ECO 426 (Market Design) - Lecture 8

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Revenue equivalence

- Model:
  - N bidders
  - Bidder i has valuation v_i
  - Each v_i is drawn independently from the same distribution F (e.g. $U[0,1]$)
- Theorem In any auction such that in equilibrium:
  - the winner with the highest valuation wins, and
  - the bidder with the lowest possible valuation pays nothing,

  the average revenue are the same, and the average bidder profits are the same.

- DP, SP, FP and AP share the properties that
  - equilibrium outcome is efficient (i.e. the highest value bidder wins the auction)
  - a bidder with a 0 valuation pays nothing.

Envelope theorem

- Consider a maximization problem

$$\max_b \; u(b, v)$$

where $u()$ is differentiable in b and v

- The solution $b^*(v)$ is a function of v and satisfies the FOC

$$u_b(b^*(v), v) = 0$$

- The value of the maximization problem is $U(v) \equiv u(b^*(v), v)$ and by the chain rule of differentiation

$$U'(v) = u_b(b^*(v), v)b^{*\prime}(v) + u_v(b^*(v), v)$$

- The envelope theorem says that

$$U'(v) = u_v(b^*(v), v)$$

Envelope theorem and auctions

- A bidder with valuation v choosing to submit a bid b solves

$$\max_{b} \; vPr(win|b) - \mathbb{E}[Payment|b]$$

- In an auction the objective function is

$$u(b,v) = vPr(win|b) - \mathbb{E}[Payment|b]$$

$$u_v(b,v) = Pr(win|b)$$

- If $b^*(v)$ is the equilibrium bidding strategy, the envelope theorem says that the bidder expected profit $U(v)$ satisfies

$$U'(v) = Pr(win|b^*(v)) = \text{Eq. Prob. value-}v\text{ bidder wins}$$

- Integrating

$$U(v) = U(0) + \int_0^v Pr(win|\tilde{v})\mathrm{d}\tilde{v}$$

Revenue equivalence theorem

$$U(v)_{=}U(0) + \int_0^v Pr(win|\tilde{v})\mathrm{d}\tilde{v}$$

- A bidder expected profit only depends on
  - his probability of winning as function of his valuation (i.e. $Pr(win|\tilde{v})$)
  - his expected profit when he has the lowest possible valuation (i.e. $U(0)$)
  - Both are identical across the four auction formats we considered
- Critical assumptions
  - bidder know their own values (their values do not depend on others private information)
  - values are statistically independent
  - bidders only care about their profit (i.e. payoff equals valuation minus price paid)

Using the Revenue equivalence theorem

- The revenue equivalence theorem implies that: in any auction where, in equilibrium, the highest valuation bidder wins the object
  - the expected revenue to the seller is constant
  - the expected surplus to each bidder is constant
- In a second price auction:
  - the highest value bidder wins the object
  - equilibrium strategies are easily characterized (dominant strategy)
  - bidders expected surplus and sellers revenue are easily characterized
- Can use the bidder expected revenue characterization in a second price auction to derive the (less obvious) equilibrium strategies of other auctions

First price auction

- Two bidders - valuations are independent draws from $U[0,1]$
- Second price auction
  - Each bidder bids his valuation
  - A bidder with valuation v
    - wins with probability v (i.e. the probability his opponent value is less than v)
    - the expected payment upon winning is $v/2$ (i.e. the expected valuation of his opponent, provided his opponent has a valuation smaller than v)
- First price auction
  - Suppose, in equilibrium, the highest valuation bidder wins
  - A bidder with valuation v
    - wins with probability v (i.e. the probability his opponent value is less than v)
    - by the revenue equivalence theorem, his expected payment upon winning must be the same as in a SP auction (i.e. $v/2$)
    - since the payment of the winner in a FP auction equals his own bid, the equilibrium bid of a bidder with valuation v must be $v/2$ (i.e. the equilibrium bidding strategy is $b(v) = v/2$.)

All pay auction

- Each bidder submits a sealed bid
- Bids are open
  - Bidder who submitted the highest bid wins the object
  - **Each bidder** pays a price to the seller equal to his own bid
- What should bidders do?
- Suppose there is an equilibrium where the highest valuation bidder wins
  - use the revenue equivalence theorem to solve for the candidate equilibrium bidding strategies
  - ex-post verify that the strategies constitute an equilibrium (i.e. no bidder has any incentive to deviate)

All pay auction

- Two bidders - valuations are independent draws from $U[0,1]$
- Suppose, in equilibrium, the highest valuation bidder wins
- A bidder with valuation v
  - wins with probability v (i.e. the probability his opponent value is less than v)
  - pays his own bid, $b(v)$, regardless of whether he wins or not
  - his expected profit is then

$$Prob(win|v) * v - b(v) = v^2 - b(v)$$

  - in a second price auction has an expected profit of

$$v(v - v/2) = v^2/2$$

  - from the revenue equivalence theorem

$$v^2 - b(v) = v^2/2$$

  - the equilibrium bidding strategy in an all pay auction must be

$$b(v) = v^2/2$$

Auction with a reserve price

- A reserve price is a price below which the seller is not willing to give up the object
- Second price auction with a reserve price r
  - the highest bidder wins the object if bid $> r$
  - the winner pays a price equal to the largest between the second highest bid and the reserve price r
  - Example 1: Two bids, 0.3 and 0.6, and reserve price $r = 0.4$. The high bidder wins and pays 0.4.
  - Example 2: Two bids, 0.5 and 0.6, and reserve price $r = 0.4$. The high bidder wins and pays 0.5.
  - Example 1: Two bids, 0.3 and 0.36, and reserve price $r = 0.4$. Nobody wins, object remains with seller.

Second price auction with reserve price

- Two bidders - valuations are independent draws from $U[0,1]$
- It is a dominant strategy to:
  - bid own valuation when $v > r$
  - not bid when $v \leq r$ (or bid own valuation)
- A bidder with valuation $v > r$
  - wins with probability v
  - when winning pays a price equal to:
    - opponent value, $\hat{v}$, if $\hat{v} > r$ (happens with probability $(v-r)/v$)
    - reserve price , r, if $\hat{v} \leq r$ (happens with probability r/v)
  - expected payment when winning

$$(r/v) * r + ((v-r)/v) * (v+r)/2 = r + (v-r)^2/(2v)$$

First price auction with reserve price

- Two bidders - valuations are independent draws from $U[0,1]$
- Suppose, in equilibrium, the highest valuation bidder wins
- A bidder with valuation v
  - does not bid if $v \leq r$ (dominant strategy)
  - bids $b(v) = r + (v-r)^2/(2v)$ if $v > r$ (by the revenue equivalence theorem)
- note that $r + (v-r)^2/(2v)$ is strictly increasing in v, so in equilibrium the highest value bidder wins

Optimal reserve price

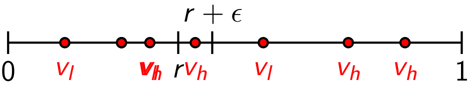
- What reserve price maximizes the seller's revenue?
- Suppose there is just one bidder, with $U[0,1]$ valuation
  - reserve price is just a posted price
  - sell at price equal r if $v > r$
  - do not sell otherwise
- Expected revenue is

$$Prob(v > r) * r = (1 - r) * r$$

  - monopolist's revenue with demand function $Q(p) = 1 - p$
- revenue maximizing reserve price $r = 1/2$
  - same as monopolist price

Optimal reserve price

- Two bidders - independent $U[0,1]$ valuations
- Compare the revenue from marginally increasing the reserve price r to $r+\epsilon$, across all possible pairs of valuations $v_l < v_h$



- $v_l < v_h < r$
- $r+\epsilon < v_l < v_h$
- $v_l < r < v_h$
- $v_l < r < v_h < r+\epsilon$

- no impact on revenue
- no impact on revenue
- R increases by ϵ (probability $2r(1-r)$)
- R decreases by r (probability $2\epsilon r$)

- Expected revenue change: $\Delta Revenue = \epsilon 2r(1-r) - r2\epsilon r$
- must be zero at the optimal reserve price $r^* = 1/2$

Optimal reserve price

- With $N > 2$ bidders same argument applies
  - The revenue only depends on the highest two bids
  - Similar calculation of impact on revenue
  - Optimal reserve price remains $r^* = 1/2$
- First price, second price, ascending price and descending price auctions all have the same optimal reserve price
- Optimal reserve price in an all pay auction?
  - Use the revenue equivalence theorem
  - Bidders with valuation below r^* bid nothing
  - For $v \geq r^*$, solve for bidding strategy, $b(v)$, using the revenue equivalence theorem
  - Reserve price must be equal to $b(r^*)$
    - if it higher, the allocation rule is not the same as in the second price auction
    - if it is lower a bidder with value r^* would have an incentive to lower its bid
  - Homework: calculate the optimal reserve price with two bidders