ANSWERS TO TEST NUMBER 5

Question 1: (25 points)

a) The critical value must be $z(.99)$ standard deviations above 0. Using XLISP-STAT, we calculate $z(.99)$ as

```
> (normal-quant .99)
2.3263478740408408
```

so the critical value becomes

$$z(.99)\frac{\sigma}{\sqrt{n}} = z(.99)\frac{1.3}{\sqrt{100}} = (2.326)(1.3)/10 = .30238$$

and we reject the null hypothesis if $\bar{X} > .30238$.

[Give yourself 5 points for knowing how to calculate the critical value and understanding the decision rule.]

b) When $\mu = 0$ the rejection probability (i.e., probability of rejecting the null) is equal to α which is .01. When $\mu = 0.5$ the critical value is to the left of the mean and the rejection probability is the area of a probability distribution centered on 0 to the right of .32038. The standardized value of .32038 is

$$z^* = \frac{.32038 - .5}{1.3/10} = \frac{.32038 - .5}{.13} = \frac{-.17962}{.13} = -1.381692307$$

The area of the standard normal distribution to the right of this value of z, which is equal to the area to the left of 1.381692307, can be calculated using XLISPSTAT as

```
> (normal-cdf 1.381692307)
0.9164665469144261.
```

The probability of rejecting the null hypothesis is about .92. The standardized value of the distance of the critical value .32038 to the left of $\mu = 1$ is

$$z^* = \frac{.32038 - 1}{.13} = \frac{-.67962}{.13} = -5.2278461538.$$

The probability mass of the normal distribution to the right of -5 standard deviations is

```
> (normal-cdf 5.2278461538)
.999999142519418
```

The β risk is, for practical purposes, zero. The rejection probability for $\mu = 1.5$ is even closer to 1.0.

[Give yourself 10 points for understanding how to calculate these rejection probabilities.]

c) The rejection curve starts at a vertical distance of .01 from the horizontal axis at the position zero on that axis—i.e., where μ, which is measured along the horizontal axis, is zero. The curve then rises at an increasing rate for some distance and then at a decreasing rate for the remaining (infinite) distance to the right along the horizontal axis, attaining a value of .92 when $\mu = .5$ and becoming extremely close to a horizontal line drawn at 1 on the vertical axis at $\mu = 1$ and tangent to that line as $\mu \to \infty$.

[Give yourself 5 points for knowing how to draw this curve.]

d) When $\mu = .60$ the correct decision is to reject the null hypothesis that $\mu < 0$. The rejection probability (probability of rejecting the null hypothesis) is virtually 1 at $\mu = .5$. It will be even closer to 1 at $\mu = .6$. The probability that the above decision rule will lead to an incorrect conclusion—i.e., accept the null hypothesis and reject the alternative hypothesis—is miniscule. This miniscule probability of erroneously rejecting the alternative hypothesis is the β risk.

[Give yourself 1 point for understanding what the correct decision is, 2 points for knowing what the rejection probability is, and 2 points for knowing that the probability that the decision rule will reach an incorrect conclusion when $\mu = .6$ is the β risk.

Question 2: (25 points)

a) Obviously, the worst error (Type I) would be to fail a student when that student has a level of understanding is above 50% and should pass. Accordingly, we let the null hypothesis be $H_0 : \mu \geq 50$ with the alternative hypothesis then being $H_1 : \mu < 50$. Erroneously rejecting the alternative hypothesis—i.e., passing a student who should have failed—is a Type II error.

[Give yourself 2 points for realizing what the correct null hypothesis is and 3 points for understanding what are the Type I and Type II errors.]

b) We have $\bar{X}_1 = \bar{X}_2 = 48$ for two students. Student 1 has a sample standard deviation of $s_1 = 8$ and student 2 has a sample standard deviation of $s_2 = 12$. Both students, it turns out, have the same level of understanding, $\mu_1 = 46$. The standard deviations of the two students' means are therefore

$$s_{\bar{X}_1} = \frac{8}{\sqrt{n}} = \frac{8}{8} = 1$$

and

$$s_{\bar{X}_2} = \frac{12}{\sqrt{n}} = \frac{12}{8} = 1.5.$$

It turns out that both students have mean scores less than 50. The question is whether the probability of observing these scores is less than .05 if the students' true understanding is 50. The .05 critical values will be

$$A_1 = 50 - (1.645)(1) = 48.355$$

for student 1 and

$$A_2 = 50 - (1.645)(2) = 50 - 3.29 = 46.71$$

for student 2.

If we center normal sampling distributions of the two students' means on their true understanding, 46, we obtain z values for the distances between their true understanding and the rejection scores of of

$$z_1 = \frac{48.355 - 46}{1} = 2.355 \qquad z_2 = \frac{46.71 - 46}{1.5} = 1.065$$

The probabilities that the two students would be passed when they should have failed (i.e. of rejecting the alternative hypothesis when it is true) are, respectively,

```
> (- 1 (normal-cdf 2.355))
0.00926135284833618
> (- 1 (normal-cdf 1.065))
0.14343796487675098
```

These areas to the right of 2.355 and 1.065 (the `normal-cdf` function gives the areas to the left of these values) are the respective β risks.

[Give yourself 3 points for understanding how to calculate the correct standard deviations of the students' mean scores and 4 points for understanding how to calculate the critical values. Give yourself 8 points for understanding how to calculate the β-risk in the two cases.]

c) The β risk (i.e., the probability of passing a student that should have failed) is greater for the student with the higher variability. Given the level of true understanding and the term average, the greater the variability of a student's performance the greater the chance of passing in spite of a failing performance. The power of test (i.e., the probability of accepting the alternative hypothesis when it is true), which equals $1 - \beta$, is thus lower when the students test scores show greater variability.

[Give yourself 5 points for understanding how to calculate the power of test and drawing the correct conclusion.]

Question 3: (25 points)

You are given the following probabilities:

$P(TG|G) = .9$ (Tests guilty conditional upon actually being guilty)
$P(TI|I) = .98$ (Tests not-guilty conditional upon actually being innocent)
$P(G) = .12$ (Probability of being innocent)

[Give yourself 2 points for recognizing each of the above.]

The right-most column in the table below can now be filled in.

Calculate

$P(TG \cap G) = P(TG|G)P(G) = (.9)(.12) = .108$
$P(TI \cap I) = P(TI|I)P(I) = (.98)(.88) = .8624$
and enter them in the appropriate cells in the table below.

[Give yourself 3 points for correctly doing each of the above two calculations (6 points in total).]

You can then fill in the rest of the table.

		Test		
		Guilty	Innocent	
Actual	Guilty	.1080	.0120	.12
	Innocent	.0176	.8624	.88
		.1256	.8744	1.00

[Give yourself 1 point for getting each of the numbers in the table correct and in the right place (9 points in total).]

The probability of a suspect being innocent, given that he tested guilty (i.e., failed the polygraph test), is

$P(I|TG) = P(I \cap TG)/P(TG) = .0176/.1256 = .14012$

[Give yourself 4 points for recognizing that the above calculation is required and doing it correctly.]

Question 4: (25 points)

The inspector can decide, acting on behalf of the seller, to control the α risk at .01 when $\mu_0 = 50$ and set the null hypotheses at $H_0 : \mu \geq 50$, calculating a critical value A at some point below 50. The alternative hypothesis thus becomes $H_1 : \mu > 50$. The inspector would then, acting on behalf of the purchaser, control the β risk at .05 when $\mu_1 = 49$ given the critical value, which must then be at a point between 49 and 50 for which the α risk is .01 and the β risk is .05. The standardized distance between A and 50 must be equal to $z(.99) = 2.326$ and the standardized distance between 49 and A must equal $z(.95) = 1.645$. Accordingly, we have

$$50 - A = 2.326 \, \frac{\sigma}{\sqrt{n}} = 2.326 \, \frac{5}{\sqrt{n}}$$

and

$$A - 49 = 1.645 \, \frac{\sigma}{\sqrt{n}} = 1.645 \, \frac{5}{\sqrt{n}}.$$

Adding these two equations, we obtain

$$50 - 49 = 1 = (2.326 + 2.645)\frac{5}{\sqrt{n}}.$$

Multiplying both sides by $\sqrt{n}$, we convert this into

$$\sqrt{n} = (2.324 + 1.645)(5) = (3.971)(5) = 19.855.$$

The required sample size is thus equal to $19.855^2 = 394.221025$ which rounds up to 395.

You could have proceeded in alternative fashion by assuming that the inspector sets the α risk at .05 and the β risk at .01, and obtained the same required sample size.

[Give yourself 10 points for understanding how to set up each of the first two equations (making a total of 20 points) and 5 points for understanding how to use those equations to determine the required sample size.]