Vector Autoregression Analysis: Estimation and Interpretation

John E. Floyd
University of Toronto*

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1 Introduction

This expositional paper lays out the mechanics of running and interpreting vector autoregressions. It proves no theorems. Rather, it sets out the basics of how VAR's 'work' and outlines some fundamentals regarding interpretation. For the theoretical details, see Walter Enders, *Applied Econometric Time Series*, John Wiley & Sons, 1995, pp. 291–353 and earlier material as required, Helmut Lütkepohl, *Introduction to Multiple Time Series Analysis*, Springer-Verlag, 1991, pp. 9–27, 43–58, and 97–117, and James D. Hamilton, *Time Series Analysis*, Princeton University Press, 1994, pp. 257–372 and earlier material as required.

2 The Underlying Economic Model

Consider the following economic model with two variables y_1 and y_2, each of which depends on itself lagged, on the current and lagged values of the other variable and on a *iid* error term:

$$y_{1(t)} = v_{10} + v_{12}\, y_{2(t)} + a_{11}\, y_{1(t-1)} + a_{12}\, y_{2(t-2)} + e_{1(t)} \tag{1}$$

$$y_{2(t)} = v_{20} + v_{21}\, y_{1(t)} + a_{21}\, y_{1(t-1)} + a_{22}\, y_{2(t-2)} + e_{2(t)} \tag{2}$$

*This is written to help students understand how to run VARs. It is not a substitute for reading the literature cited. I would like to thank John Maheu for coments.

This system can be written in matrix notation as

$$\begin{bmatrix} y_{1(t)} \\ y_{2(t)} \end{bmatrix} = \begin{bmatrix} v_{10} \\ v_{20} \end{bmatrix} + \begin{bmatrix} 0 & v_{12} \\ v_{21} & 0 \end{bmatrix} \begin{bmatrix} y_{1(t)} \\ y_{2(t)} \end{bmatrix}$$
$$+ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_{1(t-1)} \\ y_{2(t-1)} \end{bmatrix} + \begin{bmatrix} e_{1(t)} \\ e_{2(t)} \end{bmatrix} \quad (3)$$

or, in general matrix notation with m variables and p lags,

$$\mathbf{y}_t = \mathbf{v} + \mathbf{A_0}\,\mathbf{y}_t + \mathbf{A_1}\,\mathbf{y}_{t-1} + \mathbf{A_2}\,\mathbf{y}_{t-2} + \mathbf{A_3}\,\mathbf{y}_{t-3} + \cdots + \mathbf{A_p}\,\mathbf{y}_{t-p} + \mathbf{e}_t \quad (4)$$

where $\mathbf{y}_t$, $\mathbf{v}$ and $\mathbf{e}_t$ are $m \times 1$ column vectors and $\mathbf{A_0}$, $\mathbf{A_1}$, $\mathbf{A_2}$, $\cdots$ $\mathbf{A_p}$ are $m \times m$ matrices of coefficients. The vector $\mathbf{e}_t$ is a m-element vector of white noise residuals that satisfies $E\{\mathbf{e}_t\mathbf{e}_t{}'\} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix. An appropriate scaling of the elements of $\mathbf{y}$ would make $\mathbf{D}$ an identity matrix.

3 VAR estimation

Equations (1) and (2), which are called a *structural* VAR or a *primitive system* can be solved simultaneously to yield the *reduced form* or *standard form* of the VAR:

$$y_{1(t)} = b_{10} + b_{11}\,y_{1(t-1)} + b_{12}\,y_{2(t-2)} + u_{1(t)} \quad (5)$$
$$y_{2(t)} = b_{20} + b_{21}\,y_{1(t-1)} + b_{22}\,y_{2(t-2)} + u_{2(t)} \quad (6)$$

or

$$\begin{bmatrix} y_{1(t)} \\ y_{2(t)} \end{bmatrix} = \begin{bmatrix} b_{10} \\ b_{20} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} y_{1(t-1)} \\ y_{2(t-1)} \end{bmatrix} + \begin{bmatrix} u_{1(t)} \\ u_{2(t)} \end{bmatrix} \quad (7)$$

where

$$\begin{aligned} b_{10} &= \frac{v_{10} + v_{12}\,v_{20}}{1 - v_{12}\,v_{21}} \\ b_{11} &= \frac{v_{12}\,a_{11} + a_{21}}{1 - v_{12}\,v_{21}} \\ b_{12} &= \frac{a_{12} + v_{12}\,a_{22}}{1 - v_{12}\,v_{21}} \\ b_{20} &= \frac{v_{20} + v_{21}\,v_{10}}{1 - v_{12}\,v_{21}} \\ b_{21} &= \frac{a_{21} + v_{21}\,a_{11}}{1 - v_{12}\,v_{21}} \\ b_{22} &= \frac{v_{21}\,a_{12} + a_{22}}{1 - v_{12}\,v_{21}} \end{aligned}$$

and

$$u_{1(t)} = \frac{1}{1 - v_{12}\, v_{21}} \left[e_{1(t)} + v_{12}\, e_{2(t)}\right]$$
$$u_{2(t)} = \frac{1}{1 - v_{12}\, v_{21}} \left[v_{21}\, e_{1(t)} + e_{2(t)}\right].$$

In the general m variable case with p lags we have

$$(\mathbf{I} - \mathbf{A_0})\, \mathbf{y}_t = \mathbf{v} + \mathbf{A_1}\, \mathbf{y}_{t-1} + \mathbf{A_2}\, \mathbf{y}_{t-2} + \mathbf{A_3}\, \mathbf{y}_{t-3} + \cdots + \mathbf{A_P}\, \mathbf{y}_{t-p} + \mathbf{e}_t \quad (8)$$

which reduces to

$$\begin{aligned} \mathbf{y}_t \;&=\; (\mathbf{I} - \mathbf{A_0})^{-1}\mathbf{v} + (\mathbf{I} - \mathbf{A_0})^{-1}\mathbf{A_1}\, \mathbf{y}_{t-1} + (\mathbf{I} - \mathbf{A_0})^{-1}\mathbf{A_2}\, \mathbf{y}_{t-2} \\ &+ (\mathbf{I} - \mathbf{A_0})^{-1}\mathbf{A_3}\, \mathbf{y}_{t-3} + \cdots + (\mathbf{I} - \mathbf{A_0})^{-1}\mathbf{A_P}\, \mathbf{y}_{t-p} + (\mathbf{I} - \mathbf{A_0})^{-1}\mathbf{e}_t. \quad (9) \end{aligned}$$

Letting $\mathbf{b} = (\mathbf{I} - \mathbf{A_0})^{-1}\mathbf{v}$, $\mathbf{B_1} = (\mathbf{I} - \mathbf{A_0})^{-1}\mathbf{A_1}$, $\mathbf{B_2} = (\mathbf{I} - \mathbf{A_0})^{-1}\mathbf{A_2}$, $\cdots$ etc., and $\mathbf{u}_t = (\mathbf{I} - \mathbf{A_0})^{-1}\, \mathbf{e}_t$ we can write the VAR in standard form in the general case as

$$\mathbf{y}_t \;=\; \mathbf{b} + \mathbf{B_1}\, \mathbf{y}_{t-1} + \mathbf{B_2}\, \mathbf{y}_{t-2} + \mathbf{B_3}\, \mathbf{y}_{t-3} + \cdots + \mathbf{B_P}\, \mathbf{y}_{t-p} + \mathbf{u}_t. \quad (10)$$

All this assumes, of course, that the matrix $(\mathbf{I} - \mathbf{A_0})$ has an inverse. Given that $E\{\mathbf{e}_t\mathbf{e}_t{}'\} = \mathbf{D}$, the variance-covariance matrix of the vector of residuals $\mathbf{u}_t$ equals

$$\begin{aligned} \Omega \;&=\; E\{\mathbf{u}_t\mathbf{u}_t{}'\} \\ &=\; E\{[(\mathbf{I} - \mathbf{A_0})^{-1}\mathbf{e}_t][(\mathbf{I} - \mathbf{A_0})^{-1}\mathbf{e}_t]'\} \\ &=\; E\{[(\mathbf{I} - \mathbf{A_0})^{-1}]\mathbf{e}_t\mathbf{e}_t{}'[(\mathbf{I} - \mathbf{A_0})^{-1}]'\} \\ &=\; [(\mathbf{I} - \mathbf{A_0})^{-1}]E\{\mathbf{e}_t\mathbf{e}_t{}'\}[(\mathbf{I} - \mathbf{A_0})^{-1}]' \\ &=\; [(\mathbf{I} - \mathbf{A_0})^{-1}]\, \mathbf{D}\, [(\mathbf{I} - \mathbf{A_0})^{-1}]' \end{aligned}$$

The equations in (10) can be estimated by ordinary least squares—because the independent variables in all equations are the same, there is no efficiency gain by estimating these equations as a system using the seemingly unrelated regression technique.

At this point it is appropriate to perform a number of tests to determine what variables should be in the VAR, the appropriate number of lags, whether seasonal dummies should be included and, indeed, whether a VAR is even appropriate for the research problem at hand. To focus strictly on the mechanics at this point, however, these model-selection issues are postponed to a later section.

4 The Moving Average Representation

The standard form system given by (10) can be manipulated to express the current value of each variable as a function solely of the vector of residuals $\mathbf{u}_t$. This is called its *moving average representation*—$\mathbf{y}_t$ is a moving average of the current and past values of $\mathbf{u}_t$.

$$\mathbf{y}_t \quad = \quad \mathbf{C_0}\,\mathbf{u}_t + \mathbf{C_1}\,\mathbf{u}_{t-1} + \mathbf{C_2}\,\mathbf{u}_{t-2} + \cdots\cdots + \mathbf{C_s}\,\mathbf{u}_{t-s} + \mathbf{y_0}. \tag{11}$$

where $\mathbf{y_0}$ is some initial value of $\mathbf{y}_t$.

To see how we can do this, suppose for the moment that we have only one lag of each variable in the VAR (i.e., a VAR(1) process). Under this assumption, (10) reduces to

$$\mathbf{y}_t \quad = \quad \mathbf{b} + \mathbf{B}\,\mathbf{y}_{t-1} + \mathbf{u}_t. \tag{12}$$

Lagging (12) n times, we obtain

$$\begin{aligned}
\mathbf{y}_{t-1} &= \mathbf{b} + \mathbf{B}\,\mathbf{y}_{t-2} + \mathbf{u}_{t-1} \\
\mathbf{y}_{t-2} &= \mathbf{b} + \mathbf{B}\,\mathbf{y}_{t-3} + \mathbf{u}_{t-2} \\
\mathbf{y}_{t-3} &= \mathbf{b} + \mathbf{B}\,\mathbf{y}_{t-4} + \mathbf{u}_{t-3} \\
&\ldots \\
&\ldots \\
&\ldots \\
&\ldots \\
\mathbf{y}_{t-s} &= \mathbf{b} + \mathbf{B}\,\mathbf{y}_{t-s} + \mathbf{u}_{t-s-1}
\end{aligned}$$

Successive substitution into (12) yields

$$\begin{aligned}
\mathbf{y}_t &= [1 + \mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3 + \mathbf{B}^4 + \cdots + \mathbf{B}^s]\,\mathbf{b} \\
&+ \mathbf{u}_t + \mathbf{B}\,\mathbf{u}_{t-1} + \mathbf{B}^2\,\mathbf{u}_{t-2} + \mathbf{B}^3\,\mathbf{u}_{t-3} + \cdots\cdots + \mathbf{B}^s\,\mathbf{u}_{t-s} \\
&= (1 - \mathbf{B})^{-1}\,\mathbf{b} + \mathbf{u}_t + \mathbf{B}\,\mathbf{u}_{t-1} + \mathbf{B}^2\,\mathbf{u}_{t-2} \\
&\qquad\qquad + \mathbf{B}^3\,\mathbf{u}_{t-3} + \cdots\cdots + \mathbf{B}^s\,\mathbf{u}_{t-s}.
\end{aligned} \tag{13}$$

In terms of (11) this yields $\mathbf{y_0} = (1 - \mathbf{B})^{-1}\,\mathbf{b}$ and $\mathbf{C_k} = \mathbf{B}^k$, $k = 0 \cdots s$.

When there are $p > 1$ lags, we first convert the VAR(p) system into a VAR(1) system of the form

$$
\begin{bmatrix} \mathbf{y}_t \\ \mathbf{y}_{t-1} \\ \mathbf{y}_{t-2} \\ \mathbf{y}_{t-3} \\ \vdots \\ \vdots \\ \vdots \\ \mathbf{y}_{t-p+1} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{B_1} & \mathbf{B_2} & \mathbf{B_3} & \cdots & \mathbf{B_{P-1}} & \mathbf{B_P} \\ \mathbf{I_m} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{I_m} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathbf{I_m} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{I_m} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y}_{t-1} \\ \mathbf{y}_{t-2} \\ \mathbf{y}_{t-3} \\ \mathbf{y}_{t-4} \\ \vdots \\ \vdots \\ \vdots \\ \mathbf{y}_{t-p} \end{bmatrix} + \begin{bmatrix} \mathbf{u}_t \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix}
$$

which can be expressed more simply as

$$
\mathcal{Y}_t \quad = \quad \Upsilon + \mathcal{B}\,\mathcal{Y}_{t-1} + \Psi_t. \tag{14}
$$

Here, $\mathcal{Y}_t, \Upsilon$, and Ψ_t are $mp \times 1$ column vectors and $\mathcal{B}$ is an $mp \times mp$ matrix. This system is formed by taking the expression (10) as the first equation (more correctly, set of equations) and adding the $p-1$ equations (sets of equations)

$$
\begin{aligned}
\mathbf{y}_{t-1} &= \mathbf{y}_{t-1} \\
\mathbf{y}_{t-2} &= \mathbf{y}_{t-2} \\
\mathbf{y}_{t-3} &= \mathbf{y}_{t-3} \\
\cdots \\
\cdots \\
\cdots \\
\mathbf{y}_{t-p+2} &= \mathbf{y}_{t-p+2} \\
\mathbf{y}_{t-p+1} &= \mathbf{y}_{t-p+1}
\end{aligned}
$$

sequentially below.

By analogy with equation (13), the moving average representation of (14) is seen to be

$$
\begin{aligned}
\mathcal{Y}_t \quad = \quad & [\mathbf{I}_{mp} - \mathcal{B}]^{-1}\,\Upsilon + \Psi_t + \mathcal{B}\,\Psi_{t-1} + \mathcal{B}^2\,\Psi_{t-2} \\
& + \mathcal{B}^3\,\Psi_{t-3} + \cdots\cdots + \mathcal{B}^n\,\Psi_{t-s}
\end{aligned} \tag{15}
$$

where $\mathbf{I}_{mp}$ is an $mp \times mp$ identity matrix.

It turns out that the moving average representation of our original VAR(p) system is represented by selected parts of the top m equations of the system

(15). We can strip off these terms by operating on (15) with the $m \times mp$ matrix

$$\mathcal{J} = [\mathbf{I_m} \;\; 0 \;\; 0 \;\; 0 \;\; \cdots \;\; 0].$$

We thereby obtain (11), reproduced below,

$$\mathbf{y}_t \;\; = \;\; \mathbf{C_0}\,\mathbf{u}_t + \mathbf{C_1}\,\mathbf{u}_{t-1} + \mathbf{C_2}\,\mathbf{u}_{t-2} + \cdots\cdots + \mathbf{C_s}\,\mathbf{u}_{t-s} + \mathbf{y_0}. \qquad (11)$$

where

$$\begin{aligned}
\mathbf{y_0} &= \mathcal{J}\,\Upsilon \\
\mathbf{C_0} &= \mathcal{J}\mathcal{B}^0\mathcal{J}' \;\; = \;\; \mathcal{J}\,\mathbf{I}_{mp}\,\mathcal{J}' \\
\mathbf{C_1} &= \mathcal{J}\mathcal{B}^1\mathcal{J}' \;\; = \;\; \mathcal{J}\mathcal{B}\mathcal{J}' \\
\mathbf{C_2} &= \mathcal{J}\mathcal{B}^2\mathcal{J}' \\
\mathbf{C_3} &= \mathcal{J}\mathcal{B}^3\mathcal{J}' \\
&\cdots \\
&\cdots \\
&\cdots \\
\mathbf{C_s} &= \mathcal{J}\mathcal{B}^s\mathcal{J}'
\end{aligned}$$

To fully understand how the above procedures work it is necessary to apply them. Consider the age-old problem of modelling the behaviour of monetary and other aggregates over the business cycle. We focus the analysis on conditions in the United States because that is one of the few economies that one might treat, at considerable risk, as if it were a closed economy. This is probably defensible because the U.S. authorities pay little attention to the effects of their policies on the rest of the world and the country is large enough that changes in the domestic money supply and output can have a significant effect on world levels of these variables. Moreover, to the extent that other countries are concerned about the effects of U.S. monetary shocks on their exchange rates with respect to the U.S. dollar, and adjust their monetary policies to offset these effects, their monetary conditions will mimic those in the U.S., whose authorities will then effectively control world monetary policy.

For reasons that will be outlined later when issues regarding the interpretation of VARs are discussed, we select four variables—the detrended logarithms of base money and real GDP, the year over year rate of inflation of the implicit GDP deflator, and the level of the market interest rate on 90-day commercial paper.[1] Quarterly data are used with no seasonal dummy variables. The base money and real GDP variables are detrended to keep our analysis of the interactions between the variables uncontaminated by the relationships between their trends. The purpose at this point is to demonstrate the calculations—model selection criteria will be considered later.

Perhaps the best statistical package to use for VAR analysis is RATS.[2] Unfortunately, RATS is expensive and one can, with some inconvenience, program VARs using a free platform for statistical computing called Xlisp-Stat, which was written by Luke Tierney at the University of Minnesota.[3] The VAR presented immediately below is programmed in detail in the Xlisp-stat code file `vardemcd.lsp`, available from my web-site location noted in footnote 3. While VARs can also be programmed in raw fashion in RATS, it is quicker to use the canned RATS procedures for making these calculations. These are contained in the file `vardemcd.prg`.[4]

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[1] Seasonally adjusted U.S. real and nominal gross domestic product figures were obtained from the International Monetary Fund publication *International Financial Statistics* (111/99b.cvs is the mnemonic for nominal GDP and 111/99b.rzf is the mnemonic for real GDP), and the implicit GDP deflator was obtained by dividing the nominal series by the real series. A seasonally unadjusted base money series was obtained from *DRI-Citibase* (mnemonic FZMFB) and seasonally adjusted using the SAS X11 procedure. The 90-day commercial paper rate was obtained from the *CANSIM* data base (mnemonic B54412).

[2] For details about RATS and how to obtain it, see `http://www.estima.com`.

[3] Versions for all major operating systems can be obtained from Tierney's website at `http://stat.umn.ca/~luke/xls/xlsinfo/node1.html`. The MS-Windows version zipfile `wxls32zp.exe` can also be downloaded from my web-site by anonymous ftp at `carmel.economics.utoronto.ca/pub/var` along with some official documentation (`xlispstatdoc.ps`), a short manual that I have put together (`minmanls.ps or minmanls.pdf`) and some additional functions that will be helpful for time-series analysis (`newfuncs.lsp`). A complete discussion of how to use the program is contained in Luke Tierney, *Lisp-Stat: An Object Oriented Environment for Statistical Computing and Dynamic Graphics*, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, 1990. LINUX versions of XlispStat are bundled with most of the popular distributions.

[4] This RATS program file `vardemcd.prg`, along with the two data files it requires (`usm.rat, usq.rat`) and other RATS program files subsequently referred to in this paper, is available in the zip files `ratszip.exe` and `ratszip.tgz` from the ftp site noted in the previous footnote. The code in `vardemcd.prg` performs the necessary plots to the screen and writes them to file. When an output-filename is added to to the command line the entire output is redirected to that file. The output from `vardemcd.prg` is in the file `vardemcd.out`, which is also contained in the zip-files.

5 Identification

The moving average representation (11) does not give a proper indication of how the system responds to shocks to the individual structural equations. The problem is that the shocks to the equations contained in the vector $\mathbf{u}_t$ are correlated with each other. We therefore cannot determine what the effects on the m variables of a shock to an individual structural equation alone would be—an observed u_t will represent the combined shocks to a number of equations. This can be seen from the fact that from (9)

$$\mathbf{u}_t = (\mathbf{I} - \mathbf{A_0})^{-1}\,\mathbf{e}_t.$$

In order to determine the effects of a shock to an individual structural equation of the system we have to be able to solve the system for $\mathbf{A_0}$ and thereby obtain $(\mathbf{I} - \mathbf{A_0})^{-1}$. This will enable us to operate on (11) to transform the $\mathbf{u}_{t-j}$'s in into $\mathbf{e}_{t-j}$'s. In the process, of course, the matrices $\mathbf{C_j}$ will also be transformed into a useful representation of the impulse-responses.

One way to obtain the matrix $\mathbf{A_0}$ is to statistically estimate the structural model (4). Were we to do this, we would not be running a VAR. Indeed, the reason for VAR analysis is to avoid multi-equation structural models.

The approach used to identify $\mathbf{A_0}$ in VAR analysis is to find the matrix that will orthogonalize the errors—i.e., transform the $\mathbf{u}_{t-j}$'s in into the $\mathbf{e}_{t-j}$'s, which are uncorrelated with each other.

Given any matrix $\mathbf{G}$ that has an inverse, equation (11) can be rewritten

$$\begin{aligned}\mathbf{y}_t &= \mathbf{C_0}\,\mathbf{G}\mathbf{G}^{-1}\,\mathbf{u}_t + \mathbf{C_1}\,\mathbf{G}\mathbf{G}^{-1}\,\mathbf{u}_{t-1} + \mathbf{C_2}\,\mathbf{G}\mathbf{G}^{-1}\,\mathbf{u}_{t-2} \\ &\qquad + \cdots\cdots + \mathbf{C_s}\,\mathbf{G}\mathbf{G}^{-1}\,\mathbf{u}_{t-s} + \mathbf{y_0}.\end{aligned} \tag{16}$$

Our task is to find the $\mathbf{G}$ for which

$$\mathbf{G} = (\mathbf{I} - \mathbf{A_0})^{-1}.$$

Then

$$\begin{aligned}\mathbf{y}_t &= \mathbf{Z_0}\,\mathbf{e}_t + \mathbf{Z_1}\,\mathbf{e}_{t-1} + \mathbf{Z_2}\,\mathbf{e}_{t-2} \\ &\qquad + \cdots\cdots + \mathbf{Z_s}\,\mathbf{e}_{t-s} + \mathbf{y_0}\end{aligned} \tag{17}$$

where

$$\mathbf{Z_j} = \mathbf{C_j}\,\mathbf{G}$$

and

$$\mathbf{e}_{t-j} = \mathbf{G}^{-1}\,\mathbf{u}_{t-j} \quad ==\!\Rightarrow \quad \mathbf{u}_{t-j} = \mathbf{G}\,\mathbf{e}_{t-j}$$

5.1 Choleski Decompositions

Suppose that the matrix $\mathbf{A_0}$ takes the following form:

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ a^0_{21} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ a^0_{31} & a^0_{32} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ a^0_{41} & a^0_{42} & a^0_{43} & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a^0_{m1} & a^0_{m2} & a^0_{m3} & \cdots & \cdots & \cdots & \cdots & a^0_{m(m-1)} & 0 \end{bmatrix}$$

This will mean that the structural equations will take the form:

$$\begin{aligned} y_{1t} &= a^1_{11}\, y_{1(t-1)} \\ y_{2t} &= a^0_{21}\, y_{1t} + a^1_{21}\, y_{1(t-1)} + a^1_{22}\, y_{2(t-1)} \\ y_{3t} &= a^0_{31}\, y_{1t} + a^0_{32}\, y_{2t} + a^1_{31}\, y_{1(t-1)} + a^1_{32}\, y_{2(t-1)} \\ y_{4t} &= a^0_{41}\, y_{1t} + a^0_{42}\, y_{2t} + a^0_{43}\, y_{3t} + a^1_{41}\, y_{1(t-1)} + a^1_{42}\, y_{2(t-1)} \\ \cdots &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

None of the current year values of y_2, y_3, y_4, $\cdots\cdots\cdots$, y_m enter into the determination of the current year level of y_1. The current year level of y_1 enters into the determination of the current year level of y_2 and both the current levels of y_1 and y_2 enter into the determination of the current level of y_3, the current levels of y_1, y_2 and y_3 enter into the determination of the current level of y_4, and so forth. This system is a *recursive* system.

The standard approach to identifying the elements of $\mathbf{A_0}$ in VAR analysis is to decompose the matrix of reduced form residuals

$$\mathbf{u_t}\,\mathbf{u_t'} = \Omega = \mathbf{G}\,\mathbf{e_t}(\mathbf{G}\,\mathbf{e_t})' = \mathbf{G}\,\mathbf{e_t}\mathbf{e_t'}\,\mathbf{G'} = \mathbf{G}\,\mathbf{D}\,\mathbf{G'}.$$

If we choose implicit units of measurement for the variables for which the standard deviations of the structural errors are unity, $\mathbf{D} = \mathbf{I}$ and our problem is to choose the matrix $\mathbf{G}$ for which

$$\mathbf{G}\,\mathbf{G'} = \mathbf{\Omega}.$$

This simply involves doing a Choleski decomposition of the matrix $\mathbf{\Omega}$. We thus obtain

$$(\mathbf{I} - \tilde{\mathbf{A}}_{\mathbf{0}})^{-1} = \mathbf{G}$$

and, hence,

$$\tilde{\mathbf{A}}_{\mathbf{0}} = \mathbf{I} - \mathbf{G}^{-1}$$

where $\tilde{\mathbf{A}}_{\mathbf{0}}$ is a representation of $\mathbf{A}_{\mathbf{0}}$ after scaling of the variables to render $\mathbf{D} = \mathbf{I}$. Using the matrix $\mathbf{G}$ so obtained we can obtain the $\mathbf{Z}_{\mathbf{j}}$ matrices in equation (17) with the errors $\mathbf{e}_t$ having unit variance.

The upper-left-corner element of $\mathbf{Z}_{\mathbf{0}}$ gives the response of y_1 to a one standard-deviation shock to the first equation in the current period. The sum of the upper-left-corner elements of $\mathbf{Z}_{\mathbf{0}}$ and $\mathbf{Z}_{\mathbf{1}}$ gives the response of y_1 to a one standard-deviation shock to the first equation in the previous period. And sum of the of the upper-left-corner elements of $\mathbf{Z}_{\mathbf{0}}$,$\mathbf{Z}_{\mathbf{1}}$ and $\mathbf{Z}_{\mathbf{2}}$ gives the response of that variable to a one standard-deviation shock to the first equation two periods previously, and so forth. The response of the first variable to a one-standard-deviation shock to the second variable in the current and previous periods is given by the second elements from the left in the top rows of the $\mathbf{Z}_{\mathbf{j}}$ matrices. And the response of the second variable to orthogonal one-standard-deviation shocks to the other variables is given by the elements of the second rows of the $\mathbf{Z}_{\mathbf{j}}$ matrices, and so forth. These matrices are called *impulse-response functions*.

It is important to emphasize that this decomposition of $\mathbf{\Omega}$ and the impulse-response functions that are obtained from it are critically dependent on the ordering of the variables in the VAR. Had we ordered the variables differently, putting the fourth variable first and the first variable fourth, for example, the Choleski decomposition would have led to different impulse-response functions.[5] Economic theory has to be used to decide which ordering of the variables to use. In many cases, no such ordering is acceptable because the theoretical system that the VAR is being used to analyse is not recursive.

The impulse responses for our United States VAR are calculated with the following ordering: detrended log RGDP, year over year rate of growth of the GDP deflator, interest rate on 90-day commercial paper, and deviation of log base money from trend. Figure 1 plots the responses of the four variables to a first-period shock to each in turn. These responses are scaled

[5]This assumes that the error terms in the equations of the standard-form representation are correllated with each other so that the off-diagonal elements of $\mathbf{\Omega}$ are non-zero. If these off-diagonal terms are zero, the impulse-response functions obtained from all the different orderings will be the same.

by (i.e., divided by) the variance of the responding variable obtained from $\mathbf{\Omega}$.

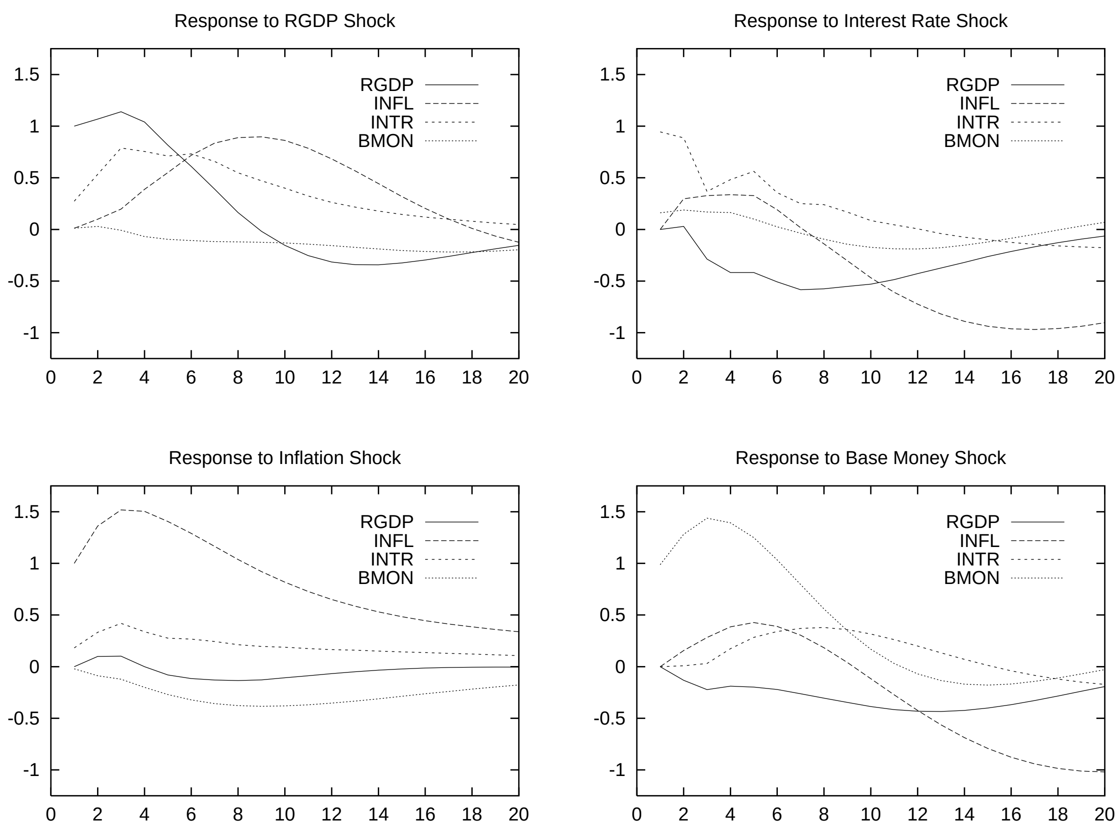

Figure 1: The responses of the four variables to first period shocks of each.

The plots in Figure 1 give superficial encouragement to those that would interpret U.S. monetary policy as operating through changes in the level at which the Federal Reserve sets short-term interest rates. An upward shock to the commercial paper rate in the upper-right panel of Figure 1 tends to result in a decline in real GDP. But its immediate effect on base money, over which the Fed. has direct control, is to cause an increase. Moreover, the rise in the short-term interest rate is accompanied during early subsequent periods by an increase in the inflation rate. Inflation does not become negative until seven quarters have elapsed. Why should tight money cause inflation in the short-run? Also, it is apparent from the lower right panel that an expansion of base money, which only the Fed. can engineer, causes short-term interest rates to rise and output to fall. And the inflation rate again eventually becomes negative, after increasing in the early quarters. A real GDP shock, in the upper-left panel, leads to increases in short-term interest rates and the inflation rate, and is associated with a slight subsequent decline in base money.

The fact that base money increases immediately in response to an upward shock to the short-term interest rate suggests that the interest rate increase may have occurred as a result of a positive shock to the demand for money, with the Fed. leaning slightly against the wind by expanding base money. We should realize, from looking at these charts, that first-period shocks to the variables could result from a variety of alternative factors. The Federal Reserve could reduce base money and deliberately raise short-term interest rates or, what seems more likely, respond to interest rate changes arising from other sources, by varying base money endogenously. It could be argued, of course, that it is the federal funds rate that the Federal Reserve controls, not the 90-day commercial paper rate. Substitution of the federal funds rate for the commercial paper rate in this VAR, however, results in no appreciable change in the impulse-responses.

As it is specified, the VAR does not permit a current-period shock to base money to affect the short-term interest rate—the ordering of the variables in the identification scheme has current RGDP affecting all the remaining variables in the current period, the year-over-year inflation rate affecting only the nominal interest rate and base money, and the nominal interest rate affecting the base money. The current level of base money cannot affect any of the other variables, the current level of the interest rate cannot affect the inflation rate or output, and the current inflation rate cannot affect output. We could reorder the variables to produce a different identification—it is not obvious that the current identification scheme would be superior to alternative ones. It turns out that placing base money ahead of the interest rate in the ordering, so that base money can affect the interest rate in the current period and the interest rate cannot affect base money, leads to impulse responses that are indistinguishable from those in Figure 1 except that the response of base money to an interest rate shock in the upper-right panel shifts upward very slightly with the pattern of response remaining virtually unchanged, and the the response of the interest rate to a base money shock in the bottom-right panel shifts upward by a similar amount, with the pattern of response also remaining essentially unchanged.

Conventional macroeconomic theory would postulate that a rise in the real interest rate should reduce output in the current period. It would also hold that there should be little effect of current output on the current inflation rate—the latter would be expected to change only with wage and price adjustments taking more than one quarter to respond. Conventional theory would also lead us to expect both that the short-term interest rate should respond to current changes in base money, current output, and the current price level (i.e., the current and past levels of the inflation rate), and that the Fed. will adjust base money in response to current period information about the level of output (actually employment), the short-term interest rate, and the rate of inflation. It may not observe the inflation rate and the level of output directly (because current estimates of these variables are subject to substantial later revision) but it will observe other contemporaneous variables that will be indicative of the current realizations of output and inflation.

We might visualize the current-period relationships between real GDP (call it o_t) the current inflation rate (call it τ_t), the current real interest rate (call it r_t) and the current level of base (or high-powered) money (call it h_t), given the predetermined past levels of these variables as follows:

$$
\begin{aligned}
o_t &= \alpha_r \, r_t && (18)\\
r_t &= \alpha_h \, h_t + \alpha_o \, o_t + \alpha_\tau \, \tau_t && (19)\\
h_t &= \gamma_r \, r_t + \gamma_o \, o_t + \gamma_\tau \, \tau_t && (20)
\end{aligned}
$$

where we would expect α_r, α_h and α_τ to be negative and α_o to be positive. Equation (18) is the traditional real goods market equilibrium equation, (19) a rearrangement of the demand function for money, and (20) is the Fed.'s reaction function.

5.2 Structural VARs

The problem, of course, is that the system we are trying to model may not be recursive, making a Choleski decomposition inappropriate. Sims[6] and Bernanke[7] model the innovations using economic analysis that postulates non-recursive relationships between the variables.

As in the Choleski decomposition, the object is to extract the coefficients of $\mathbf{G}$ from the variance-covariation matrix of the reduced-form system $\mathbf{\Omega}$.

[6] "Chrisopher Simms, "Are Forecasting Models Usable for Policy Analysis?" *Federal Reserve Bank of Minneapolis Quarterly Review* (Winter 1986), 3–16.

[7] Ben Bernanke, "Alternative Explanations of Money-Income Correlation," *Carnegie-Rochester Conference Series on Public Policy 25* (1986), 49–100.

The latter matrix is symmetric with the variances of the shocks along the diagonal and the covariances in the off-diagonal elements. It thus contains only $(m^2+m)/2$ distinct elements. We are therefore able to identify the same number of elements of $(\mathbf{I}-\mathbf{A_0})^{-1}$ in $\mathbf{G}$. This is precisely the number of elements identified by the Choleski decomposition. In a structural VAR, however, we can place the elements we want to identify anywhere in the $m \times m$ grid in accordance with the dictates of economic theory.

Bernanke's paper provides us with a method of exact identification. To keep the argument simple, let us assume that $m = 4$, as in our U.S. example. We can represent $\mathbf{\Omega}$ as

$$\mathbf{\Omega} = E\{\mathbf{u}_t\mathbf{u}_t'\} = \frac{\sum \mathbf{u}_t\mathbf{u}_t'}{n} = \mathbf{M} \tag{21}$$

where n is the number of observations. Then, from (10) we have

$$(\mathbf{I}-\mathbf{A_0})\mathbf{\Omega}(\mathbf{I}-\mathbf{A_0})' = (\mathbf{I}-\mathbf{A_0})\mathbf{M}(\mathbf{I}-\mathbf{A_0})' = \mathbf{D}. \tag{22}$$

Since $\mathbf{D}$ is diagonal, all the off-diagonal elements of the matrix on the left-hand side must be zero. We can take advantage of this fact to solve for the elements of $\mathbf{A_0}$.

Under the Choleski decomposition the matrix $\mathbf{A_0}$ was configured as follows, according to the way the variables were ranked.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{bmatrix} \quad = \quad \begin{array}{c|cccc} & o & \tau & r & h \\ \hline o & 0 & 0 & 0 & 0 \\ \tau & a_{21} & 0 & 0 & 0 \\ r & a_{31} & a_{32} & 0 & 0 \\ h & a_{41} & a_{42} & a_{43} & 0 \end{array}$$

The decomposition identifies six coefficients plus the variances of the four equations. Equations (18) through (20) above suggest the configuration

$$\begin{array}{c|cccc} & o & \tau & r & h \\ \hline o & 0 & 0 & a_{13} & 0 \\ \tau & 0 & 0 & 0 & 0 \\ r & a_{31} & a_{32} & 0 & a_{34} \\ h & a_{41} & a_{42} & a_{43} & 0 \end{array}$$

where $\alpha_r = a_{13}$, $\alpha_h = a_{34}$, $\alpha_o = a_{31}$, $\alpha_\tau = a_{32}$, $\gamma_o = a_{41}$, $\gamma_\tau = a_{42}$ and $\gamma_r = a_{43}$. This configuration attempts to estimate one too many parameters and, even if it did not, could not be identified because the last two equations are structurally identical, differing only according to which variable is put on the left-hand-side. We need to abandon one coefficient from either the third or fourth equation, imposing an additional restriction on the system by inserting a zero in its place. The zeros in the matrix represent restrictions imposed on the system while the a_{ij} represent coefficients to be identified. As it stands, the system is underidentified because there is too little information to identify all seven coefficients. Moreover, there is no basis for distinguishing the magnitudes of the parameters in the third equation from the magnitudes of the corresponding parameters in the fourth equation.

It would seem reasonable to replace the parameter a_{42} with a zero on the grounds that the authorities are likely to base their view of the current equilibrium inflation rate on the history of past inflation and pay little attention to current within-period inflation estimates, while holders of money will base their decision on how much money to hold based on a 'feel' for the current situation that will be influenced by price movements experienced in the current period. If one wanted to assume the reverse, that the authorities pay attention to current period inflation information while the public does not, then the third equation would be interpreted as the authorities reaction function and the fourth equation as the private sector's demand for money function. Accordingly, we arrive at the following configuration of $\mathbf{A_0}$:

$$\begin{bmatrix} 0 & 0 & a_{13} & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & 0 & a_{43} & 0 \end{bmatrix}$$

Letting s_{ij} be the ij-th element of $\mathbf{M}$ and expanding $(\mathbf{I}-\mathbf{A_0})\,\mathbf{M}\,(\mathbf{I}-\mathbf{A_0})'$, we obtain

$$\begin{bmatrix} 1 & 0 & -a_{13} & 0 \\ 0 & 1 & 0 & 0 \\ -a_{31} & -a_{32} & 1 & -a_{34} \\ -a_{41} & 0 & -a_{43} & 1 \end{bmatrix} \begin{bmatrix} s_{11} & s_{21} & s_{31} & s_{41} \\ s_{21} & s_{22} & s_{32} & s_{42} \\ s_{31} & s_{32} & s_{33} & s_{43} \\ s_{41} & a_{42} & s_{43} & s_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 & -a_{31} & -a_{41} \\ 0 & 1 & -a_{32} & 0 \\ -a_{13} & 0 & 1 & -a_{43} \\ 0 & 0 & -a_{34} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 & -a_{31} & -a_{41} \\ 0 & 1 & -a_{32} & 0 \\ -a_{13} & 0 & 1 & -a_{43} \\ 0 & 0 & -a_{34} & 1 \end{bmatrix}$$

where

$$
\begin{aligned}
g_{11} &= s_{11} - a_{13}s_{31} \\
g_{12} &= s_{21} - a_{13}s_{32} \\
g_{13} &= s_{31} - a_{13}s_{33} \\
g_{14} &= s_{41} - a_{13}s_{43} \\
g_{21} &= s_{21} \\
g_{22} &= s_{22} \\
g_{23} &= s_{32} \\
g_{24} &= s_{42} \\
g_{31} &= -a_{31}s_{11} - a_{32}s_{21} + s_{31} - a_{34}s_{41} \\
g_{32} &= -a_{31}s_{21} - a_{32}s_{22} + s_{32} - a_{34}s_{42} \\
g_{33} &= -a_{31}s_{31} - a_{32}s_{32} + s_{33} - a_{34}s_{43} \\
g_{34} &= -a_{31}s_{41} - a_{32}s_{42} + s_{43} - a_{34}s_{44} \\
g_{41} &= -a_{41}s_{11} - a_{43}s_{31} + s_{41} \\
g_{42} &= -a_{41}s_{21} - a_{43}s_{32} + s_{42} \\
g_{43} &= -a_{41}s_{31} - a_{43}s_{33} + s_{43} \\
g_{44} &= -a_{41}s_{41} - a_{43}s_{43} + s_{44}
\end{aligned}
$$

Multiplying together the above two matrices, we then obtain

$$
\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix}
$$

where

$$
\begin{aligned}
d_{11} &= g_{11} - g_{13}a_{13} \\
&= s_{11} - a_{31}s_{31} - a_{13}s_{31} + a_{13}^2 s_{33} && (23) \\
d_{22} &= g_{22} \;=\; s_{22} && (24) \\
d_{33} &= -a_{31}g_{31} - a_{32}g_{32} + g_{33} - g_{34}a_{34} \\
&= a_{31}^2 s_{11} + a_{31}a_{32}s_{21} - a_{31}s_{31} + a_{31}a_{34}s_{41} \\
&\qquad + a_{32}a_{31}s_{21} + a_{32}^2 s_{22} - a_{32}s_{32} + a_{32}a_{34}s_{42} \\
&\qquad\quad - a_{31}s_{31} - a_{32}s_{32} + s_{33} - a_{34}s_{43} + a_{34}a_{31}s_{41} \\
&\qquad\qquad + a_{34}a_{32}s_{42} - a_{34}s_{43} + a_{34}^2 s_{44} && (25)
\end{aligned}
$$

$$
\begin{aligned}
d_{44} &= -a_{41}g_{41} - a_{43}g_{43} + g_{44} \\
&= +a_{41}^2 s_{11} + a_{41}a_{43}s_{31} - a_{41}s_{41} + a_{43}a_{41}s_{31} + a_{43}^2 s_{33} \\
&\qquad - a_{43}s_{43} - a_{41}s_{41} - a_{43}s_{43} + s_{44} \qquad (26) \\
d_{12} &= d_{21} \;=\; g_{12} \\
&= s_{21} - a_{13}s_{32} \;=\; 0 \qquad (27) \\
d_{13} &= d_{31} \;=\; -a_{31}g_{11} - a_{32}g_{12} + g_{13} - a_{34}g_{14} \\
&= -a_{31}s_{11} + a_{31}a_{13}s_{31} - a_{32}s_{21} + a_{32}a_{13}s_{32} \\
&\quad + s_{31} - a_{13}s_{33} - a_{34}s_{41} + a_{34}a_{13}s_{43} \;=\; 0 \qquad (28) \\
d_{14} &= d_{41} \;=\; -a_{41}g_{11} - a_{43}g_{13} + g_{14} \\
&= -a_{41}s_{11} + a_{41}a_{13}s_{31} - a_{43}s_{31} + a_{43}a_{13}s_{33} \\
&\quad + s_{41} - a_{13}s_{43} \;=\; 0 \qquad (29) \\
d_{23} &= d_{32} \;=\; -a_{31}g_{21} - a_{32}g_{22} + g_{23} - a_{34}g_{24} \\
&= -a_{31}s_{21} - a_{32}s_{22} + s_{32} - a_{34}s_{42} \;=\; 0 \qquad (30) \\
d_{24} &= d_{42} \;=\; -a_{41}g_{21} - a_{43}g_{23} + g_{24} \\
&= -a_{41}s_{21} - a_{43}s_{32} + s_{42} \;=\; 0 \qquad (31) \\
d_{34} &= d_{43} \;=\; -a_{41}g_{31} - a_{43}g_{33} + g_{34} \\
&= a_{41}a_{31}s_{11} + a_{41}a_{32}s_{21} - a_{41}s_{31} + a_{41}a_{34}s_{41} + a_{43}a_{31}s_{31} \\
&\quad + a_{43}a_{32}s_{32} - a_{43}s_{33} + a_{43}a_{34}s_{43} - a_{31}s_{41} \\
&\qquad - a_{32}s_{42} + s_{43} - a_{34}s_{44} \;=\; 0 \qquad (32)
\end{aligned}
$$

The last six of these equations, (27) through (32) can be solved for a_{21}, a_{23}, a_{31}, a_{41}, a_{42} and a_{43}. This gives us an estimate of the matrix $\mathbf{A_0}$ and, by matrix substitution, $\mathbf{I} - \mathbf{A_0}$. These solutions for the a_{ij} can then be plugged into equations (23) through (26) to obtain the diagonal elements d_{11}, d_{22}, d_{33} and d_{44} and, hence, the matrix $\mathbf{D}$. The actual calculations for our U.S. VAR can be found in the XlispStat code file `vardemst.lsp`.[8]

Calculation of $(\mathbf{I} - \mathbf{A_0})$ by itself is insufficient to obtain the matrix $\mathbf{G}$ and the impulse-response functions. The matrix $(\mathbf{I} - \mathbf{A_0})$ has to be scaled to reduce $\mathbf{D}$ to an identity matrix to ensure that the orthogonalized residuals have unit variance. Thus,

$$
\begin{aligned}
\mathbf{G} &= (\mathbf{I} - \mathbf{A_0})^{-\mathbf{1}}\,\mathbf{D}^{\mathbf{1/2}} \\
&= (\mathbf{I} - \tilde{\mathbf{A}}_{\mathbf{0}})^{-\mathbf{1}} \qquad (33)
\end{aligned}
$$

[8]This file differs from `vardemcd.lsp` only in that an additional function, `structural-decomposition`, is added to the code file and called in the calculations instead of the previously used `choleski-decomposition` function.

where $\tilde{\mathbf{A}}_{\mathbf{0}}$ is the representation of $\mathbf{A}_{\mathbf{0}}$ for which the $\mathbf{e}_t$ have unit variance.

Structural VARS can be calculated in RATS using code shown in the file `vardemst.prg`.[9] The procedure first performs the above calculations for a just-identified structural VAR using our U.S. data. The resulting matrix of coefficients can then be fed to the RATS procedure in the accompanying file `bernanke.src` (also contained in the above noted zip files) which will calculate confidence intervals for the parameters. Alternatively, within `vardemst.prg` one can rely exclusively on the RATS bernanke procedure by simply feeding it a pattern matrix containing ones to represent the coefficients that are to be determined and zeros elsewhere. For exact identification there should be $(m^2 + m)/2$ ones and $(m^2 - m)/2$ zero restrictions. The pattern matrix for our U.S. VAR is

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

RATS then attempts to calculate the coefficients of $\mathbf{I} - \mathbf{A}_{\mathbf{0}}$ and compute standard errors for these coefficients. To do this it maximizes a likelihood function through a process of iteration. Quite often it is unable to obtain a maximum, in which case it indicates that convergence has not occurred and fails to calculate the standard errors of some of the coefficients. In these cases it is necessary to prompt RATS by passing it an input matrix of guesses as to levels of the coefficients at which it should begin iteration. For a just-identified system we would pass it the matrix of values previously calculated. To estimate an over-determined system we change the appropriate ones in the pattern matrix to zeros. If RATS cannot successfully calculate the coefficients we could try feeding it the matrix for the just-identified system after setting the appropriate element or elements equal to zero.

The advantage of using RATS for just-identified structural VARs is that it will calculate standard errors of the coefficients. Also, in the case of over-identified systems (i.e., ones with more than $(m^2 - m)/2$ zero restrictions) RATS performs a likelihood ratio test of whether the overidentifying restrictions are statistically significant.[10] Only point estimates of the coefficients can be calculated using our XlispStat code.

[9] This file and the corresponding output file `vardemst.out` are contained in above noted zip-files `ratszip.exe` and `ratszip.tgz` obtainable from my web-site.

[10] The most recent versions of RATS contain a function called `cvmodel` which can be used instead of the procedure `bernanke.src` and makes the above calculation of the just-identified system unnecessary. For purposes of learning what is happening, the approach outlined here is best. To repeatedly run many different structural VARs in RATS one

The impulse-responses for our U.S. data using the above just-identified structural VAR are plotted in Figure 2. They appear to be much the same as those obtained from the earlier choleski decomposition.

5.3 Blanchard-Quah Decompositions

For certain types of problems it is useful to decompose $\mathbf{\Omega}$ by a method developed by Blanchard and Quah.[11] Their method assumes a two variable VAR with two equations and two types of shock, real and nominal, that are statistically independent of each other and affect both equations. Here we take the two variables to be the real and nominal exchange rates. A Blanchard-Quah decomposition identifies the real and nominal shocks under the assumption that one type of shock, in our case the nominal shock, has a temporary but no permanent effect on the level of one of the variables, in our case the real exchange rate, and a permanent effect on the level of the other variable, the nominal exchange rate. The other type of shock, a real shock, has permanent effects on the levels of both variables. Neither the real nor the nominal shocks have permanent long-run effects on the first differences of either of the variables. Decomposition is accomplished by imposing one restriction on the two-variable VAR—the restriction, in our case, that nominal shocks can have no permanent effects on the level of the real exchange rate. In using the Blanchard-Quah method, our interest is directed more toward decomposition of the standard-form errors into orthogonal structural errors than to obtaining the elements of the matrix $\mathbf{A_0}$.

We are still interested in the matrix $\mathbf{G}$ which will reduce the moving average representation (16) to (17) and in particular enable us to obain the orthogonal errors

$$\mathbf{e}_{t-j} = \mathbf{G}^{-1}\mathbf{u}_{t-j}.$$

Now, however, the identifying restriction is that the sum of the upper left corner elements of the (now 2×2) matrices $\mathbf{Z_0} = \mathbf{C_0 G}$, $\mathbf{Z_1} = \mathbf{C_1 G}$, $\mathbf{Z_2} = \mathbf{C_2 G}$, $\cdots\cdots$, $\mathbf{Z_n} = \mathbf{C_n G}$, be equal to zero. This assumes that the real exchange rate equation is the first equation and that the nominal shock is the first shock. Had we wanted to assume that the nominal shock is the second shock the identifying restriction would have been that the sum of the upper right corner elements of the above matrices is zero. There is no requirement that the nominal shock be identified with the nominal exchange

would benefit from using the more modern approach.

[11] Olivier Jean Blanchard and Danny Quah, "The Dynamic Effects of Aggregate Demand and Supply Disturbances," *American Economic Review,* 79, September 1989, 655–73.

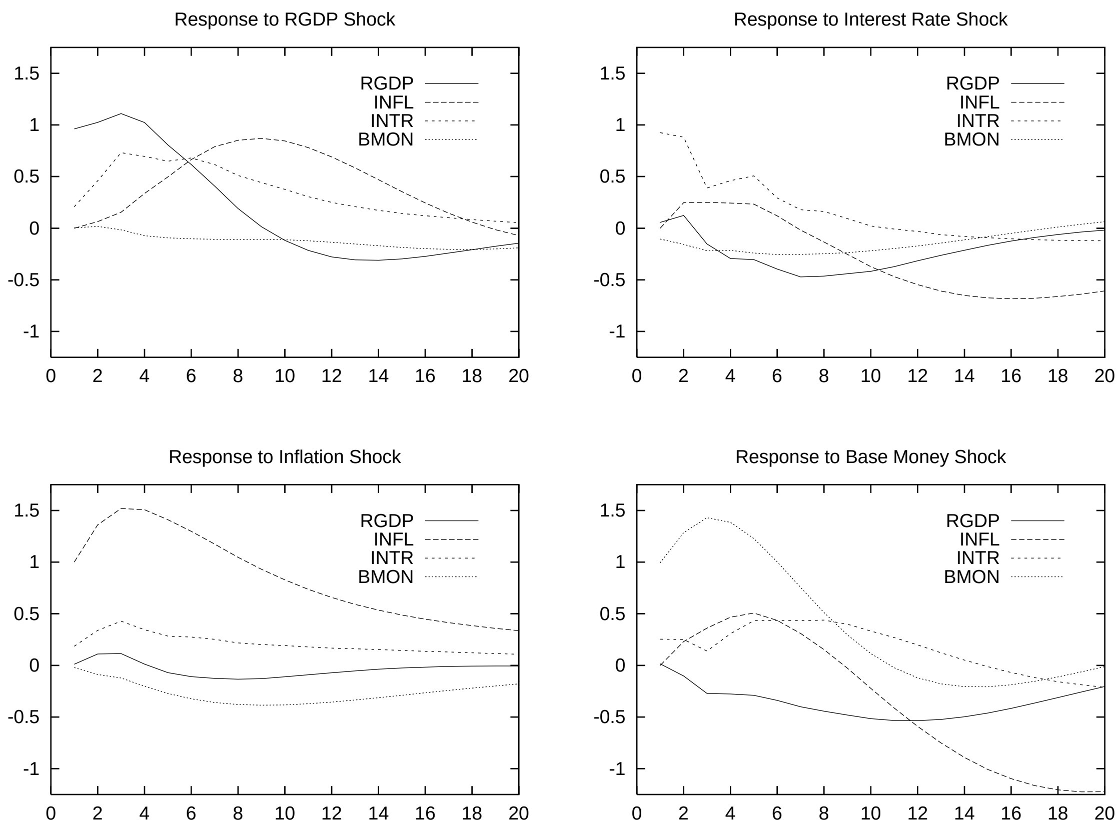

Figure 2: The responses of the four variables to first period shocks of each under a structural-decomposition of $\mathbf{\Omega}$.

rate variable. The statistical results will be the same whether the nominal shock is the first or the second shock.

The sum of the matrices $\mathbf{C_0}$, $\mathbf{C_1}$, $\mathbf{C_2}$, $\cdots\cdots$, etc. can be obtained by calculating the sum of the corresponding elements in equation (15) using the relationship

$$\begin{aligned}\mathcal{S} &= \mathbf{I}_{mp} + \mathcal{B} + \mathcal{B}^2 + \mathcal{B}^3 + \mathcal{B}^4 + \cdots\cdots\cdots\cdots \\ &= (\mathbf{I}_{mp} - \mathcal{B})^{-1}.\end{aligned} \tag{34}$$

and stripping off the upper-left 2×2 matrix of elements from $\mathcal{S}$ using the (now $2 \times 2p$) matrix $\mathcal{J}$ used to obtain equation (11) from equation (17). We thus obtain

$$\mathbf{S} = \mathcal{J}\mathcal{S}\mathcal{J}'. \tag{35}$$

The Blanchard-Quah condition is that the upper left element of the 2×2 matrix $\mathbf{SG}$ (which is the sum of the upper-left elements of the $\mathbf{C_j}$ matrices after each has been post-multiplied by $\mathbf{G}$) be zero.

In addition, the $\mathbf{G}$ matrix must have the property $\mathbf{GG'} = \mathbf{\Omega}$. A Choleski decomposition of $\mathbf{\Omega}$ produces a matrix with this property, as does the transformation of $(\mathbf{I} - \mathbf{A})^{-1}$ by (33) we made in the structural VAR calculations. It turns out that any orthogonal transformation of a matrix obtained from a Choleski decomposition will also possess this property. The procedure here is therefore to make a Choleski decomposition of $\mathbf{\Omega}$ to obtain some matrix $\mathbf{E}$ which we can then transform using some orthogonal matrix $\mathbf{P}$ to impose upon it the Blanchard-Quah condition and thereby obtain the desired matrix $\mathbf{G}$. The two requirements are that

$$\begin{aligned}\mathbf{GG'} &= \mathbf{EP}(\mathbf{EP})' \\ &= \mathbf{EPP'E'} \\ &= \mathbf{EE'} = \mathbf{\Omega}\end{aligned} \tag{36}$$

since orthogonality of $\mathbf{P}$ implies $\mathbf{PP'} = \mathbf{I}$, and that the upper left corner element of the (2×2) matrix

$$\mathbf{SG} = \mathbf{SEP} = \mathbf{SEP} = \mathbf{HP} \tag{37}$$

equal zero, where $\mathbf{H} = \mathbf{SE}$. Expanding the latter condition we have[12]

$$h_{11}p_{11} + h_{12}p_{21} = 0. \tag{38}$$

[12] Were we to designate the second shock as the nominal shock, this condition would become

$$h_{11}p_{21} + h_{12}p_{22} = 0.$$

From the fact that $\mathbf{PP}' = \mathbf{I}$ we obtain

$$\begin{bmatrix} p_{11}^2 + p_{12}^2 & p_{11}p_{21} + p_{12}p_{22} \\ p_{21}p_{11} + p_{22}p_{12} & p_{21}^2 + p_{22}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which yields the three conditions

$$p_{11}^2 + p_{12}^2 = 1 \tag{39}$$

$$p_{11}p_{21} = -p_{12}p_{22} \tag{40}$$

$$p_{21}^2 + p_{22}^2 = 1 \tag{41}$$

Given the values of $\mathbf{S}$ and $\mathbf{E}$ and $\mathbf{H}$, calculated from the data, the four equations (38) through (41) solve for the four elements of $\mathbf{P}$. The latter matrix can then be multiplied by $\mathbf{E}$ to obtain $\mathbf{G}$. Using this matrix, the impulse-response functions can be calculated from (17).[13]

These calculations are illustrated using monthly data on the Canada/U.S. real and nominal exchange rates. Following Enders, the first differences of the logarithms of the two series are taken.[14] A careful programming of calculations in RATS from the ground up are presented in the file `vardembq.prg` while calculations exclusively based on RATS functions are undertaken in the file `bqvarcau.prg`. These files, along with the corresponding output files, `vardembq.out` and `bqvarcau.out` and the RATS data file `vardem.rat` which both programs use, are contained in `ratszip.exe` and `ratszip.tgz`.[15] The impulse-responses are plotted later on in Figure 3, where confidence intervals are also shown. Nominal shocks of unit variance have very small

[13] In calculating the elements of the matrix $\mathbf{P}$ it is useful to rearrange (38) to obtain

$$p_{11} = -\frac{h_{12}}{h_{11}} p_{21} = w\, p_{21}$$

and then square both sides of this equation and substitute the resulting expression for p_{11}^2 into the square of equation (40). From there it can be shown that

$$p_{12} = p_{21} = \sqrt{\frac{1}{1+w^2}},$$

which will be positive regardless of the sign of w. Multiplication of p_{21} by w then yields p_{11}. Using these results along with (40), it can then be shown that

$$p_{22} = -p_{11}.$$

[14] See page 338 of the book cited in the Introduction.

[15] The code in `bqvarcau.prg` will work only with version 5 of RATS or higher, since it utilizes improvements made in that version. The code and results in `vardembq.out` should be in large part understandable to readers unfamiliar with the RATS program.

effects on the real exchange rate and effects on the nominal exchange rate about half as big as the effects of real shocks.

6 Forecast Error Variance Decomposition

One way to determine how important the different exogenous shocks are in explaining the dependent variables is to calculate the fractions of the forecast error variance of these variables attributable to the respective orthogonal shocks. The variance of any given dependent variable in response to the orthogonal shocks to it can be thought of as the variance of the errors in forecasting it using (17) because without the shocks we would forecast the variable to remain unchanged. The central question is: What fractions of these forecast errors are due to the individual shocks?

Consider the forecast error for period t obtained from (17) which is reproduced below.

$$\begin{aligned} \mathbf{y}_t &= \mathbf{Z_0}\,\mathbf{e}_t + \mathbf{Z_1}\,\mathbf{e}_{t-1} + \mathbf{Z_2}\,\mathbf{e}_{t-2} \\ &\qquad\qquad + \cdots\cdots + \mathbf{Z_n}\,\mathbf{e}_{t-n} + \mathbf{y_0} \end{aligned} \tag{17}$$

The vector of one step ahead forecast errors is given by $\mathbf{Z_0}\,\mathbf{e}_t$. Consider the simple case where there are only two equations. Letting z^0_{ij} be the ij-th element of $\mathbf{Z_0}$, we can express the current-period forecast errors as

$$\begin{aligned} y_{1t} &= z^0_{11} e_{1t} + z^0_{12} e_{2t} \\ y_{2t} &= z^0_{21} e_{1t} + z^0_{22} e_{2t} \end{aligned}$$

from which it follows that

$$\begin{aligned} Var\{y_1\} &= (z^0_{11})^2 Var\{e_1\} + (z^0_{12})^2 Var\{e_2\} &= (z^0_{11})^2 + (z^0_{12})^2 \\ Var\{y_2\} &= (z^0_{21})^2 Var\{e_1\} + (z^0_{22})^2 Var\{e_2\} &= (z^0_{21})^2 + (z^0_{22})^2 \end{aligned}$$

since e_1 and e_2 are independent shocks with unit variance. The standard errors of y_1 and y_2 are therefore

$$Std\{y_1\} = \sqrt{(z^0_{11})^2 + (z^0_{12})^2} \qquad \text{and} \qquad Std\{y_2\} = \sqrt{(z^0_{21})^2 + (z^0_{22})^2}$$

and the fraction of the error variance attributable to the shock to the first and second equations are, respectively,[16]

[16] In a Blanchard-Quah decomposition the shocks are simply labelled 1 and 2 and are not visualized as 'tied' to any particular equation. Nevertheless, the calculations are the same.

$$\frac{(z_{11}^0)^2}{(z_{11}^0)^2 + (z_{12}^0)^2} \qquad \text{and} \qquad \frac{(z_{12}^0)^2}{(z_{11}^0)^2 + (z_{12}^0)^2}.$$

Now consider the two step ahead forecast. In this case the forecast errors in response to the two period's shocks are

$$\begin{aligned} y_{1t} &= z_{11}^0 e_{1t} + z_{12}^0 e_{2t} + z_{11}^1 e_{1(t-1)} + z_{12}^1 e_{2(t-1)} \\ y_{2t} &= z_{21}^0 e_{1t} + z_{22}^0 e_{2t} + z_{21}^1 e_{1(t-1)} + z_{22}^1 e_{2(t-1)} \end{aligned}$$

where z_{ij}^1 is the ij-th element of $\mathbf{Z_1}$. The variances of the respective two-period forecast errors are

$$(z_{11}^0)^2 + (z_{12}^0)^2 + (z_{11}^1)^2 + (z_{12}^1)^2$$

and

$$(z_{21}^0)^2 + (z_{22}^0)^2 + (z_{21}^0)^2 + (z_{22}^0)^2$$

and the standard errors of the two-period forecasts are

$$\sqrt{(z_{11}^0)^2 + (z_{12}^0)^2 + (z_{11}^1)^2 + (z_{12}^1)^2}$$

and

$$\sqrt{(z_{21}^0)^2 + (z_{22}^0)^2 + (z_{21}^0)^2 + (z_{22}^0)^2}.$$

The fraction of the two-step ahead forecast error variance of y_1 attributable to the shock to the first shock is

$$\frac{(z_{11}^0)^2 + (z_{11}^1)^2}{(z_{11}^0)^2 + (z_{12}^0)^2 + (z_{11}^1)^2 + (z_{12}^1)^2}$$

and the fraction attributable to the second shock is

$$\frac{(z_{12}^0)^2 + (z_{12}^1)^2}{(z_{11}^0)^2 + (z_{12}^0)^2 + (z_{11}^1)^2 + (z_{12}^1)^2}$$

And the fractions of the two-step ahead forecast error variance of y_2 attributable to the respective shocks are

$$\frac{(z_{21}^0)^2 + (z_{21}^1)^2}{(z_{21}^0)^2 + (z_{22}^0)^2 + (z_{21}^1)^2 + (z_{22}^1)^2}$$

and

$$\frac{(z_{22}^0)^2 + (z_{22}^1)^2}{(z_{21}^0)^2 + (z_{22}^0)^2 + (z_{21}^1)^2 + (z_{22}^1)^2}.$$

The derivations of the forecast error variances and the fractions attributable to the two shocks for forecasts greater than two-steps ahead are straightforward extensions of the calculations above.

The forecast-error variance decompositions for the Blanchard-Quah VAR in `vardembq.prg` are presented in Table 1. It is evident that over 93% of the forecast-error variance of the Canada/U.S. real exchange rate can be attributed to real shocks. And nominal shocks account for less than 16% of the forecast-error variance of the Canada/U.S. nominal exchange rate.

7 Obtaining Confidence Intervals

There are a number of ways to obtain confidence intervals for the impulse-responses and the forecast-error variance decompositions and there is controversy about which way is best.[17] A bootstrap method is adopted here.[18]. The vector of residuals $\mathbf{u}_t$ from the standard-form estimation equation (10) can be viewed as a random draw from the population of true residuals. By repeated sampling with replacement from these residuals we can obtain additional samples from that population. These samples differ from each other not only in the ordering of the residuals but in the number of times individual residuals appear. Starting with the initial lagged values of the vector $\mathbf{y}_t$, and using the vector of coefficients obtained from our initial standard form estimation, we can reconstruct new series of $\mathbf{y}_t$ for each sample of residuals drawn. Then using each of these new series in turn we can rerun the VAR and calculate new impulse-responses and forecast-error variance decompositions that can be viewed as draws from the true data generating process for the series. The 5th and 95th percentiles of the resulting set of impulse-responses and forecast-error variance decompositions for each step can then be viewed as the lower and upper 90% confidence interval at that step. The confidence intervals calculated for our Blanchard-Quah Canada/U.S. real exchange rate VAR are shown in Figure 3. The actual calculations are outlined in the RATS program file `bqvarcau.prg`. Not surprisingly, the confidence bounds on the response of the real exchange rate to the nominal shock in our Blanchard-Quah VAR bracket zero.

[17] Enders, in the book cited above, does not even discuss the issue.

[18] For a discussion of bootstrap methods and their application to the problem at hand, see Bradley Efron and Robert J. Tibshirani, *An Introduction to the Bootstrap*, Chapman and Hall/CRC, 1998, David E. Runkle, "Vector Autoregressions and Reality," *Journal of Business and Economic Statistics*, Vol. 5, No. 4 (October), 1987, and Hongyi Lee and G. S. Maddala, "Bootstrapping Time Series Models," *Econometric Reviews*, Vol. 15, No. 2, pp. 115-158, 1996.

Table 1: Forecast-Error-Variance Decomposition for the Blanchard-Quah VAR in `vardembq.prg`

Step /	Real Exchange Rate		Nom. Exchange Rate	
Shock	Nominal	Real	Nominal	Real
1	0.484	99.516	13.416	86.584
2	1.106	98.894	14.154	85.846
3	1.962	98.038	14.302	85.698
4	2.310	97.690	14.851	85.149
5	2.763	97.237	15.484	84.516
6	3.113	96.887	15.709	84.291
7	3.150	96.850	15.705	84.295
8	3.341	96.659	15.701	84.299
9	3.695	96.305	15.519	84.481
10	4.034	95.966	15.497	84.503
11	4.025	95.975	15.188	84.812
12	3.969	96.031	14.828	85.172
13	4.203	95.797	14.838	85.162
14	4.236	95.764	14.553	85.447
15	4.253	95.747	14.554	85.446
16	4.309	95.691	14.552	85.448
17	4.298	95.702	14.648	85.352
18	5.051	94.949	15.208	84.792
19	5.000	95.000	15.030	84.970
20	5.375	94.625	15.486	84.514
21	5.376	94.624	15.456	84.544
22	5.837	94.163	15.573	84.427
23	5.937	94.063	15.591	84.409
24	5.934	94.066	15.589	84.411
25	6.036	93.964	15.591	84.409
26	6.030	93.970	15.588	84.412
27	5.994	94.006	15.505	84.495
28	5.993	94.007	15.499	84.501
29	6.000	94.000	15.489	84.511
30	6.027	93.973	15.519	84.481
31	6.105	93.895	15.575	84.425
32	6.175	93.825	15.597	84.403
33	6.177	93.823	15.614	84.386
34	6.179	93.821	15.636	84.364
35	6.229	93.771	15.752	84.248
36	6.262	93.738	15.786	84.214
37	6.310	93.690	15.788	84.212
38	6.360	93.640	15.806	84.194
39	6.360	93.640	15.804	84.196
40	6.412	93.588	15.828	84.172

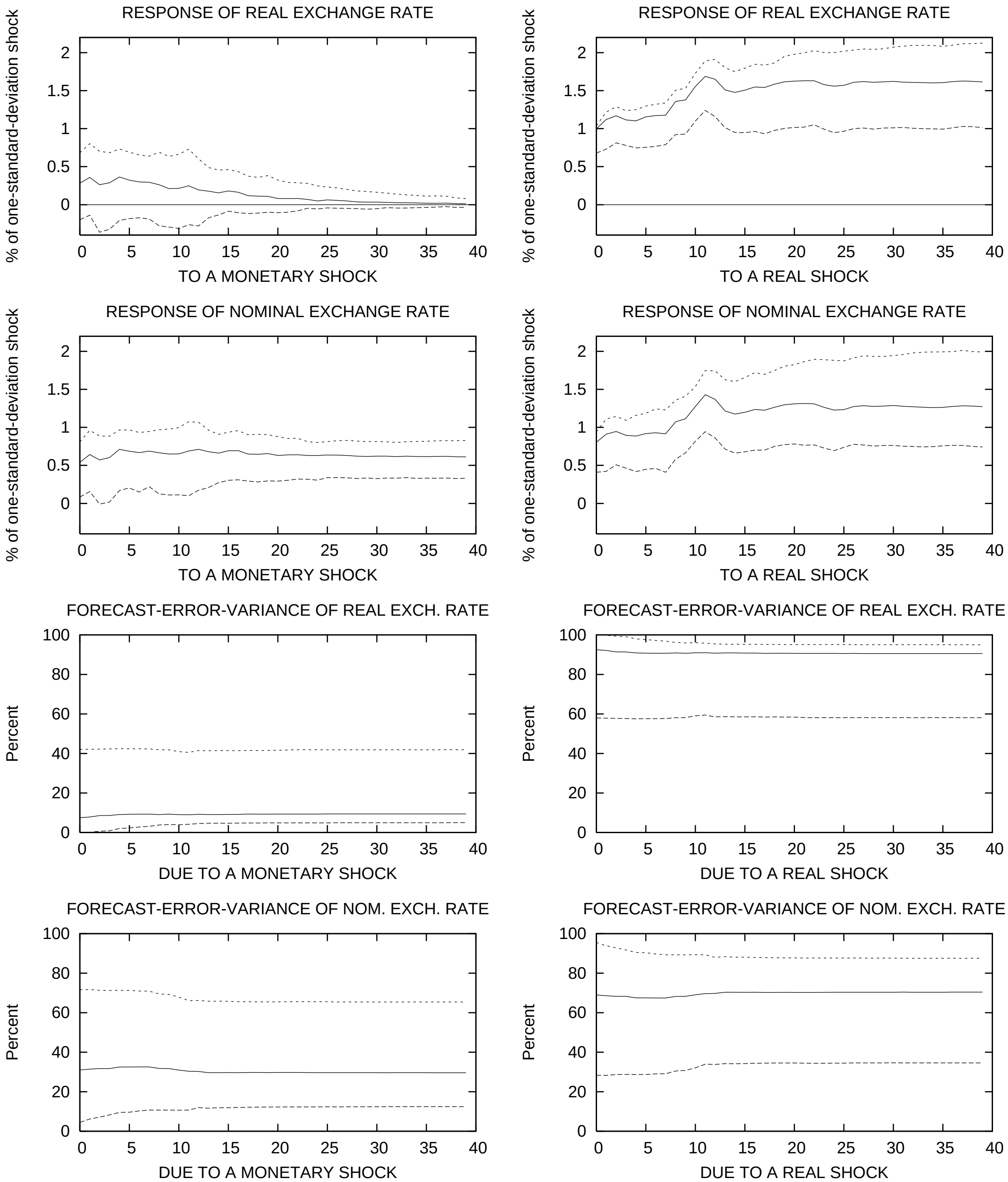

Figure 3: Blanchard-Quah VAR impulse-responses and forecast-error-variance decompositions for Canada's real and nominal exchange rates with respect to the U.S. dollar. The confidence intervals are 90 percent.

8 Historical Decompositions

It is often useful to decompose the actual movements in a series into the movements that occurred on the basis of each individual shock. For example, in our Blanchard-Quah VAR above it is important to be able to determine what movements in the real exchange rate occurred as a result of nominal shocks and what movements occurred as a result of real shocks, and similarly for the nominal exchange rate. First we obtain the actual orthogonalized shocks from the reduced-form residuals using the relationship

$$\mathbf{e}_t = \mathbf{G}^{-1}\mathbf{u}_t.$$

Then the effect of shock one (the nominal shock) in the first period on the real exchange rate in that period is

$$z^0_{11}\, e_{11}$$

and the effect of that shock on the nominal exchange rate in that period is

$$z^0_{21}\, e_{11},$$

where e_{ij} gives the value of the i-th shock in the j-th period. Similarly, the effects of shock two (the real shock) in the first period on the real and nominal exchange rates are

$$z^0_{12}\, e_{21} \qquad \text{and} \qquad z^0_{22}\, e_{21}.$$

In the next period, the effect of the nominal shock on the real exchange rate will be

$$z^0_{11}\, e_{12} + z^1_{11}\, e_{11}$$

and in the third period the influence of the nominal shock will be

$$z^0_{11}\, e_{13} + z^1_{11}\, e_{12} + z^2_{11}\, e_{11}.$$

The influences of the nominal shock on the nominal exchange rate and the real shocks on the real and nominal exchange rate can be layed out correspondingly. If we have calculated 40-period responses, as we did in the above example, there will be 40 terms in the expressions determining the response of each variable in the period 40 and all subsequent periods to each particular shock.

The calculations above give simply the influences of the real and nominal shocks on the change in each series in each period. To obtain the effects of the shocks on the levels of the real and nominal exchange rates we have to add, in turn, each period's change to the level at the beginning of that period.[19]

Since the data were first-differenced prior to estimation in our real and nominal exchange rate VAR, the responses calculated above will contain no trend, contrary to what is observed in the undifferenced series. It is therefore appropriate when plotting the historical decompositions against the levels of the actual series to appropriately assign the trends in the actual series to the decomposed series. Since the Blanchard-Quah decomposition assumed that nominal shocks have no long-run influence on the level of the real exchange rate, the trend in this variable must be assigned entirely to the real shock. This can be accomplished by adding the mean of the differenced real exchange rate series to each year's calculation before adding the incremental effect of current and past shocks in that period to the level calculated for the previous period. In decomposing the nominal exchange rate, it would seem reasonable to assign the excess of the trend of the nominal exchange rate (mean of the differenced nominal exchange rate series) over the trend of the real exchange rate (mean of the differenced real exchange rate series) to the nominal shock. This can be accomplished by adding the excess of the mean difference of the actual nominal exchange rate series over the mean difference of the actual real exchange rate series to the incremental change in the nominal exchange rate resulting from current and previous shocks before adding that incremental change to the level obtained for the previous period. The historical decompositions are calculated in the RATS program file `bqvarcau.prg` and the results are plotted in Figure 4. It is clear that nominal shocks had very little influence on the real exchange rate and a modest though not predominant influence on the nominal exchange rate. Real shocks explain all the turning points and major variations in both series.

[19] In the example here is is also useful to eventually take the antilogs of the decomposed series.

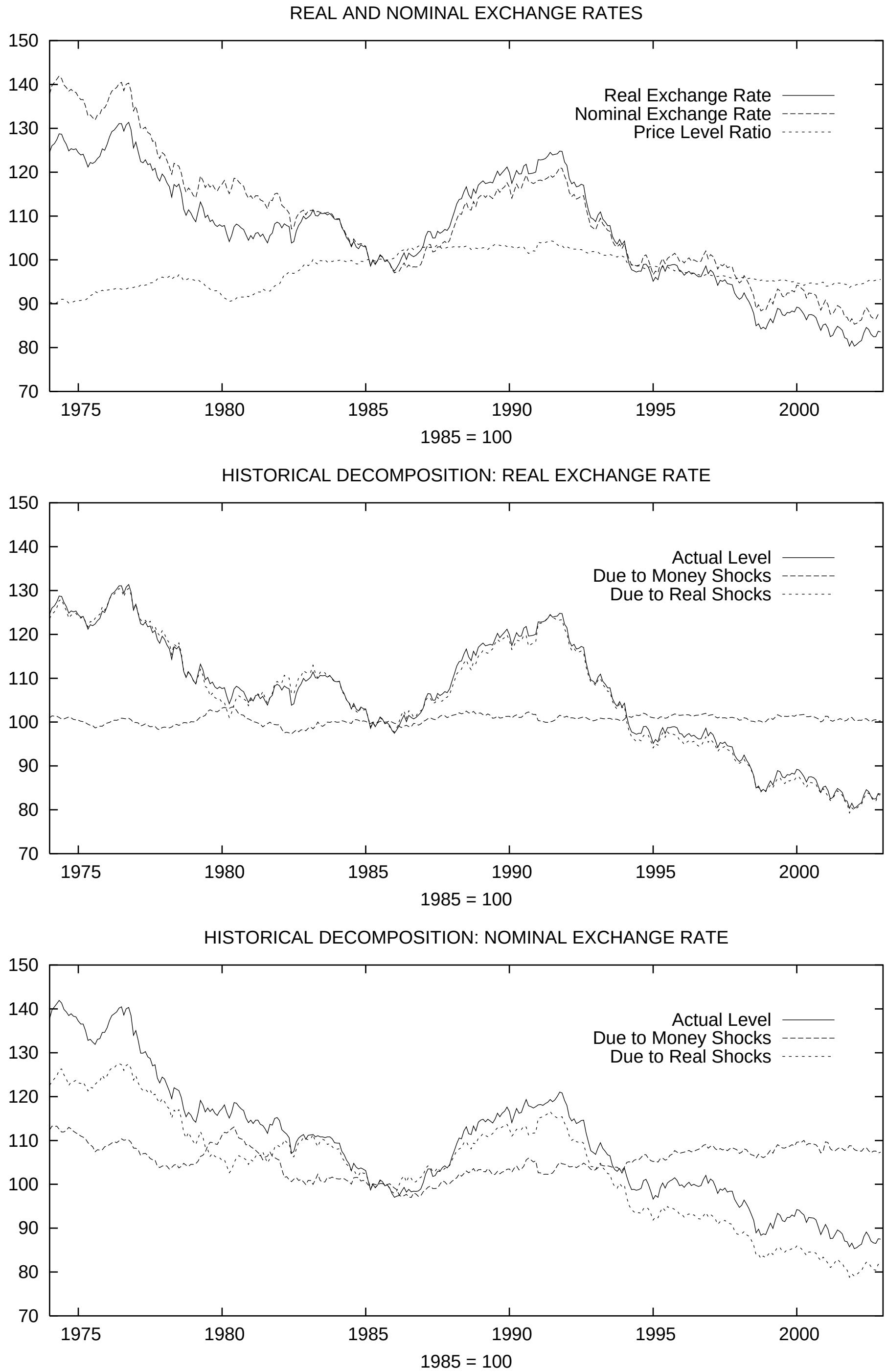

Figure 4: Blanchard-Quah-VAR historical decompositions of Canada's real and nominal exchange rates with respect to the U.S. dollar into the movements attributable to real and money shocks.

9 Choosing Lag-Length and Model Specification

In the examples above we set the lag-length arbitrarily, focussing entirely on the calculations involved in running a VAR. Choosing the lag-length is an important consideration, and even before that we must decide what variables to include in the VAR. Deciding what variables to include is fundamentally a matter of economics, the main considerations being the question that is being asked and the economic theory being used and tested. Within this framework, the causality and exogeniety of the variables and groups of variables has to be investigated.

We say that a particular variable, say variable two, Granger causes another variable, say variable three, if the coefficients of the lagged values of variable two in the variable three regression (the regression having variable three as the dependent variable) are statistically significant. This in effect says that variable two affects variable three in subsequent periods—or precedes it. A simple F-test for Granger causality can be used. We simply run the regression in question with the lagged values of variable two and then again without them. We then obtain

$$F = \frac{\sum(u^R_{i(t)})^2 - \sum(u_{i(t)})^2}{v} / \frac{\sum(u_{i(t)})^2}{df}$$

where the $u^R_{i(t)}$ are the residuals from the *restricted* regression—the one with the lagged values of variable two omitted—and the $u_{i(t)}$ are the residuals from the unrestricted regression where all the lags are included. The number of restrictions imposed (i.e., the number of lagged variables omitted in calculating the restricted regression) is equal to v, and df is the number of degrees of freedom in the unrestricted regression. The latter is equal to the total number of observations minus the number of variables on the right-hand side of that regression including the constant term—this will equal $(mp + 1)$ where m is the number of variables in the VAR and p is the number of lags of those variables. This F-statistic can then be compared with the appropriate critical value of $F(v, df)$.

In addition to performing the test for the lagged values of all the other variables in each equation we can use the same test to determine whether the dependent variables in each of the regression are significantly related to their own lagged values—we are interested in whether the current values of a variable are dependent on its own past as well as on the past values of the other variables. It may be that a particular variable is statistically related to its own past values but not to the past values of the other variables.

Granger causality is a weaker condition than exogeneity. A variable is exogenous with respect to another if its *current* and past values do not affect the other variable. It Granger causes the other variable if its past values affect the current level of that variable. The fact that a variable does not Granger cause another does not necessarily imply that it is exogenous with respect to the other variable—their current-period values may be related.

In deciding whether to include a particular variable in the VAR it may be useful to determine whether that variable Granger causes any of or any combination of other variables in the system. To test whether the variable x is causally related to either or both of variables y and z we run the system with the lags of x included in the y and z equations and then again with the lags of x omitted from these equations. We then do a likelihood-ratio test to see if removing the lags of x has a statistically significant effect on the two-equation system. This test is customarily called a block-exogeneity test, although the term block-causality would be a more appropriate description.

Likelihood ratio tests use the fact that maximizing the likelihood of observing a sample is equivalent to minimizing the residual sum of squares of the regression(s).[20] The likelihood ratio is

$$\lambda = \left[\frac{RRSS}{URSS}\right]^{-n/2}$$

where *RRSS* is the residuals sum of squares of the restricted model (with the relevant lags omitted) and *URSS* is the residual sum of squares of the unrestricted model and n is the number of observations. If the restriction has no effect we would expect λ to be unity. Otherwise, we would expect it to be less than unity. The question is whether the observed λ is less than unity by an amount greater than could reasonably be accounted for on the basis of random chance—i.e., whether the restriction is statistically significant. To apply the test, we take the logarithms to obtain

$$-2log_e\lambda = n(log_e\Sigma_R - log_e\Sigma_U).$$

General applications of the test use the statistic

$$n(log|\Sigma_R| - log|\Sigma_U|),$$

where $log|\Sigma_R|$ and $log|\Sigma_U|$ are the natural logarithms of the determinants of the variance-covariance matrices of the residuals of the restricted and

[20]See G. S. Maddala, *Introduction to Econometrics*, Macmillan, 1988, pp. 83–86, for a simple discussion of the statistical foundations of these tests. See also Enders, pp. 312–316.

unrestricted systems. Given the sample sizes found in economic analysis, Sims recommends using a modified form of this statistic

$$(n-c)(log|\Sigma_R| - log|\Sigma_U|) \tag{39}$$

where c is the number of parameters estimated in each equation of the unrestricted system.[21] When there are p lags and m variables with a constant term,

$$c = p\,m + 1.$$

This statistic is distributed according to the χ^2 distribution with degrees of freedom equal to the number of restrictions in the system. In the case where the restriction involves eliminating v lags from each of two equations the degrees of freedom would be

$$df = 2v.$$

The question of whether to include seasonal dummies when using quarterly or monthly data can be resolved using this same approach. Run the all the regressions in the system with seasonal dummies included and then impose the restriction of no seasonality by running the same regressions without them. Then test the restriction using (39).

Likelihood-ratio tests are one of three general methods of determining the appropriate lag length. Run the system of equations first with long lags and then repeat with progressively shorter lags. Then perform likelihood ratio tests of the restriction of leaving out lags in comparing each set of lag lengths, dropping lags where they are statistically insignificant.

Two additional criteria that are used to determine the appropriate lag length, and also seasonality, are the Akaike and Schwartz-Bayesian criteria. To apply these criteria calculate the following statistics for the system for each lag length (and with and without seasonal dummies)

$$\begin{aligned} AIC &= log|\Sigma| + 2\,g/n && (40)\\ SBC &= log|\Sigma| + g\,log(n)/n && (41) \end{aligned}$$

where $|\Sigma|$ is the determinant of the variance covariance matrix of the residuals of the system, g is the total number of parameters estimated in all equations and n is the number of observations. Thus, if we have m equations and p lags and an intercept in each equation,

$$g = m^2 p + m.$$

[21] Christopher Sims, "Macroeconomics and Reality," *Econometrica*, 48, 1980, 1–49.

Select the model with the lowest *AIC* or *SBC* value. The same sample period must be used for all models being compared. If in doubt using any of these criteria for choosing optimal lag length one should err on the side of including too many lags although close attention must be paid to the number of degrees of freedom remaining for estimation.

All of the above calculations assume that the same number of lags are included for all variables in all equations. This is normally done to ensure symmetry of the system and permit ordinary-least-squares estimation. If one wishes to have different numbers of lags in different equations the seemingly unrelated regression technique must be used for estimation.

Table 2: Tests for Lag-Length and Seasonality

	Likelihood Ratio Test: P-Value	
8 vs. 12 Lags	0.15340089	
6 vs. 8 Lags	0.65553269	
4 vs. 8 Lags	0.00002649	
Seasonal with 4 Lags	0.83247273	
	AIC	SBC
12 Lags	-21.9255	-17.9382
8 Lags	-22.0255	-19.3402
6 Lags	-22.2128	-20.1785
4 Lags	-22.0337	-20.6504
Seasonal with 4 Lags	-21.0169	-20.3010

Table 2 presents the results of the above tests for our U.S. VAR. These results are produced by the RATS code file `vardemll.prg` and are shown in the resulting output file `vardemll.out`. The likelihood-ratio test strongly suggests four lags while the Akaike Information criterion (AIC) suggests six and the Schwartz-Bayesian criterion (SBC) suggests four. Because the complete lack of ambiguity of the likeliood-ratio test, 4 lags were chosen in the preceding analysis. The likelihood ratio test for seasonality indicates that the seasonal dummies would be statistically significant at only at the

83% level, confirming our decision not to include seasonal dummies. With 4 lags, the AIC and SBC are both smaller when the seasonal dummies are not included.

Table 3: Model Selection Tests

Variable	Equation – P-Values				
	Other Three	Real GDP	Inflation Rate	Interest Rate	Base Money
Real GDP	0.0304732	0.0000000	0.5793367	0.0001040	0.4767769
Inflation Rate	0.9427256	0.0375855	0.0000000	0.0223877	0.1522231
Interest Rate	0.0676488	0.0000384	0.0355060	0.0000000	0.0183484
Base Money	0.4770359	0.8504577	0.7292598	0.8798984	0.0000000
	AIC			SBC	
	Included	Excluded		Included	Excluded
Real GDP	-22.1960	-21.8995		-20.4646	-20.5755
Inflation Rate	-11.2911	-11.1821		-9.5598	-9.8581
Interest Rate	-13.3325	-13.0629		-11.6012	-11.7389
Base Money	-21.9113	-21.7288		-20.1769	-20.4848

The results of model-selection tests for our U.S. VAR are presented in Table 3.[22]. The P-values in the column labelled 'other three' indicate that only the lagged values of the real GDP variable are statistically significant as a block in the overall model, and then at only the 10% level. The AIC suggests that the block lags of all the variables belong in the model while the SBC suggests the exact opposite—that none of them do. The inflation rate, the interest rate and the lagged values of real GDP itself are statistically significant in the real GDP equation. The interest rate is the only variable other than the inflation rate itself whose lagged values are significant in the inflation rate equation. The lagged values of everything but base money are statistically significant in the interest rate equation. And the interest rate

[22]These are produced by the RATS code file `vardemll.prg` and are contained in the resulting output file `vardemll.out`

is the only variable, other than base money itself, that is significant in the base money equation.

10 Some Concluding Issues

Economic theory is the basis for choosing the variables in the U.S. VAR. The fact that the lagged values of the variables are not statistically significant in some equations suggests that a 'near VAR', where different numbers of lags are included in the different equations, might be worth constructing. In this case the standard-form model would have to be estimated using seemingly-unrelated-regression analysis.

Also, preliminary results such as those in Table 3 provide us with some insight into the possible nature of the relationships between the variables present in the real world. For example, it is interesting that lagged values of base money appear to have no effect on the interest rate variable, but lagged values of the interest rate variable affect the current level of the monetary base. This, together with the fact that lagged levels of the interest rate are significantly related to the current levels of all the other variables, provides a basis for contemporary formulations that leave the base money variable out of the VAR, treating the interest rate as the instrument of monetary policy. The problem with that approach, however, is that the only way that the authorities can affect interest rates is by varying the stock of base money—it is the only instrument over which the monetary authority has complete control. One can easily see how a structural VAR can improve upon a simple recursive system that requires only a Choleski decomposition.

A limitation of the Blanchard-Quah approach is that there must be the same number of shocks as there are variables—in our example, there can only be two shocks. This precludes the possibility that there could be two types of nominal shock—fully anticipated nominal shocks that immediately affect the nominal exchange rate but have no effect on the real exchange rate, and unanticipated nominal shocks that temporarily affect the real exchange rate and ultimately permanently affect the nominal exchange rate. This is likely to be less of a problem in the current example where the monetary policies of the two countries have been similar and shown no diversions that could have been reasonably predicted by agents.

These issues provide food for thought. But it should be kept in mind that the purpose of the present exercise is simply to help the reader understand how VARs are constructed. For one intent upon working with VARs, this is only a beginning.