ECO 426 (Market Design) - Lecture 8

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- Integrating

$$U(v) = U(0) + \int_0^v Pr(win|\tilde{v})\mathrm{d}\tilde{v}$$

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  - values are statistically independent
  - bidders only care about their profit (i.e. payoff equals valuation minus price paid)

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    - bidders expected surplus and sellers revenue are easily characterized
- Can use the bidder expected revenue characterization in a second price auction to derive the (less obvious) equilibrium strategies of other auctions

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    - since the payment of the winner in a FP auction equals his own bid, the equilibrium bid of a bidder with valuation v must be $v/2$ (i.e. the equilibrium bidding strategy is $b(v) = v/2$.)

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  - use the revenue equivalence theorem to solve for the candidate equilibrium bidding strategies
  - ex-post verify that the strategies constitute an equilibrium (i.e. no bidder has any incentive to deviate)

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  - Example 2: Two bids, 0.5 and 0.6, and reserve price $r = 0.4$. The high bidder wins and pays 0.5.

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  - Example 2: Two bids, 0.5 and 0.6, and reserve price $r = 0.4$. The high bidder wins and pays 0.5.
  - Example 1: Two bids, 0.3 and 0.36, and reserve price $r = 0.4$. Nobody wins, object remains with seller.

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    - opponent value, $\hat{v}$, if $\hat{v} > r$

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  - expected payment when winning

$$(r/v) * r + ((v-r)/v) * (v+r)/2 = r + (v-r)^2/(2v)$$

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- note that $r + (v-r)^2/(2v)$ is strictly increasing in v, so in equilibrium the highest value bidder wins

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- revenue maximizing reserve price $r = 1/2$
  - same as monopolist price

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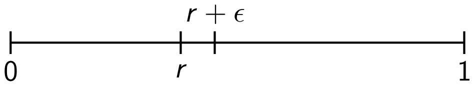

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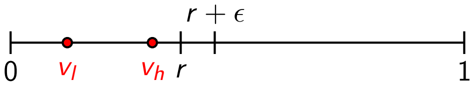

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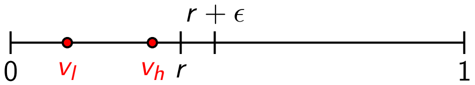

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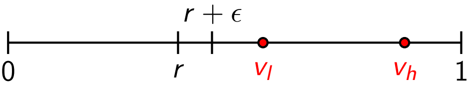

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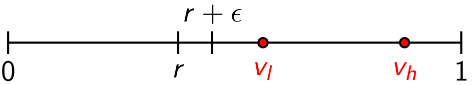

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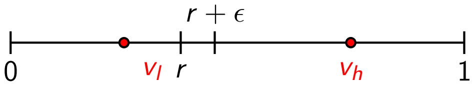

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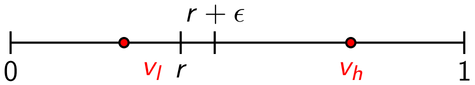

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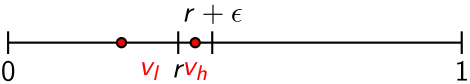

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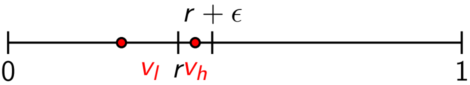

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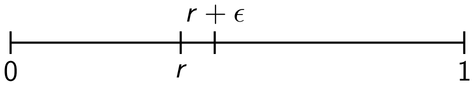

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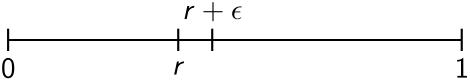

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  - Homework: calculate the optimal reserve price with two bidders