ANSWERS TO TEST NUMBER 3

Question 1: (25 points)

The probability that both engines on a two-engine airplane will fail is π^2. [If you can demonstrate understanding of this, give yourself 2 points.] Engine failures on a four-engine plane can be analyzed by letting $(1-\pi)=\vartheta$ and then taking $(\pi+\vartheta)$ to the power $n=4$ to obtain

$$(\pi+\vartheta)^4 = \pi^4 + 4\pi^3\vartheta + 6\pi^2\vartheta^2 + 4\pi\vartheta^3 + \vartheta^4$$

which is the sum of the probabilities that X engines will fail where $X = 0, 1, 2, 3, 4$. The probability that all 4 engines will fail is π^4. The probability that 3 specific engines will fail is $\pi^3\vartheta$ and this can happen in four different ways so the probability that any three engines will fail is $4\pi^3\vartheta = 4\pi^3(1-\pi)$. The probability that the 4-engine plane will crash is thus

$$\pi^4 + 4\pi^3(1-\pi).$$

So the 2-engine plane will be safer if

$$\pi^2 < \pi^4 + 4\pi^3(1-\pi) = \pi^4 + 4\pi^3 - 4\pi^4 = 4\pi^3 - 3\pi^4$$

which, after dividing both sides by π^2, reduces to

$$4\pi - 3\pi^2 > 1.$$

We can also obtain this expression from a tree-diagram or directly from the binomial probability function determining $P(X=x)$—the relevant calculations can be organized in a table like the one used in the answer to the next question.

By substituting various values of π into the above expression it can be easily seen that any value of $\pi > 1/3$ will maintain the inequality. The 2-engine airplane is thus safer when the probability of engine failure is greater than 1/3. This probability vastly exceeds the probability of an engine failure on a typical modern airplane.

[Give yourself 18 points for demonstrating how to organize your calculations of the probability that a 4-engine plane will crash, and another 5 points for knowing how to compare the probabilities of a 2-engine plane crashing vs. a 4-engine plane crashing and get the correct answer.]

Question 2: (35 points)

You are given the following:

The probability that the defendant is actually guilty–

$$P(AG) = .65$$

The probability that a juror will vote a guilty person innocent–

$$P(VN|AG) = .20$$

The probability that a juror will vote an innocent person guilty–

$$P(VG|AN) = .10$$

[Give yourself 1 point for correctly recognizing each of the above.]

You can then immediately calculate

$$P(VN \cap AG) = P(VN|AG)P(AG) = .20 \times .65 = .13$$

and

$$P(VG \cap AN) = P(VG|AN)P(AN) = .10 \times .35 = .315$$

[Give yourself 7 points for recognizing how to do the above calculations correctly.]

Using these results you can fill in the table below—

		Juror's Vote		
		Guilty (VG)	Not Guilty (VN)	
Actually	Guilty (AG)	.52	.13	.65
	Not Guilty (AN)	.035	.315	.35
		.555	.445	1.00

—and determine that the probability that an individual juror will vote a defendant guilty is .555.

[Give yourself 10 points for correctly setting up the above table (or for doing an appropriate tree-diagram or some other appropriate demonstration of your understanding of the calculations.]

The problem is then to determine the probability that 10 or more of 12 jurors, acting independently, will vote a defendant guilty. Let Z be the proportion of defendants that are voted guilty and X be the number of jurors that will vote a typical defendant guilty. X is distributed according to the binomial distribution with $p = .555$ and $n = 12$.

The *binomial probability function*, which gives the probabilities that X will take values $(0, \ldots, n)$, is

$$P(x) = \binom{n}{x} p^x (1-p)^{n-x} \tag{1}$$

The probability that $X \geq 10$ can be calculated as illustrated in the table below.

x	$n!/(x!(n-x)!)$	p^x	$(1-p)^{n-x}$	$P(x)$
12	$12!/(12!0!) = 1$	$.555^{12} = .0008541$	$.445^0 = 1$	.0008541
11	$12/(11!1!) = 12$	$.555^{11} = .0015389$	$.445^1 = .445$	.0082177
10	$12/(10!2!) = 66$	$.555^{10} = .0027729$	$.445^2 = .198$	.0362000

The probability of 10 or more jurors voting the defendant guilty is equal to

$$P(X \geq 10) = .0008541 + .0082177 + .0362 = .0453$$

This means that 4.53% of defendants will be found guilty.

The above probabilities can also be obtained by looking up the appropriate entries in a table of binomial probabilities. Alternatively, we can fire up XLISP-STAT and enter the command (the answer it gives appears below the command line)

```
> (- 1 (binomial-cdf 9 12 .555))
0.045312542105687026
```

[Give yourself 5 points for knowing that you are dealing with a binomial distribution and 10 points for demonstrating that you know how to organize and do the calculations.]

Question 3: (15 points)

The *poisson probability function* is

$$P(X = x) = \frac{\lambda^x \, e^{-\lambda}}{x!}$$

where X is the number of customers per minute and λ, the mean number of customers, equals 3. The probability that there are less than three customers in a given minute is equal to $P(X = 0) + P(X = 1) + P(X = 2)$. From the Poisson Probability Tables in the back of any book in statistics we can find, for $\lambda = 3$,

$$P(X = 0) = .050 \quad P(X = 1) = .149 \quad P(X = 2) = .224$$

which, added together, yields $P(X < 3) = .050 + .149 + .224 = .423$.

The probability of there being less than 3 customers in each of two successive minutes is equal to the product of the probabilitities of there being less than 3 customers for each minute, or $.423^2 = .179$. Using XLISPSTAT yields the calculation

```
> (^ (poisson-cdf 2 3) 2)
0.1790898447641445
```

[Give yourself 10 points for demonstrating that you know how to calculate the probability of there being less than 3 customers in any given minute, and 5 points for knowing that you have to square this number to get the probability of there being less than 3 customers in each of the two successive minutes.]

Question 4: (15 points)

Let X_i be the weight of the i^{th} football player. The sum of the weights of 25 of them thus will equal

$$W = \sum_{i=1}^{25} X_i = X_1 + X_2 + \ldots + X_{25}$$

The mean of Z will equal

$$E\{W\} = \sum_{i=1}^{25} E\{X_i\} = (25)(120) = 3000$$

Since the weights of the players are statistically independent, the variance of Z will equal

$$\sigma^2\{W\} = \sum_{i=1}^{25} \sigma^2\{X_i\} = (25)(80^2) = 160000$$

so that $\sigma\{W\} = 400$. The break-weight for the cable is

$$z = \frac{2500 - 3000}{400} = -1.25$$

standard deviations from the sum of the mean weights of the 25 football players. From the Normal Probability Tables we find that

$$P(z > -1.25) = P(z < 1.25) = .8944.$$

Or, from XLISPSTAT,

```
> (normal-cdf 1.25)
0.8943502263331448
```

There is almost a 90% chance that the elevator cable will snap.

[Give yourself 2 points for recognizing that you have to sum the 25 normally distributed variables, 2 points for calculating the mean of the summed variables, 3 points for knowing how to calculate the standard deviation of the sum of normally distributed values, 6 points for knowing how to calculate z and 2 points for correct use of the correct Statistical Table.]

Question 5: (10 points)

The distribution of the location of accidents in the tunnel is a continuous uniform distribution. One-quarter of the total number of accidents will thus occur in each of the four miles of the tunnel. The probability that any given accident will occur in the first mile of the tunnel is thus .25. There is a zero possibility that an accident will occur at the mid-point of the tunnel because a vertical line drawn at the exact mid-point of the tunnel (or any uniform distribution) will have zero width.

[Give yourself 10 points for demonstrating your recognition and understanding of the continuous uniform distribution. To demonstrate this you need to do more than simply write down the numbers .25 and 0 —i.e., you need to show your work!]