Investment coordination

Model This application of the Nash equilibrium concept models a situation where an investment opportunity requires a certain (possibly large) number of participants to be successful. Participation here can be interpreted as making a monetary investment into the project as well as providing a costly effort to the project's success. In the example illustrated in class we considered the following scenario. A group of N agents must each choose whether or not to invest a strictly positive amount of money, $\$X$, in a given project. The project is successful if at least and fraction α of all investors contribute and it is unsuccessful otherwise. A successful project yields a $r\%$ return on the investment of each participant where $r > 0$. An unsuccessful project yields a -100% return (i.e. the entire investment is lost).

This story describes a strategic situation because the outcome of an individual investment decision depends on the choices of all other investors. Assuming that investment decisions are simultaneous, (or alternatively that each individual chooses without observing other agents' decisions,) we can model this scenario with the following strategic game:

1. **Players:** The set of players is the collection of N potential investors.
2. **Strategies:** Each player can choose one of two actions: invest $\$X$, or not invest.
3. **Payoffs:** The payoff to a player who does not invest is always 0. The payoff from investing is rX if the total number of investors is at least αN and is $-X$ otherwise.

In the example considered in class N was the number of students in the class that day, X was \$10, $r = 50\%$ and $\alpha = 0.9$.

Analysis There are two obvious candidates for Nash equilibrium. First the strategy profile where every single player chooses to invest is a Nash equilibrium. Since everybody is investing, the project is successful and each player receives a payoff of rX. If a player chooses not to invest his payoff is 0, which by assumption is strictly less than rX, thus each player is best responding to the other players' strategies. Let's consider the strategy profile where every player chooses not to invest. A player not investing receives a payoff of 0, if the same player chooses to invest the fraction of players who are investing is given by $1/N$. If $1/N < \alpha$ the project is unsuccessful which gives a negative payoff to the investor. In this case not investing yields a strictly better payoff and each player is best responding to the other players' strategies. Note that if $1/N \geq \alpha$, then a single investor is sufficient for the project to be successful. In this case, the strategy profile where no player invest is not a Nash equilibrium. In the example considered in class $1/N < \alpha$ and both the strategy profile where everybody invest and the strategy profile where nobody invest are Nash equilibria.

Are there equilibria other than those already considered? To answer this question it is useful to note that the payoffs to each player from investing and not investing depend only on the number of other players investing and not on their identity. Consider a strategy profile in which a fraction greater or equal to α but smaller than 1 of players has invested in the project. Any strategy profile with this property is not a Nash equilibrium. Any player who chose not to invest would strictly prefer to invest since the project is successful. Similarly, a strategy profile in which a fraction smaller than α but greater than 0 of players invest is also not a Nash equilibrium. Now it is the players who choose to invest who would be strictly better off by not investing since the project is unsuccessful. These two cases cover all strategy profiles other than the perfectly coordinated outcomes we have considered earlier. Thus we can conclude that no other Nash equilibrium exists in this game.

Comments In the case when more than one investor is needed for the project to succeed (which is the interesting case), we have identified two "coordinated" Nash equilibria. We can rank the two equilibrium outcomes using the Pareto efficiency criterion. In the equilibrium where everyone invests each player receives a strictly positive payoff, in the equilibrium where nobody invests every player receives a payoff of 0. The first equilibrium Pareto dominates the latter because every player is better off when everybody invests.

On which of the two equilibria will the population of players end up coordinating? In our class experiment the first round of play resulted in an outcome which was not a Nash equilibrium. A positive fraction of students choose to invest but not sufficiently many for the project to be successful (i.e. $< 90\%$). In the following round of play every single student choose not to invest, thus the play "converged" to the Pareto inferior Nash equilibrium. A possible explanation of why the play converged to the non-investment equilibrium is that in the first round the investment was unsuccessful. Those students who did not invest were right and happy not to have done so, the students who did invest in the first round regretted doing so. If players who regret their past actions are more likely to change their play in the future it would explain convergence to the non-investment equilibrium in our class experiment. We could add that in the level of coordination on investment required for success was much higher (90% or higher) than the level of coordination on non-investment required for failure (higher than 10%). This might explain why in the first round of play the outcome was a failure of project which then led to convergence toward the non-investment equilibrium outcome.

In principle, some form of pre-play communication between agents might help coordination on the Pareto efficient equilibrium outcome. A public invitation to invest might be credible and facilitate coordination on the good equilibrium even if there is no reward (penalty) for (not) following the suggestion. This is because all players have the same incentives and have no reason to mistrust the suggestion of the proponent. There is no guarantee that this would work though, and in fact it did not in our class experiment.

Location choice and segregation

Model This second application of the Nash equilibrium concept models a situation where a large population of individuals choose where to live (or go to school, or which club to join etc. etc.). The population is heterogeneous and individuals care about the composition of the population of the city they live in (or the school they attend, or the club they join etc. etc.) These two factors introduce a strategic component in the interaction among agents. Where an individual would like to live depends on the choices of other agents because those choices determine the population composition across the available locations.

To study this type of scenario, we considered a stylized model in which agents differ with respect to a single characteristics (in reality there are many dimensions of heterogeneity). In our abstract class example that characteristic was whether an agent is right-handed or left-handed. We assumed a total population of 1 million individuals (players) equally split between right-handed and left-handed people. These individuals choose simultaneously to live in one of two cities: North city (N) and South city (S), so each player has two strategies: N and S. Each city can host half of the population. In case of over-demand for one of the two cities, a random draw determines which individuals will have their demand satisfied and the remaining individuals are forced in the other city. The preferences of each individual exhibits the following characteristic. Every individual likes diversity in the population, and the ideal city is one were the populations in perfectly diverse (i.e. 50% of the population is right-handed and 50% is left-handed.) Between two cities one with K right-handed residents and the other with K left-handed residents, each individual prefers the city where they are among the majority. The following payoff functions for right-handed (π_r) and left-handed (π_l) individuals respectively, satisfy these assumptions:

$$\pi_r(x) = \begin{cases} \frac{x}{250{,}000} & \text{if } x \le 250,000 \\ \frac{750{,}000 - x}{500{,}000} & \text{otherwise} \end{cases} \qquad \pi_l(x) = \begin{cases} \frac{250{,}000 + x}{500{,}000} & \text{if } x \le 250,000 \\ \frac{500{,}000 - x}{250{,}000} & \text{otherwise} \end{cases}$$

In the above, x denotes the number of right-handed people in the city that the given individual joins.

There is one last assumption we need to complete our game description. Because sometimes there will be more individuals demanding to live in a city than space in that city, the residence of a player might be determined by a random draw. In those cases, we assume that the payoff to that player is described by his **expected payoff**. Fora example, if a left-sided player will end up in S with probability p_s and in N with probability p_n and further there are x_s and x_r right-handed residents in S and N respectively, the payoff to the player is given by the formula

$$p_s \pi_l(x_s) + p_n \pi_l(x_n),$$

which describes his expected payoff.

Analysis In the class discussion, we identified three candidates strategy profiles and we proceeded to verify that they were indeed Nash equilibria. The first candidate Nash equilibrium is the strategy profile in which exactly half of the right-handed population and half of the left-handed population choose N. This strategy profiles yields that both cities will host exactly 250,000 right handed players and 250,000 left-handed player. The payoff to every player is 1, which is the highest payoff any player can achieve (both π_l and π_r are equal 1 and maximized when $x = 250,000$.) No player by changing his strategy can obtain a larger payoff, thus this is a Nash equilibrium. Note that the equilibrium outcome is that the population composition of the two cities is identical, thus no-segregation occurs in this equilibrium.

The second candidate Nash equilibrium is the "perfectly segregated" strategy profile in which all right-handed players choose N and all left-handed players choose S. In this outcome, all players receive a payoff of 1/2: all right-handed players live in a city with 500,000 right-handed residents, and all left-handed players live in a city with no right handed resident. From the formula of the payoff functions we have $\pi_r(500,000) = 1/2 = \pi_l(0)$. To verify that this is a Nash equilibrium, we need to study the payoff that a right-handed player would obtain from changing his choice from N to S, and the payoff that a left-handed player would obtain from changing his choice from S to N, assuming that all other players maintain their choices. If a right-handed player switches to choosing S, the city S is over-demanded because now 500,001 individuals want to live there. With a very small probability (1/500,001) our right-handed player will be forced to remain in N, which does not change his payoff. With probability close to 1 (500,000/500,001) the right-handed player ends up in a city in which he is the only right-handed resident and his payoff will be $\pi_r(1) = 1/250,000$, which is strictly smaller than 1/2. This shows that a right handed player does not benefit from changing his choice. A similar argument shows that also left-handed players do not benefit from changing their choice from S to N. Thus the strategy profile in which all right-handed players choose N and all left-handed players choose S is a Nash equilibrium.

The third candidate Nash equilibrium is the other "perfectly segregated" strategy profile where all right-handed players choose S and all left-handed players choose N. This is also a Nash equilibrium by the same argument as the one employed in the previous case.

Are there any other Nash equilibria?

A complete answer to this question is somewhat involved, but the following simple argument provides most of the intuition of why the answer is NO. Consider a strategy profile where x right-handed and $500,000 - x$ left-handed players choose N, where x is a positive number strictly smaller than 250,000. Given this strategy profile, N hosts a minority of x right-handed residents and S hosts a majority of $500,000 - x$ right-handed residents. A right-handed resident of N would be strictly better off by changing his choice to S. This is because if he succeeds in moving to S (his attempt to move to S implies that S is over-demanded and he may be forced to remain in N) he will be living in a city with either $500,000 - x$ or $500,000 - x - 1$ right-handed residents depending of whether he displaces a right-handed or a left-handed resident of S. In both cases the move to S improves his payoff (use the payoff function π_r to verify this claim). Any strategy profile of this type is not a Nash equilibrium, and a similar argument can be used that the same is true if x is strictly between 250,000 and 500,000.

Comments Similarly to the investment coordination game, we can Pareto rank equilibrium outcomes. The "segregated" equilibria are Pareto dominated by the "no-segregation" equilibrium. However, despite being inferior from the welfare standpoint, the two segregated equilibria can be thought of as more "robust" in the following sense. Start from any distribution of the players among the two cities that is not an equilibrium. That is in one city, say N, there is a positive minority, x, of right-handed residents, and in the other city there is the same minority x of left-handed residents. Consider the incentives to move location: the right-handed residents of N have an incentive to move to S as similarly the left-handed residents of S have an incentive to move to N. On the contrary, neither the left-handed residents of N nor the right-handed residents of S have an incentive to move. If it is more likely that people who would benefit from moving will try to do so, there will be an outflow of right-handed agents from N toward S and an outflow of left-handed agent from S toward N. This would further increase the majority of left-handed residents in N and of right-handed in S. In turn this further increases the incentives of the minorities to move to a different city (they are now an even smaller group), and so on until a perfectly segregated outcome emerges. The weakness of the non-segregated equilibrium outcome is that unless the initial population distribution corresponds to that equilibrium (which is unlikely because it requires precise coordination by a large group of agents), then the simple (and plausible) adjustment mechanism we have described moves the outcome away from the non-segregated equilibrium outcome. The important conclusion here is that segregation can arise as the (likely) outcome in a strategic environment despite the fact every individual prefers unsegregated outcomes. This result, which is due to to Thomas Schelling (Schelling, T. (1969). "Models of segregation", The American Economic Review, 1969, 59(2), 488-493), also cautions us about concluding that segregates outcomes indicate some form of individual preferences for segregation.