ECO 426 (Market Design) - Lecture 1

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  - products are exchanged for a price
  - prices adjust so that supply=demand

Competitive Markets

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1. the price is the only relevant variable in the economic decision to exchange
   - the identity of the counterparty is irrelevant
2. commodity markets are "liquid"

Failures of prices as allocation mechanisms

Sometimes prices are not “all that matter”

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- job finding (allocating jobs to workers)

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  - being willing to work for an employer offering a given wage does not guarantee employment
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- college admissions (allocating students to colleges)
  - being willing to pay ongoing tuition does not guarantee admission
  - admitting a student does not guarantee enrollment

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- kidney transplants (allocating kidneys from donors to patients)
  - legal (and moral) constraints prevents exchanges for valuable consideration

market liquidity concerns

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- trading fine art (thin supply side of van Gogh's Starry Night)

Need institutions different from competitive markets to address their shortcomings

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- matching markets: do not use (only) prices as allocation mechanism
- auction markets: allocation (and price formation) mechanism for "thin markets"

Marriage model

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  - A set of "men" M, with a typical man $m \in M$
  - A set of "women" W, with a typical woman $w \in W$
- Allocation: each man can be matched to one woman (or stay single), and vice-versa (one-to-one matching)

Matching

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such that:

  - if $\mu(m) \neq m$ then $\mu(m) \in W$ (each man is either single or matched to a woman)
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  - $\mu(\mu(x)) = x$ (if $\mu(x)$ is x's partner, then x is $\mu(x)$'s partner)

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  - The matching $(m_1, w_1), (m_2, w_2)$ is the unique stable matching

Stable matchings examples

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  - Note both men prefer the first matching and the opposite is true for the women
- Question: can we always find a stable matching?

Gale and Shapley '62 - deferred acceptance algorithm

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  - Each man rejected in the previous round proposes to the most preferred woman who has not yet rejected him

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- Round T - no proposal is rejected
    - algorithm ends
    - each woman is matched to the currently held proposal

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Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

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m_3	w_1	w_3	w_2

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m_2	w_3	w_1	
m_3	w_1	w_3	w_2

w_1	m_1	m_2	m_3
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w_3	m_3	m_1	m_2

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m_3	w_1	w_3	w_2

w_1	m_1	m_2	m_3
w_2	m_1	m_2	m_3
w_3	m_3	m_1	m_2

$m_1 \quad m_2 \quad m_3$

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m_2	w_3	w_1	
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w_1	m_1	m_2	m_3
w_2	m_1	m_2	m_3
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$$\begin{array}{c|ccc} m_1 & w_1 & w_2 & w_3 \\ m_2 & w_3 & w_1 & \\ m_3 & w_1 & w_3 & w_2 \end{array} \qquad \begin{array}{c|ccc} w_1 & m_1 & m_2 & m_3 \\ w_2 & m_1 & m_2 & m_3 \\ w_3 & m_3 & m_1 & m_2 \end{array}$$

m_1	m_2	m_3
w_1	w_3	w_1
w_1	w_3	$\emptyset$

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m_1	w_1	w_2	w_3
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m_3	w_1	w_3	w_2

w_1	m_1	m_2	m_3
w_2	m_1	m_2	m_3
w_3	m_3	m_1	m_2

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m_3	w_1	w_3	w_2

w_1	m_1	m_2	m_3
w_2	m_1	m_2	m_3
w_3	m_3	m_1	m_2

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w_2	m_1	m_2	m_3
w_3	m_3	m_1	m_2

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w_2	m_1	m_2	m_3
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w_1	w_3	w_1
w_1	w_3	$\emptyset$
		w_3
w_1	$\emptyset$	w_3
	w_1	
w_1	$\emptyset$	w_3
	m_2	

Gale and Shapley '62 - deferred acceptance algorithm

Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

m_1	w_1	w_2	w_3
m_2	w_3	w_1	
m_3	w_1	w_3	w_2

w_1	m_1	m_2	m_3
w_2	m_1	m_2	m_3
w_3	m_3	m_1	m_2

m_1	m_2	m_3
w_1	w_3	w_1
w_1	w_3	$\emptyset$
		w_3
w_1	$\emptyset$	w_3
	w_1	
w_1	$\emptyset$	w_3
	m_2	
w_1	m_2	w_3

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m_2	w_3	w_1	
m_3	w_1	w_3	w_2

w_1	m_1	m_2	m_3
w_2	m_1	m_2	m_3
w_3	m_3	m_1	m_2

m_1	m_2	m_3
w_1	w_3	w_1
w_1	w_3	$\emptyset$
		w_3
w_1	$\emptyset$	w_3
	w_1	
w_1	$\emptyset$	w_3
	m_2	
w_1	m_2	w_3
w_1	m_2	w_3

Existence of a stable matching

The deferred acceptance (DA) algorithm end in finitely many rounds

- No man ever proposes twice to the same woman
- The outcome of the DA algorithm is a matching

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- Every time he makes a new proposal, a man proposes to the next most desirable woman
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- For each man m, every woman more desirable than his match prefers her current match to m.

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one-sided market - the roomate problem

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- Example (the roomate problem) Three students A, B, C can form a pair to share a two-people room
  - A prefers sharing with B to sharing with C to not sharing
  - B prefers sharing with C to sharing with A to not sharing
  - C prefers sharing with A to sharing with B to not sharing
  - there is no stable matching

- When multiple stable matching exist, the DA algorithm with man proposing yields a different outcome from the DA algorithm with women proposing

Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

multiple stable matching - conflicting preferences

Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

$$\begin{array}{c|ccc} m_1 & w_1 & w_2 & w_3 \\ m_2 & w_1 & w_2 & w_3 \\ m_3 & w_1 & w_3 & w_2 \end{array}$$

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Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

$$\begin{array}{c|ccc}
m_1 & w_1 & w_2 & w_3 \\
m_2 & w_1 & w_2 & w_3 \\
m_3 & w_1 & w_3 & w_2
\end{array}
\qquad
\begin{array}{c|ccc}
w_1 & m_1 & m_2 & m_3 \\
w_2 & m_1 & m_3 & m_2 \\
w_3 & m_1 & m_2 & m_3
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m_1	w_1	w_2	w_3
m_2	w_1	w_2	w_3
m_3	w_1	w_3	w_2

w_1	m_1	m_2	m_3
w_2	m_1	m_3	m_2
w_3	m_1	m_2	m_3

$m_1 \quad m_2 \quad m_3$

DA m

Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

m_1	w_1	w_2	w_3
m_2	w_1	w_2	w_3
m_3	w_1	w_3	w_2

w_1	m_1	m_2	m_3
w_2	m_1	m_3	m_2
w_3	m_1	m_2	m_3

m_1	m_2	m_3
w_1	w_1	w_1

DA m

Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

$$\begin{array}{c|ccc}
m_1 & w_1 & w_2 & w_3 \\
m_2 & w_1 & w_2 & w_3 \\
m_3 & w_1 & w_3 & w_2
\end{array}
\qquad
\begin{array}{c|ccc}
w_1 & m_1 & m_2 & m_3 \\
w_2 & m_1 & m_3 & m_2 \\
w_3 & m_1 & m_2 & m_3
\end{array}$$

	m_1	m_2	m_3
	w_1	w_1	w_1
DA m	w_1	$\emptyset$	$\emptyset$

Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

$$\begin{array}{c|ccc} m_1 & w_1 & w_2 & w_3 \\ m_2 & w_1 & w_2 & w_3 \\ m_3 & w_1 & w_3 & w_2 \end{array} \qquad \begin{array}{c|ccc} w_1 & m_1 & m_2 & m_3 \\ w_2 & m_1 & m_3 & m_2 \\ w_3 & m_1 & m_2 & m_3 \end{array}$$

$$\text{DA } m \quad \begin{array}{ccc} m_1 & m_2 & m_3 \\ \hline w_1 & w_1 & w_1 \\ w_1 & \emptyset & \emptyset \\ \hline & w_2 & w_3 \end{array}$$

Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

$$\begin{array}{c|ccc} m_1 & w_1 & w_2 & w_3 \\ m_2 & w_1 & w_2 & w_3 \\ m_3 & w_1 & w_3 & w_2 \end{array} \qquad \begin{array}{c|ccc} w_1 & m_1 & m_2 & m_3 \\ w_2 & m_1 & m_3 & m_2 \\ w_3 & m_1 & m_2 & m_3 \end{array}$$

$$\text{DA } m \quad \begin{array}{ccc} m_1 & m_2 & m_3 \\ \hline w_1 & w_1 & w_1 \\ w_1 & \emptyset & \emptyset \\ \hline & w_2 & w_3 \\ w_1 & w_2 & w_3 \end{array}$$

multiple stable matching - conflicting preferences

Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

m_1	w_1	w_2	w_3
m_2	w_1	w_2	w_3
m_3	w_1	w_3	w_2

w_1	m_1	m_2	m_3
w_2	m_1	m_3	m_2
w_3	m_1	m_2	m_3

	m_1	m_2	m_3
	w_1	w_1	w_1
	w_1	$\emptyset$	$\emptyset$
DA m		w_2	w_3
	w_1	w_2	w_3
	w_1	w_2	w_3

multiple stable matching - conflicting preferences

Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

$$\begin{array}{c|ccc} m_1 & w_1 & w_2 & w_3 \\ m_2 & w_1 & w_2 & w_3 \\ m_3 & w_1 & w_3 & w_2 \end{array} \qquad \begin{array}{c|ccc} w_1 & m_1 & m_2 & m_3 \\ w_2 & m_1 & m_3 & m_2 \\ w_3 & m_1 & m_2 & m_3 \end{array}$$

DA m

m_1	m_2	m_3
w_1	w_1	w_1
w_1	$\emptyset$	$\emptyset$
	w_2	w_3
w_1	w_2	w_3
w_1	w_2	w_3

DA w

w_1	w_2	w_3

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Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

m_1	w_1	w_2	w_3
m_2	w_1	w_2	w_3
m_3	w_1	w_3	w_2

w_1	m_1	m_2	m_3
w_2	m_1	m_3	m_2
w_3	m_1	m_2	m_3

DA m

m_1	m_2	m_3
w_1	w_1	w_1
w_1	$\emptyset$	$\emptyset$
	w_2	w_3
w_1	w_2	w_3
w_1	w_2	w_3

DA w

w_1	w_2	w_3
m_1	m_1	m_1
m_1	$\emptyset$	$\emptyset$

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m_1	w_1	w_2	w_3
m_2	w_1	w_2	w_3
m_3	w_1	w_3	w_2

w_1	m_1	m_2	m_3
w_2	m_1	m_3	m_2
w_3	m_1	m_2	m_3

DA m

m_1	m_2	m_3
w_1	w_1	w_1
w_1	$\emptyset$	$\emptyset$
	w_2	w_3
w_1	w_2	w_3
w_1	w_2	w_3

DA w

w_1	w_2	w_3
m_1	m_1	m_1
m_1	$\emptyset$	$\emptyset$
	m_3	m_2
m_1	m_3	m_2

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Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

m_1	w_1	w_2	w_3
m_2	w_1	w_2	w_3
m_3	w_1	w_3	w_2

w_1	m_1	m_2	m_3
w_2	m_1	m_3	m_2
w_3	m_1	m_2	m_3

DA m

m_1	m_2	m_3
w_1	w_1	w_1
w_1	$\emptyset$	$\emptyset$
	w_2	w_3
w_1	w_2	w_3
w_1	w_2	w_3

DA w

w_1	w_2	w_3
m_1	m_1	m_1
m_1	$\emptyset$	$\emptyset$
	m_3	m_2
m_1	m_3	m_2
m_1	m_3	m_2

multiple stable matching - conflicting preferences

- Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

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  - DA men proposing: $(m_1, w_1), (m_2, w_2), (m_3, w_3)$
  - DA women proposing $(m_1, w_1), (m_2, w_3), (m_3, w_2)$

multiple stable matching - conflicting preferences

- Example $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$

$$\begin{array}{c|ccc} m_1 & w_1 & w_2 & w_3 \\ m_2 & w_1 & w_2 & w_3 \\ m_3 & w_1 & w_3 & w_2 \end{array} \qquad \begin{array}{c|ccc} w_1 & m_1 & m_2 & m_3 \\ w_2 & m_1 & m_3 & m_2 \\ w_3 & m_1 & m_2 & m_3 \end{array}$$

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  - all men prefer (weakly) the DA men proposing outcome

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m_1	w_1	w_2	w_3
m_2	w_1	w_2	w_3
m_3	w_1	w_3	w_2

w_1	m_1	m_2	m_3
w_2	m_1	m_3	m_2
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  - DA men proposing: $(m_1, w_1), (m_2, w_2), (m_3, w_3)$
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multiple stable matching - conflicting preferences

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m_1	w_1	w_2	w_3	w_1	m_1	m_2	m_3
m_2	w_1	w_2	w_3	w_2	m_1	m_3	m_2
m_3	w_1	w_3	w_2	w_3	m_1	m_2	m_3

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- Theorem For two stable matchings μ, μ', all men (weakly) prefer μ if and only if all women (weakly) prefer μ'.

multiple stable matching - conflicting preferences

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  - DA men proposing: $(m_1, w_1), (m_2, w_2), (m_3, w_3)$
  - DA women proposing $(m_1, w_1), (m_2, w_3), (m_3, w_2)$
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  - all women prefer (weakly) the DA women proposing outcome
- Theorem For two stable matchings μ, μ', all men (weakly) prefer μ if and only if all women (weakly) prefer μ'.
- The DA algorithm with men proposing yields the best stable matching for men and the worst stable matching for women.