University of Toronto
Faculty of Arts and Sciences

DECEMBER EXAMINATIONS 2012

ECO316H1F

Duration - 2 hours

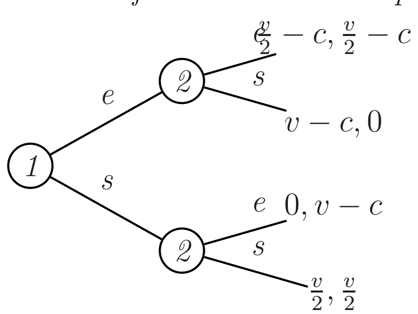
No aids are allowed

There are 4 questions in 2 pages. **You must give arguments to support your answers.**

Question 1. [30 points] Two players, 1 and 2, participate in a contest to win a prize v. The rules of the contest state that player 1 moves first and chooses whether to exert effort (action E) or shirk (action S). After observing player 1's choice, player 2 chooses whether to exert effort of shirk. The prize is assigned to the player who exerted more effort. In the case of a tie the prize is divided equally between the two players. The payoff to a player is equal to the amount won by the player minus the cost of effort, c, if the player exerted effort. For example, if player 1 exerts effort and player 2 does not, the payoff to player 1 is $v - c$, while the payoff to player 2 is 0. For all questions below assume that $2c > \frac{1}{2}v > c$.

a) Model the strategic situation described above through an appropriate game.

In the game tree below, e stands for exerting effort and s for shirking. In each terminal node the first number is the payoffs of player 1, the second number the payoff of player 2



b) Which strategies are strictly dominated for player 1? Which strategies are strictly dominated for player 2?

No strategy of player 1 is strictly dominated. If player 2 always shirk, exerting effort gives a strictly larger payoff to player 1, $(v - c)$, than shirking, $v/2$. However, if player 2's strategy is to match player 1's action (i.e. exert effort if player 1 did so and shirk otherwise), then by shirking player 1 gets a payoff of $v/2$ which is strictly larger than the payoff from exerting effort $v/2 - c$.

For player 2, the strategy of always exerting effort strictly dominates the strategy of always shirking. If player 1 shirks, the former yields a payoff of $v - c$ which is larger than the payoff from always shirking, $v/2$. If player 1 exert efforts, the payoff from always exerting effort is $v/2 - c$ which is strictly larger than the payoff from always shirking, 0. No other strategy is strictly dominated. Matching player 1's actions does as well as always exerting *effort when player 1 exerts effort. The strategy of ding the opposite of player 1 (i.e. exert effort is 1 shirks and shirk is 1 exerts effort) does as well as always exerting effort when player 1 shirks.*

c) Find a subgame perfect Nash equilibrium (i.e. backward induction equilibrium).

In any sub-game perfect NE, player 2 exerts effort both if player 1 chooses s and if player 1 chooses e. Player 1 anticipates that player 2 will exert effort regardless of his choice and expects a payoff of $v/c-2$ from e and a payoff of 0 from s. SInce $v/2-c>0$, the strategy profile where player 1 chooses e and player 2 chooses e regardless of player 1's action is the only sub-game perfect Nash equilibrium of the game.

Suppose now that after player 2 makes his choice the contest continues for a second round. In the second round player 2 goes first and chooses one more time between exerting effort and shirking. After observing player 2's choice, player 1 also chooses between exerting effort and shirking. As before, the prize is assigned to the player who exerted the most effort over the two rounds, and in the case of a tie the prize is divided equally between the two players. The payoff to a player is equal to the amount won by the player minus c for each instance of effort exerted by the player. For example, if player 1 exerts effort in both rounds and player 2 exerts effort in the second round only, the payoff to player 1 is $v-2c$ and the payoff to player 2 is $-c$.

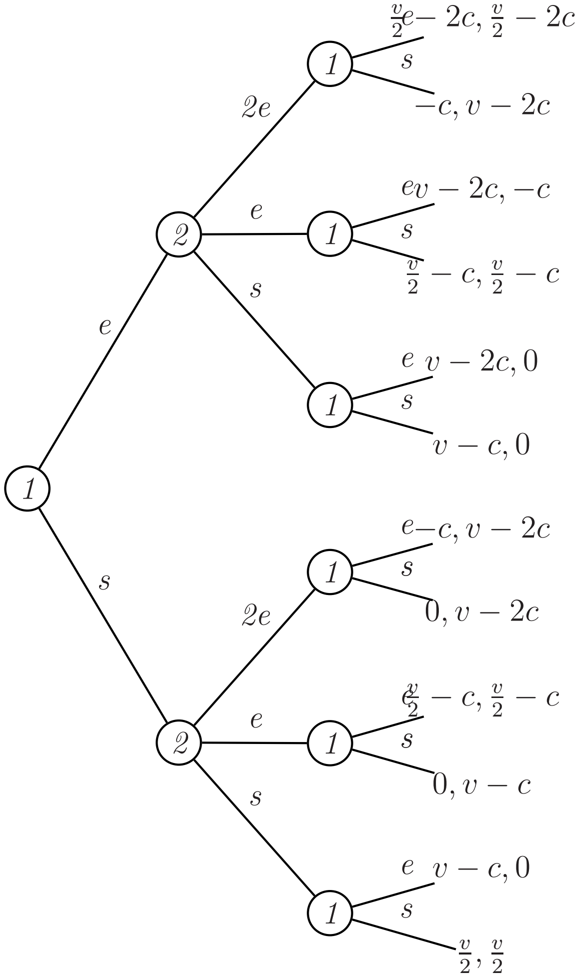
d) Find a sub-game perfect Nash Equilibrium of the game.

Because player 2 chooses to exert effort twice in a row, it is as if player 2 chooses once but has three actions available: no effort, one unit of effort or two units of effort. The game is described by the game tree below (writing the game three is useful but not necessary to answer the question)

If in the first round player 1 choose e, in the second round player 1 will again choose e unless player 2 does not exert any effort, in which case player 1 wins without exerting further effort. This implies that after observing player 1 choose e in the first round, player 2 anticipate he will loose if he exerts less than two units of effort and will tie if he exerts two units of effort. Since the payoff from shirking and loosing, 0, is strictly larger than the payoff from tying with two units of effort, $\frac{v}{2}-2c$, in any sub-game perfect NE player 2 shirks after observing player 1 choose e in the first round.

If in the first round player 1 choose s, in the second round player 1 will choose e unless player 2 has exerted two units of effort, in which case player looses regardless and strictly prefer to shirk. Thus, player 2 anticipates two win with two units of effort (payoff $v-2c$), to tie with one unit of effort (payoff $\frac{v}{2-c}$) and to loose with no effort (payoff 0). Since $v-2c>\frac{v}{2}-c>0$ player 2 exerts two units of effort if player 1 chooses s in the first period.

Player 1 anticipates that, after he chooses e in the first round, player 2 will choose to shirk, so one wins with no additional effort and a payoff of $v-c$. Player 1 anticipates that, after he chooses s in the first round, player 2 will exert two units of effort, hence player 1 will loose with a payoff at most of 0. So in the sub game perfect NE of the game player 1 chooses e in the first period, and no effort is exerted by any player afterwards. The complete strategy profile is as follows. Player 1: i) exerts effort in the first round; ii) shirks in the second round if player 2 has exerted less that he has already exerted, or player 2 has exerted two units of effort more that he has; and iii) exert effort in the second round *if the amount of effort exerted by player 2 is equal or one unit more that the amount of effort he has already exerted. Player 2: exerts no effort after observing e in the first round and exerts two units of efforts after observing s in the first round.*



e) Compare the equilibrium of this two-round contest with the equilibrium you found in question c) with respect to the total effort provided by the two players.

The total effort provided by all players is strictly smaller in the two round game than it is in the one round game (i.e. one instance of effort by player 1 vs. one instance of effort each by the two players).

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Question 2. [25 points] Consider the two-player zero-sum game described below, where the numbers refer to the payoff of the row player (i.e. the player choosing between T and B).

	l	r
T	2	3
B	4	1

a) Find a mixed strategy Nash equilibrium **by using the indifference principle.**

Let p denote strategy of the row player when he chooses T with probability p, and q the strategy for the column player when he chooses l with probability q. By the indifference principle, in a mixed strategy Nash equilibrium p must equalize the expected payoff from the action l and the action r for the column player, or

$$2p + 4(1-p) = 3p + 1(1-p).$$

Similarly q must equalize the expected payoff from T and B for the row player, or

$$2q + 3(1-q) = 4q + 1(1-q).$$

Solving for q and p in the equations above, we have that the strategy profile ($p^* = 3/4$, $q^* = 1/2$) is the mixed strategy Nash equilibrium of the game.

b) Find all Nash equilibria (in pure or mixed strategies) **by finding the intersections of the best response functions.**

The expected payoff from choosing T is given by $2q + 3(1-q)$, the expected payoff from choosing B is $4q + 1(1-q)$. The former is larger for $q < 1/2$ and the opposite is true for $q > 1/2$. For $q = 1/2$ the two actions give the same expected payoff. Choosing T (i.e. $p = 1$)is the unique best response to any strategy $q < 1/2$, choosing B (i.e. $p = 0$) is the unique best response to any strategy $q > 1/2$, and any mixed strategy p is a best response to the mixed strategy $q = 1/2$.

Similarly, for the column player, the expected payoff from choosing l is $2p+4(1-p)$, which is strictly larger (smaller) that the payoff $3p + (1-p)$ from choosing r whenever $p < 3/4$ ($p > 3/4$). The two expected payoffs are equal when $p = 3/4$. SInce the column player's objective is to minimize his payoff, choosing l (i.e. $q = 1$) is the unique best response to any strategy $p > 3/4$, choosing r (i.e. $q = 0$) is the unique best response to any strategy $p > 3/4$, and any mixed strategy q is a best response to the mixed strategy $p = 3/4$.

The two best response functions have a unique intersection at the mixed strategy NE of the game $p^* = 3/4$, $q^* = 1/2$.

c) Find all Nash equilibria **by using the min-max method**.

A mixed strategy p, achieves the worst (i.e. smallest) payoff to the row player when the column player is best responding to p. Since l is a best response to p when $p \geq 3/4$ and r is a best response when $p < 3/4$, the minimum payoff to the row player associated with strategy p is $2p + 4(1-p)$ for $p \geq 3/4$ and equals $3p + (1-p)$ for $p < 3/4$. The $\max\min$ value is achieved by $p = 3/4$ and equals $5/2$.

Similarly, a mixed strategy q achieves the worst (i.e. highest) payoff to the column player when the row player is beset responding to q. Since T is a best response for $q \leq 1/2$ and R is a best response for $q > 1/2$, the maximum payoff associated with strategy q is $2q + 3(1-q)$ for $q \leq 1/2$ and $4q + (1-q)$ for $q > 1/2$. The min max *value is achieved by $q = 1/2$ and equals $5/2$.*

Thus the mixed strategy NE of the game is the strategy profile $(p^* = 3/4, q^* = 1/2)$.

Question 3. [20 points] Consider the symmetric two-player game described below.

	A	B
A	α, α	γ, δ
B	δ, γ	β, β

Find restrictions on the payoffs α, β, γ and δ such that the game has two symmetric Nash equilibria (in pure strategy) and only strategy A is evolutionary stable.

For (A, A) to be a Nash equilibrium A must be a best response to A or $\alpha \geq \delta$. For (B, B) to be a Nash equilibrium B must be a best response to B or $\beta \geq \gamma$. If $\alpha > \delta$, (A, A) is a strict NE, hence A is evolutionary stable. For B not to be evolutionary stable it must be that $\beta = \gamma$ (i.e. A is also a best response to B) and the payoff of B against A must be smaller than the payoff of A against itself, or $\delta < \alpha$, which is already satisfied. To summarize, $\alpha > \delta$ and $\beta = \gamma$ are the needed payoff restrictions.

Question 4. [25 points] Twelve single people, five women and seven men, simultaneously choose to join one of two online dating sites, X and Y. The payoff to each individual from joining a site depends on the total number of single men and single women who join that site. Precisely, a single man who joins site i (where i can be either X or Y) receives a payoff

$$\alpha w_i - m_i.$$

In the above expression, α is a non-negative number, m_i is the total number of men who join site i, and w_i is the total number of women who join site i. A single woman who joins site i receives a payoff

$$\alpha m_i - w_i.$$

For example, if three men and one woman join site X and the remaining four men and four women join site Y then: i) each man in X receives a payoff $\alpha - 3$; ii) the woman in X receives a payoff $\alpha 3 - 1$; and iii) each man and woman in Y receives a payoff $\alpha 4 - 4$.

a) Assume that $\alpha = 1$. Find a Nash equilibrium of the game.

   Consider a strategy profile that leads the two online dating sites to be as similar as possible, for example 4 men and 3 women join one site and 3 men and 2 women join the other site. A man in the larger site does not want to switch site, otherwise he would end up in a site with the same number of men (4) and fewer women (2). Similarly for any woman in the larger site. A man in the smaller site has a payoff of -1, by switching he would join a site *with 5 men and 3 women thus receiving a -2 payoff. Similarly a woman in the smaller site receives a payoff of 1, by switching she would be joining a site with 4 men and 4 women thus receiving a payoff of 0. So also agents in the smaller site are best responding. Any strategy profile in which 4 men and 3 women choose one site and the remaining agents choose the other site is a Nash equilibrium.*

b) For what values of the parameter α is there an equilibrium in which every agent joins the same same dating site?

Consider a strategy profile in which all agents join one site, say X. The payoff to each man is $\alpha 5 - 7$ and the payoff to each woman is $\alpha 7 - 5$. If a man switches to Y, his payoff would be -1 (i.e. he would be the only agent in the site). Thus, in the strategy profile where all agents join the same site, men are best responding if $\alpha 5 - 7 \geq -1$ or $\alpha \geq 6/5$. If a woman switches to Y, her payoff would be -1 (i.e. she would be the only agent in the site). Thus, in the strategy profile where all agents join the same site, women are best responding if $\alpha 7 - 5 \geq -1$ or $\alpha \geq 4/7$. For every agent to be best responding, hence to have a Nash equilibrium with all agents join the same site, it is necessary that α be at least as large as $6/5$.

c) Find a profile of mixed strategies that is a Nash equilibrium regardless of the value of α.

Consider the strategy profile in which every agent join X or Y with equal probabilities. For each agent, the expected number of men and women who join X is the same as the expected number of men and women joining Y. Thus the expected payoff from choosing X is the same as the expected payoff from choosing Y. Since, given the strategy profile of everyone else, each agent is indifferent between choosing X and choosing Y, any randomization between the two choices is a best response. So the strategy profile in which every agent chooses X or Y with equal probability is a Nash equilibrium,