret it to reach the correct conclusions at the 5 and 1 percent levels.]

Question 3: (25 points)

If the probability distribution is uniform, the hose is equally likely to break at any point along its 1.5-meter length. We need to distribute the data among classes and test whether the observed number of occurrances in each class equals the number expected if the distribution is uniform. We must have as many classes as possible subject to the requirement that there be at least 5 expected occurrances in each class. This suggests 5 classes ranging from 0 to 1.5, implying that the width of each class interval will be 0.3. The null hypothesis is that the data are distributed according to a uniform probability distribution. We have to see if the deviations of the actual numbers of occurrances from the expected numbers in the five classes are greater than would reasonably be due to sampling error.

The calculations can be organized into the following table.

i	f_i	F_i	$(f_i - F_i)^2/F_i$
0.0 to 0.3	3	5	0.8
0.3 to 0.6	2	5	1.8
0.6 to 0.9	3	5	0.8
0.9 to 1.2	6	5	1.2
1.2 to 1.5	11	5	7.2
Sum			11.8

The number of degrees of freedom is $k - m - 1 = 5 - 0 - 1 = 4$. There are 5 classes ($k = 5$) and we calculated no statistics from the data ($m = 0$). The quantiles of the $\chi^2(4)$ distribution beyond which less than 10%, 5% and 1% of the probability weight lies are 7.78, 9.48, and 13.28, respectively. These quantiles are obtained using the following Xlispstat commands:

```
> (chisq-quant .90 4)
7.779440339734859
> (chisq-quant .95 4)
```

```
9.487729036781156
> (chisq-quant .99 4)
13.276704135987622
```

Since our χ^2 statistic is 11.8 we can reject the null hypothesis at an α risk of .05 but not at an α risk of .01. The P-value lies a bit more than half way between .01 and .05.

[Give yourself 10 points for setting up correct classes, 10 points for knowing how to calculate the test statistic and recognizing that it is distributed as $\chi^2(4)$ and 5 points for knowing how to complete the test.]

Question 4: (25 points)

a) The table can be extended to give the total number of food items checked from each type of store and the proportion correctly priced as follows:

Store Type	Number Correctly Priced	Number Incorrectly Priced	Total Number Checked	Proportion Correctly Priced
National Chain	89	10	99	.899
Regional Chain A	53	14	67	.791
Regional Chain B	45	12	57	.789
Regional Chain C	38	13	51	.745
Independent	40	7	47	.851
Total	265	56	321	.825

The null hypotheses is that the proportion correctly priced in the population is the same for all stores—a point estimate of this proportion is .825. If the null hypothesis is true, the numbers correctly priced in the 5 store types should be equal to .825 times the number of stores checked in each category.

Store Type	Actual Number Correctly Priced f_i	Expected Number Correctly Priced F_i	$(f_i - F_i)^2/F_i$
National Chain	89	81.675	.657
Regional Chain A	53	55.275	.094
Regional Chain B	45	47.025	.087
Regional Chain C	38	42.075	.395
Independent	40	38.775	.039
Sum	265		1.272

Since there are $k = 5$ classes and we estimated no statistics in setting up the table, so $m = 0$, the number of degrees of freedom is 4. The sum is thus distributed as $\chi^2(4)$. The critical value of $\chi^2(4)$ for $\alpha = .10$ is 7.78 so our statistic 1.272 clearly falls in the acceptance region.

[Give yourself 5 points for establishing the correct null hypothesis and knowing how to calculate the correct point estimate of the proportion correctly priced by all store types under the null. Then give yourself 5 points for correctly calculating the test statistic and a further 5 points for understanding that it is distributed as $\chi^2(4)$ and correctly performing the test.]

b) We want to estimate the proportion of correctly priced items in the national chain category. The point estimate is $\bar{p} = .657$. Since there is no hypothesized value of p the variance of $\bar{p}$ is

$$s_{\bar{p}}^2 = \frac{(\bar{p})(1-\bar{p})}{n-1} = \frac{(.657)(.343)}{46} = \frac{.22535}{46} = .0049,$$

which makes the standard deviation equal to $s_p = .07$. The confidence interval will thus be

$$.657 \pm z(1-\alpha/2)\, s_p = .657 \pm z(.975)\, .07 = .657 \pm (1.96)(.07) = .657 \pm .1372.$$

The interval is thus [.5198, .7942].

[Give yourself 1 point for understanding what the correct point estimate is, 5 points for knowing how to calculate the variance of $\bar{p}$ and 4 points for knowing how to calculate the correct confidence interval.]