ECO 426 (Market Design) - Lecture 1

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Market: *A medium that allows buyers and sellers of a specific good or service to interact in order to facilitate an exchange.*

- markets are institutions that determine how resources are allocated
- competitive markets
  - allocation problem is solved by a price system
  - products are exchanged for a price
  - prices adjust so that supply=demand

Competitive Markets

Competitive Markets are “good” institutions for the exchange of “commodities”

1. the price is the only relevant variable in the economic decision to exchange
   - the identity of the counterparty is irrelevant
2. commodity markets are “liquid”

Failures of prices as allocation mechanisms

Sometimes prices are not "all that matter"

- job finding (allocating jobs to workers)
  - being willing to work for an employer offering a given wage does not guarantee employment
  - being willing to hire a worker demanding a given wage does not guarantee hiring
- college admissions (allocating students to colleges)
  - being willing to pay ongoing tuition does not guarantee admission
  - admitting a student does not guarantee enrollment

Failures of prices as allocation mechanisms

Sometimes prices do not “matter at all”

- marriage “market”
  - social norms (sometimes) prevent contracting on a price for the exchange
- kidney transplants (allocating kidneys from donors to patients)
  - legal (and moral) constraints prevents exchanges for valuable consideration

market liquidity concerns

Sometimes prices are the relevant variable in the transaction but markets are "illiquid"

- allocation of radio spectrum (both supply and demand side are "thin") - also seller's objective might be different from max profit
- trading fine art (thin supply side of van Gogh's Starry Night)

Need institutions different from competitive markets to address their shortcomings

- matching markets: do not use (only) prices as allocation mechanism
- auction markets: allocation (and price formation) mechanism for "thin markets"

Marriage model

- Market participants are divided into two separate groups (two-sided market)
  - A set of "men" M, with a typical man $m \in M$
  - A set of "women" W, with a typical woman $w \in W$
- Allocation: each man can be matched to one woman (or stay single), and vice-versa (one-to-one matching)

Matching

- A matching is a collection of pairs such that:
  - each individual has one partner - (m, w) - or
  - has no partner - (m, m) - (m is "matched with self")
- A matching can be described by a function

$$\mu : M \cup W \longrightarrow M \cup W$$

such that:

  - if $\mu(m) \neq m$ then $\mu(m) \in W$ (each man is either single or matched to a woman)
  - if $\mu(w) \neq w$ then $\mu(w) \in M$ (each woman is either single or matched to a man)
  - $\mu(\mu(x)) = x$ (if $\mu(x)$ is x's partner, then x is $\mu(x)$'s partner)

Preferences

- Each agent has a (strict) preferences over "acceptable" partners
  - matching with an acceptable partner is preferred to staying unmatched
- Example: $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3\}$
  - m_1 preferences $w_2 \succ w_3 \succ w_1 \succ m_1$ (or simply w_2, w_3, w_1)
  - w_1 preferences $m_1 \succ m_3 \succ w_1$ (or simply m_1, m_3)

"Equilibrium"

- A matching μ is stable if
  - no agent is matched to an unacceptable mate - individual rationality
  - there is no pair of agents who would prefer to match with each other rather then their assigned partner - no blocking
- Example: $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$
  - Both m_1 and m_2 prefer w_1 to w_2
  - Both w_1 and w_2 prefer m_1 to m_2
  - The matching (m_1, w_2), (m_2, w_1) is not stable m_1 and w_1 prefer each other to their assigned partner
  - The matching $(m_1, w_1), (m_2, w_2)$ is the unique stable matching

Stable matchings examples

- Example: $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$
  - m_1 prefers w_1 to w_2 and m_2 prefers w_2 to w_1
  - w_1 prefers m_1 to m_2 and w_2 prefers m_2 to m_1
  - The matching $(m_1, w_1), (m_2, w_2)$ is the unique stable matching

- Example: $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$
  - m_1 prefers w_1 to w_2 and m_2 prefers w_2 to w_1
  - w_1 prefers m_2 to m_1 and w_2 prefers m_1 to m_2
  - The matching $(m_1, w_1), (m_2, w_2)$ is stable
  - The matching $(m_1, w_2), (m_2, w_1)$ is also stable
  - Note both men prefer the first matching and the opposite is true for the women
- Question: can we always find a stable matching?

Gale and Shapley '62 - deferred acceptance algorithm

- Round 1
  - Each man proposes to his most preferred woman
  - Each woman reject all but the most preferred proposal received
  - Most preferred proposal is "held" not accepted
- Round 2
  - Each man rejected in the previous round proposes to the most preferred woman who has not yet rejected him
  - Each woman rejects all but the most preferred among new and held proposals
  - Most preferred proposal is "held" not accepted
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- Round T - no proposal is rejected
  - algorithm ends
  - each woman is matched to the currently held proposal

Gale and Shapley '62 - deferred acceptance algorithm

Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

m_1	w_1	w_2	w_3
m_2	w_3	w_1	
m_3	w_1	w_3	w_2

w_1	m_1	m_2	m_3
w_2	m_1	m_2	m_3
w_3	m_3	m_1	m_2

m_1	m_2	m_3
w_1	w_3	w_1
w_1	w_3	$\emptyset$
		w_3
w_1	$\emptyset$	w_3
	w_1	
w_1	$\emptyset$	w_3
	m_2	
w_1	m_2	w_3
w_1	m_2	w_3

Existence of a stable matching

The deferred acceptance (DA) algorithm end in finitely many rounds

- No man ever proposes twice to the same woman
- The outcome of the DA algorithm is a matching

Theorem. The outcome of the DA algorithm is a stable matching.

- The proposal "held" by each woman improves (weakly) in each round
  - Each woman is matched to the most desirable man who has ever proposed to her
- Every time he makes a new proposal, a man proposes to the next most desirable woman
  - Each man has proposed to, and has been rejected by, all women more desirable than his match
- For each man m, every woman more desirable than his match prefers her current match to m.

one-sided market - the roomate problem

- Existence of a stable matching relies on the two-sided nature of the market
- Example (the roomate problem) Three students A, B, C can form a pair to share a two-people room
    - A prefers sharing with B to sharing with C to not sharing
    - B prefers sharing with C to sharing with A to not sharing
    - C prefers sharing with A to sharing with B to not sharing
    - there is no stable matching

- When multiple stable matching exist, the DA algorithm with man proposing yields a different outcome from the DA algorithm with women proposing

multiple stable matching - conflicting preferences

Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

$$\begin{array}{c|ccc} m_1 & w_1 & w_2 & w_3 \\ m_2 & w_1 & w_2 & w_3 \\ m_3 & w_1 & w_3 & w_2 \end{array} \qquad \begin{array}{c|ccc} w_1 & m_1 & m_2 & m_3 \\ w_2 & m_1 & m_3 & m_2 \\ w_3 & m_1 & m_2 & m_3 \end{array}$$

$$\text{DA } m \quad \begin{array}{ccc} m_1 & m_2 & m_3 \\ \hline w_1 & w_1 & w_1 \\ w_1 & \emptyset & \emptyset \\ \hline & w_2 & w_3 \\ w_1 & w_2 & w_3 \\ \hline w_1 & w_2 & w_3 \end{array} \qquad \text{DA } w \quad \begin{array}{ccc} w_1 & w_2 & w_3 \\ \hline m_1 & m_1 & m_1 \\ m_1 & \emptyset & \emptyset \\ \hline & m_3 & m_2 \\ m_1 & m_3 & m_2 \\ \hline m_1 & m_3 & m_2 \end{array}$$

multiple stable matching - conflicting preferences

- Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

$$\begin{array}{c|ccc} m_1 & w_1 & w_2 & w_3 \\ m_2 & w_1 & w_2 & w_3 \\ m_3 & w_1 & w_3 & w_2 \end{array} \qquad \begin{array}{c|ccc} w_1 & m_1 & m_2 & m_3 \\ w_2 & m_1 & m_3 & m_2 \\ w_3 & m_1 & m_2 & m_3 \end{array}$$

  - DA men proposing: $(m_1, w_1), (m_2, w_2), (m_3, w_3)$
  - DA women proposing $(m_1, w_1), (m_2, w_3), (m_3, w_2)$
  - all men prefer (weakly) the DA men proposing outcome
  - all women prefer (weakly) the DA women proposing outcome
- Theorem For two stable matchings μ, μ', all men (weakly) prefer μ if and only if all women (weakly) prefer μ'.
- The DA algorithm with men proposing yields the best stable matching for men and the worst stable matching for women.