ANSWERS TEST NUMBER 10

A medical doctor wished to compare two different methods of weight reduction. He took six of his patients who weighed over 300 pounds and applied method A to three of them, picked at random, and method B to the other three. After one year of treatment, he weighed the six patients and calculated for each patient the weight loss that had been attained. The results were as follows:

```
Method A: 73 110 65
Method B: 54  74 51
```

To test the relative merits of the two methods he ran a regression in which he inserted a dummy variable for method B. The regression results were as follows:

```
Least Squares Estimates:

                         Coefficient  Standard-Error
Constant                   82.6667      (11.0504)
Dummy                     -23.0000      (15.6276)

R Squared:                0.351288
Sigma hat:                 19.1398  (= square root of MSE)
Number of cases:                 6
Degrees of freedom:              4
```

The doctor concluded that both methods were effective but that method A was better than method B.

a) The t-statistic for testing the null hypothesis that the true coefficient of the dummy variable is positive is

$$t = \frac{b_d}{s_{b_d}} = \frac{-23}{15.6276} = -1.472$$

and the 5% critical value of t with $(n - k - 1) = 6 - 1 - 1 = 4$ degrees of freedom for a (lower) one-tailed test is, using XlispStat,

```
> (t-quant .05 4)
-2.1318468961236765
```

so we cannot reject the null hypothesis. It was inappropriate for the doctor to conclude that method B did worse than method A. The doctor was reasonable in concluding that the methods work because the t-statistic for testing the null hypothesis that the intercept is zero is

$$t = \frac{b_o}{s_{b_o}} = \frac{82.6667}{11.0504} = 7.48$$

which is clearly bigger than the critical value of 2.13.

[Give yourself 10 points for understanding what the correct null hypothesis is, 10 points for knowing how to calculate the t statistic, and 5 points for interpreting the results correctly.]

b) The mean square error is `Sigma hat` squared or $(19.1398)^2 = 366.33$ and the error sum of squares (sum of squared residuals) is

$$SSE = (n - k - 1)MSE = (4)(366.33) = 1465.32$$

Since the R^2 is .351288, the total sum of squares is

$$SSTO = \frac{SSE}{1 - R^2} = \frac{1465.32}{1 - .351288} = 2258.81.$$

The sum of squares due to regression is therefore equal to

$$SSR = 2258.8 - 1465.32 = 793.48.$$

The total degrees of freedom of the regression in calculating the mean square error is (n - k - 1) = 4 and the degrees of freedom in calculating the regression coefficient is 1 (= k) because two parameters are calculated and a degree of freedom is lost in calculating them because

$$\sum_{i=1}^{n} (\hat{Y} - \bar{Y}) = 0.$$

This means that the degrees of freedom for the total sum of squares is 4 + 1 = 5.

[Give yourself 5 points for calculating each of understanding how to correctly calculate each of *SSE*, *SSTO*, and *SSR*, and understanding what the degrees of freedom are in the three cases.]

We can now fill in the first two columns of the ANOVA table below.

Analysis of Variance: Weight Reduction

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square
Regression	793.48	1	793.48
Error	1465.32	4	366.33
Total	2258.81	5	

We complete the third column by dividing the first two numbers in the first column by their associated degrees of freedom in the second column. The F-statistic for significance of the regression is simply

$$F(1,4) = \frac{MSR}{MSE} = \frac{793.48}{366.33} = 2.166$$

Here, the null hypothesis is that there is no difference between *SSTO* and *SSE* in which case the ratio of *MSR* to *MSE* should be zero. The 5% critical value of F with 1 degree of freedom in the numerator and 4 degrees of freedom in the denominator is

```
> (f-quant .95 1 4)
7.708647422176785
```

so we cannot reject the null hypothesis that the degree of weight loss is not affected by whether method A or method B is used. Note that when you take the square root of the F statistic you obtain the t statistic you calculated in a) above.

[Give yourself 6 points for understanding how to and interpret the F statistic and 4 points for filling in the table correctly.]

c) The means for method A and method B are, respectively,

$$\bar{A} = (73 + 110 + 65)/3 = 82.6667 \quad \text{and} \quad \bar{B} = (54 + 74 + 51)/3 = 59.6666$$

and the overall mean is

$$\bar{X} = (73 + 110 + 65 + 54 + 74 + 51)/6 = 71.1667$$

[Give yourself 1 point for knowing how to calculate each of the above means.]

The sum of squares for treatments is therefore

$$SST = (3)(82.6667 - 71.1667)^2 + (3)(59.6666 - 71.1667)^2 = 793.48$$

and the sum of squares for error is

$$\begin{aligned} SSE &= (73-82.6667)^2+(110-82.6667)^2+(65-82.6667)^2 \\ &\quad +(54-59.6666)^2+(74-59.6666)^2+51-59.6666)^2 \\ &= 93.44+747.11+312.11+32.11+205.45+75.11 \\ &= 1152.6999+312.67=1465.32 \end{aligned}$$

To complete the test we can create an ANOVA table identical to the one above.

[Give yourself 7 points for knowing how to calculate each of SST and SSE and 8 points for knowing how everything fits into tha ANOVA table in part a).]

d) Here we want to test the null hypotheses that $\mu_A - \mu_B = 0$.

A point estimate of $\mu_A - \mu_B$ is $\bar{A} - \bar{B} = 82.6667 - 59.6666 = 23.0001$.

[Give yourself 5 points for knowing how to set up the null hypothesis correctly and 3 points for recognizing that the difference in the sample means is a point estimate of the different in the population means.]

The variance of the difference between the means is

$$\sigma^2_{\bar{A}} + \sigma^2_{\bar{B}} = \sigma^2_C\left[\frac{1}{n_A}+\frac{1}{n_b}\right] = \sigma^2_C\left[\frac{1}{3}+\frac{1}{3}\right] = \frac{2\,\sigma^2_C}{3}$$

where σ^2_C is the pooled estimate of

$$\begin{aligned} \sigma^2_A &= \frac{\sum_{i=1}^3 (A_i-\bar{A})^2}{n_A-1} \\ &= \frac{(73-82.6667)^2+(110-82.6667)^2+(65-82.6667)^2}{2} \\ &= \frac{1152.6999}{2} \quad = \quad 576.3499 \end{aligned}$$

and

$$\begin{aligned} \sigma^2_B &= \frac{\sum_{i=1}^3 (B_i-\bar{B})^2}{n_B-1} \\ &= \frac{(54-59.6666)^2+(74-59.6666)^2+(51-59.6666)^2}{2} \\ &= \frac{312.67}{2} \quad = \quad 156.33. \end{aligned}$$

We thus have

$$
\begin{aligned}
\sigma_C^2 &= \frac{(n_A - 1)\,\sigma_A + (n_B - 1), \sigma B}{n_A + n_B - 2} \\
&= \frac{(2)(576.35) + (2)(156.33)}{4} = \frac{1152.6999 + 312.67}{4} \\
&= \frac{1465.32}{4} = 366.33.
\end{aligned}
$$

yielding a value for σ_C of 19.139.

[Give yourself 5 points for understanding how to calculate the variance of the difference in the sample means, 3 points for knowing how to calculate σ_A^2 and σ_B^2, and 3 points for knowing how to calculate σ_C.]

The variance of the difference between the two means is therefore

$$\sigma^2_{\bar{A}-\bar{B}} = \frac{(2)(366.33)}{3} = \frac{732.66}{3} = 244.22$$

and $\sigma_{\bar{A}-\bar{B}} = 15.6276$. The t statistic for the null hypothesis that $\mu_A - \mu_B = 0$ is therefore

$$t = \frac{23}{15.6276} = 1.472$$

which is the same as our t-statistic in part a).

[Give yourself 5 points for correctly calculating the t statistic and 1 point for recognizing that it is the same as the one you calculated in part a).]