# ECO 426 (Market Design) - Lecture 3

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## one-sided strategy proofness

- A stable mechanism can be strategy proof for one-side of the market

  Theorem: The men (women) proposing deferred acceptance algorithm is strategy-proof for the men (women).

- If the true preferences are such that there is only one stable matching, no agent can benefit from misreporting their preferences
  - the unique stable matching is the outcome of both the DA men and DA women proposing algorithm
- When there are multiple stable matchings, how much can a woman gain by manipulating her preferences in the DA men proposing mechanism?

# benefits from preference manipulation

- An agent can truncate his/her preference list by reporting as unacceptable one or more acceptable partner, starting from the least desirable

| | | | | | |
|---|---|---|---|---|---|
| $m_1$ | $w_1$ | $w_2$ | $w_4$ | $w_3$ | $w_6$ |
| $m_1$ | $w_1$ | $w_2$ | $w_4$ | | |

- Theorem Provided all other participants are truthful, in the DA men proposing mechanism, a woman can achieve her best possible match by truncating her preference list and stopping with the man who is the best achievable in any stable matching.
  - Limits to preference manipulation: can yield at most the best partner across stable matchings
  - Bound is tight: there exists a “simple” manipulation strategy that achieves the best possible match
  - The manipulation strategy is informationally demanding

# benefits from preference manipulation - sketch of proof

1. The woman optimal matching is still a stable matching after the manipulation
2. The set of matched agents is the same under any stable matching (rural hospital theorem)
   - Thus the manipulating woman is matched in every stable matching under the new preferences
3. Any matching that gives the manipulating woman an even better partner is blocked under the true preferences. The pair that blocked under the true preferences still blocks after the manipulation
4. Therefore, the manipulating woman is getting her best possible match after the manipulation.

- Agents who are unmatched in a stable matching, are unmatched in all stable matchings (rural hospital theorem), hence they cannot gain from preference manipulation.

- We implicitly assumed no externalities
  - Preferences of each agents are defined over own partners only
- If agents care about others' matches a stable matching might not exist
- Example: **Couples** might care about joint location when looking for jobs.
  - $m$ and $w$ are two medical students in a couple, $s$ is a single medical students. $h_1$ and $h_2$ are two hospitals in the same area, each with one opening.
  - Both $m$ and $w$ find unacceptable a job at one hospital if the partner is not hired by the other hospital
  - $h_1$ prefers $m$ or $w$ to $s$ (i.e. $m \succ w \succ s$ or $w \succ m \succ s$)
  - $h_2$ prefers $s$ to $m$ or $w$
  - $s$ prefers $h_1$ to $h_2$
  - No stable matching:
    - if $m$ and $w$ are employed, $s$ and $h_2$ block the matching
    - if $s$ is employed by $h_2$, $s$ and $h_1$ block the matching
    - if $s$ is employed by $h_1$ the couple and the two hospitals "block"

- Incentive to manipulate preferences are “small” in large markets
  - The proportion of woman who can benefit from manipulation shrinks to zero in the DA men proposing mechanism as the number of agents grows (and agents preferences are independent uniform draws over all possible rankings)
  - The loss from switching from DA men proposal to DA women proposal does not make a big difference (1998 change in the NRMP)
- Probability that a stable matching exists with a fixed number of couples converges to one as the number of agents grows.
  - Consistent with practice - NRMP has always been able to find a stable matching

# many-to-one matching

- Firms often have multiple openings to fill (while workers are still looking for one job)
- Matching is a pairing of a firm to (possibly) many workers
- Need to define preferences of firms over multiple workers
- Simplest possible extension (responsive preferences):
  - Each firm $f$ has a quota $q$ of jobs to fill
  - Each firm $f$ (strictly) ranks workers
  - Replacing a worker with a higher ranked worker (or a vacancy with an acceptable worker) makes $f$ better off.
- Stable matching definition changes:
  - each firm does not exceed its quota;
  - there is not a worker and a firm pair such that: i) the worker prefers the firm to his current match; and ii) the firm prefers the worker to one of its current workers (or vacancy).

# many-to-one matching

- In the DA algorithm can treat each firm as "multiple" firms, one for each vacancy, with identical preferences over workers.
- Some results still hold
  - The DA algorithm yields a stable matching (a stable matching exists)
  - Firms proposing DA results in best stable matching for firms
  - All firms fill the same number of position and the same workers find jobs, across all stable matchings Rural hospital theorem
    - Vacancy rate in each hospital is constant across all stable mechanism
    - Cannot change the vacancy rate in rural hospital if sticking to stable mechanism
- Some results do not hold
  - No stable mechanism is strategy proof for the hospital (no stable mechanism is collusion proof)

# more general preferences

- More generally, firms might care about the composition of their workforce (e.g. an hospital might not want to hire two neurosurgeons)
- Preferences of a firm are described by an ordered list of subsets of workers

Example $F = \{f_1, f_2\}$ and $W = \{w_1, w_2, w_3, w_4, w_5\}$

- Firm 1 has a quota of 2 and "responsive" preferences $(w_1, w_2, w_3)$

$$f_1 \mid \{w_1, w_2\} \quad \{w_1, w_3\} \quad \{w_2, w_3\} \quad \{w_1\} \quad \{w_2\} \quad \{w_3\} \quad \emptyset$$

- Firm 2 has arbitrary preferences

$$f_2 \mid \{w_1, w_3, w_5\} \quad \{w_2, w_4\} \quad \{w_1, w_2, w_3\} \quad \{w_1\} \quad \{w_1 w_2\} \quad \emptyset$$

# substitutable preferences

- With arbitrary preferences a stable matching might not exists
  - cf. non existence of stable matching with externalities
- Restrict to "substitutable preferences"
- Given a set of workers $A$, the set of workers rejected by firm $f$ if it were able to choose freely is denoted $R_f(A)$

  Definition: A firm $f$ has substitutes preferences if,

$$A' \subset A \quad \Rightarrow \quad R_f(A') \subseteq R_f(A)$$

  - the set of workers rejected does not shrink when the set of workers available for choosing expands
  - rules out complementarities among workers
  - responsive preferences are always substitutes, the reverse is not true

# substitutable preferences

Example $F = \{f_1, f_2, f_3\}$ and $W = \{w_1, w_2, w_3, w_4, w_5\}$

- Firm 1 has "responsive" as well as "substitutes" preferences

$$f_1 \;\Big|\; \{w_1, w_2\} \quad \{w_1, w_3\} \quad \{w_2, w_3\} \quad \{w_1\} \quad \{w_2\} \quad \{w_3\} \quad \emptyset$$

- Firm 3's preferences are "substitutes" but not "responsive"

$$f_3 \;\Big|\; \{w_2, w_3\} \quad \{w_1, w_3\} \quad \{w_1, w_2\} \quad \{w_1\} \quad \{w_2\} \quad \{w_3\} \quad \emptyset$$

  - never reject $w_2$ and $w_3$; reject $w_1$ only if both $w_2$ and $w_3$ are available

- Firm 2 has arbitrary preferences

$$f_2 \;\Big|\; \{w_1, w_3, w_5\} \quad \{w_2, w_4\} \quad \{w_1, w_2, w_3\} \quad \{w_1\} \quad \{w_1 w_2\} \quad \emptyset$$

  - $w_1$ is rejected if all workers but $w_3$ are available, and is not rejected when all workers are available.

# substitutable preferences - existence of stable matching

Theorem Suppose firms have substitutes preferences. Then the DA algorithm yields a stable matching.

- A stable matching exists.
- DA algorithm with workers proposing
  - In each round a firm **"holds"** the **favorite set** of workers among those proposing and those held from previous round and rejects the remaining
  - If a worker is rejected by a firm in a given round, a new offer by the same worker to the same hospital would be rejected in any later round
  - When the algorithm ends the outcome is stable (no workers offer to an hospital that he prefers would be accepted)
- A firm never "regrets" making a rejection.

# Matching when only one side has preferences

Some allocation problems can be modelled as two-sided matching markets but with one side not having any preferences over the possible allocations

- Housing market (allocating houses to individuals)
  - A collection of individuals, $A$ (agents)
  - each agent $a \in A$:
    - owns a "house," $h_a$, ($H$ is the set of all houses);
    - has (strict) preferences over the set of houses in the economy
  - the initial allocation might not be efficient (i.e. Pareto efficient)
    - mutually beneficial trades might be possible

- Housing market vs. marriage market
  - one side of the market (houses) has no preferences over matches;
  - agents have an initial endowment (i.e. each agent owns a house)
    - the market starts from a default allocation where each agent is matched to his own house
  - Goal: find a matching that cannot be improved
    - it is not possible to reassign houses making some agent better off and making no agent worse off

# Housing Market - CORE

- An allocation is an assignment (matching) of agents to houses such that
  - each agent is assigned exactly one house; and
  - each house is assigned to exactly one agent.
- An allocation in an housing market is described by a **"bijection"** $\mu : A \longrightarrow H$.
- In a housing market, each agent is endowed (owns) one house (e.g. $a$ owns $h_a$)
- What allocations would we expect to arise if agents can freely dispose of their endowment?

# Housing Market - CORE

- Agents in a group $S \subseteq A$ own together (in a "coalition") a subset of the houses in the market $H_S$
- The agents in a coalition $S$ can "independently" distribute the houses they own, $H_S$, among themselves.
  - An assignment of the houses in $H_S$ to agents in $S$, is an allocation in the housing market where the set of agents is $S$ and the set of houses is $H_S$

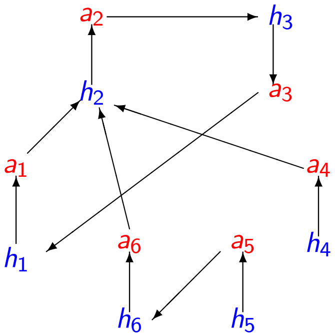
$$\mu_S : S \to H_S.$$

# Housing Market - CORE

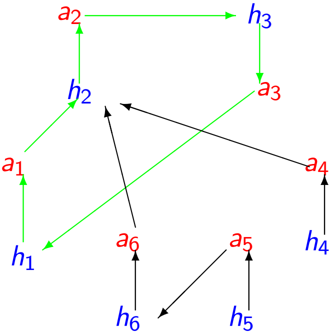
- Definition (Blocking) A coalition of agents $S$ **blocks** an allocation $\mu$, if there is an assignment $\mu_S$ of the houses owned by the coalition to the members of the coalition $S$, such that: i) some member of $S$ prefers $\mu_S$ to $\mu$; ii) no member of $S$ prefers $\mu$ to $\mu_S$.
  - A blocking coalition can find a mutually beneficial trade (i.e. an exchange of houses among members of the coalition that improves all members' welfare with respect to the allocation $\mu$)
- Definition (Core) An allocation is in the core of the housing market if it is not blocked by any coalition.
  - At a core allocation benefits from trade are exhausted
  - In a marriage market, core matchings and stable matchings coincide

Gale's Top trading cycle algorithm

- each agent points to his/her preferred house
- each house points to its owner



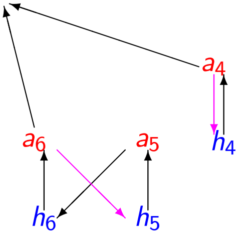
# Housing Market - TTC




- there is at least one cycle

- remove all cycles assigning houses to agents
  - agents within a cycle exchange houses among each others

- each remaining agent points to his/her preferred remaining house




- remove all cycles assigning houses to agents
- continue until no agent/house is left

# TTC and core

Theorem The outcome of the TTC mechanism is the unique core allocation of the housing market.

- The outcome of the TTC mechanism cannot be blocked
  - cannot make any agent matched in the first round better off (they are getting their favourite house)
  - cannot make any agent matched in the second round better off without making some of the agents matched in the first round worse off
  - cannot make any agent matched in round $n$ better off without making some agents matched in earlier rounds worse off.

# TTC and strategic incentives

Theorem The TTC algorithm is a strategy proof mechanism.

- an agent matched in round $n$ cannot, by manipulating his/her preferences, break any of the cycles that form before round $n$
  - preference manipulation cannot give the agent a house that was assigned earlier than round $n$.
- getting an house that was assigned in a round later than $n$ does not make the agent better off.

- Housing allocation - agents have no claim on the set of houses
  - allocating students to dorms
  - allocating students to schools
- House allocation with existing tenants - some agents have a claim on some houses others do not
  - how do we ensure wide participation?
- Applications in market design:
  - Kidney exchange
  - School assignment