University of Toronto
Faculty of Arts and Sciences

DECEMBER EXAMINATIONS 2011

ECO316H1F

Duration - 2 hours

No aids are allowed

There are 4 questions in 3 pages. You must give arguments to support your answers.

Question 1. [25 points] A police officer patrolling the highway can stop motorists and search for drugs. Searching has a cost c. A successful search yields a positive benefit b. An unsuccessful search or no search yields no benefit. The police officer's payoff is the difference between the benefit received and the search cost incurred. Motorists choose whether to carry drugs or not. The payoff to a motorist who does not carry drugs is 0 whether or not they are searched. The payoff to a motorist who carries drugs depends on whether he is searched or not. If searched, the contraband will be discovered and the motorist will go to jail, which gives a payoff of $-J < 0$. A motorist who carries drugs and is not searched will be able to sell the drugs on the market making a profit $\pi > 0$.

a) Model the situation described above as a strategic game between a police officer and one motorist.

In the strategic game below the police officer is the row player and the motorist is the column player. S stands for searching and N stands for not searching for drugs. C stands for carrying and H stands for not carrying drugs.

	C	H
S	$b-c,\ -J$	$-c,\ 0$
N	$0,\ \pi$	$0,\ 0$

b) For what values of the parameters b and c is the game dominance solvable? In that case, what is the Nash equilibrium?

The column player (i.e. the motorist) does not have any dominated action. For the row player (i.e. the police officer) the action N yields a strictly higher payoff than S when the motorist chooses H, thus the only action that can be dominated in the game is S, this happens if N gives a higher payoff to the police officer when the motorist chooses C. This is equivalent to

$$b - c \leq 0 \quad \text{or} \quad b \leq c.$$

If $b \leq c$ the action S is dominated by the action N for the police officer. After the action S is deleted from the game, the action H is dominated by the action C for the motorist. Thus the game is dominance solvable and the only outcome that survives iterated deletion of dominated strategies is the outcome where the police officer does not search and the motorists carries drugs.

c) Assume there is no dominated strategy and find a Nash equilibrium of the game in mixed strategies.

Since there is no dominated strategy, from part b) we know that $b > c$. First note that in a mixed strategy Nash equilibrium both the motorist and the police officer must be playing a mixed strategy. This is because for both of them if the other player chooses a pure strategy the best response is also a pure strategy. Since both players are mixing in equilibrium, they must be indifferent between the two actions available to them, which happens if the expected payoff from the two actions are the same. If we denote with p^* the equilibrium probability that the police officer chooses S and with q^* the equilibrium probability that the motorist carries drugs, we have that p^* and q^* must solve the system of equations

$$q(b-c) - (1-q)c = 0 \quad \textit{and} \quad -pJ + (1-p)\pi = 0.$$

Solving the above we have that $q^* = \frac{c}{b}$ and $p^* = \frac{\pi}{\pi+J}$.

Suppose you observe the following data: i) In the state of Maryland, African Americans constituted 63 percent of motorists searched by state police on Interstate 95 between January 1995 and January 1999 but only 18 percent of motorists on the road; ii) the percentage of African American drivers found carrying drugs when searched is similar to the percentage of white motorists found carrying drugs when searched.

d) In light of the game in part c), does the evidence suggest that police officers enjoy a larger benefit (b) when searching African American motorists? (i.e. racial bias). Using no more than 50 words, explain why or why not.

No. The evidence suggest that police officers search African american motorists more often than other motorists. In the language of the game we considered this means that the value of p^* is higher in a game between a police officer and an African american motorist than it is in a game between a police office and a non-African american motorist. But, as the analysis of part c) shows, the value of p^* does not depend on b thus, for the game considered, higher p^* is not evidence of racial bias.

Question 2. [25 points] Consider the two-player zero-sum game described below, where the numbers refer to the payoff of the row player.

	l	c	r
T	5	0	1
M	4	1	0
B	3	5	2

a) Find a Nash equilibrium **using the min-max method**. Clearly explain your work.

For the row player, the minimum payoff from action T is 0, the minimum payoff from M is 0 and the minimum payoff from B is 2. The largest of the minimums is 2. Thus the $\max\min$ *is 2 and is achieved by action B. For the column player, the maximum payoff* *from l is 5, the maximum payoff from c is 5 and the maximum payoff from r is 2. The minimum of these three payoffs is 2. Thus the* min max *is 2 and is achieved by action r. Since the* min max *and the* max min *coincide, the game has a Nash equilibrium in which the row player chooses B and the column player chooses r.*

b) Is any of the row-player actions strictly dominated by another action? By a mixed strategy? Explain.

No action of the row player is strictly (nor weakly) dominated by another action. However, action M is strictly dominated by a mixed strategy in which action T and B are played with positive probability. To see this note that a mixed strategy in which T and B are each played with probability $\frac{1}{2}$ already weakly dominates M. This is because the (expected) payoff from such strategy would be: 4 if the column player chooses l; 2.5 if the column player chooses c; and 2.5 if the column player chooses r. A mixed strategy that puts a bit more than half weight on T would give an expected payoff larger than 4 against l while still doing strictly better than the action M against c and r.

Question 3. [25 points] Two bidders compete for a single object in an all-pay auction. They simultaneously submit one of two possible bids p_H, p_L with $p_H > p_L > 0$. The object is worth \$$v$ to both and it is assigned to the highest bidder. Ties are broken by a fair coin flip. Each bidder pays a price equal to his submitted bid whether he receives the object or not. The payoff to each bidder is equal to the value of the object times the probability of receiving it minus the price paid in the auction.

a) Model this situation as a strategic form game.

	p_l	p_h
p_l	$v/2 - p_l,\ v/2 - p_l$	$-p_l,\ v - p_h$
p_h	$v - p_h,\ -p_l$	$v/2 - p_h,\ v/2 - p_h$

b) Find the range of values of v for which one action in the game is dominated.

Since the game is symmetric we can consider just one player, the row player. If the column player chooses p_l, the row player's payoff from p_l is larger than the payoff from p_r if

$$v/2 - p_l > v - p_h \quad \text{or} \quad p_h - p_l > v/2.$$

If the column player chooses p_h, the row player's payoff from p_l is larger than the payoff from p_h if

$$-p_l > v/2 - p_h \quad \text{or} \quad p_h - p_l > v/2.$$

Thus p_l strictly dominates p_h if $p_h - p_l > v/2$, and p_h strictly dominates p_l if $p_h - p_l < v/2$. If $p_h - p_l = v/2$ the two strategies are identical.

c) For every value of v find the Nash equilibria of the game.

It follows immediately from part b) that both player choosing p_l is the unique Nash equilibrium when $p_h - p_l > v/2$, both players choosing p_h is the unique Nash equilibrium when $p_h - p_l < v/2$, and any profile of actions is a Nash equilibrium when $p_h - p_l = v/2$.

Hoping to generate higher bids and hence more revenue, the auctioneer decides to add a consolation prize worth $r < v$ to the auction. In this modified auction the highest bidder (or the winner of the coin flip in case of equal bids) receives the object worth v and the other bidder receives the object worth r.

d) For a fixed r and every value of v larger than r study what happens to the revenue generated by the auction after the consolation prize is added.

With the consolation prize the game becomes

	p_l	p_h
p_l	$(v+r)/2 - p_l,\ (v+r)/2 - p_l$	$r - p_l,\ v - p_h$
p_h	$v - p_h,\ r - p_l$	$(v+r)/2 - p_h,\ (v+r)/2 - p_h$

Now p_l strictly dominates p_h if $p_h - p_l > (v-r)/2$ and p_h strictly dominates p_l if $p_h - p_l < (v-r)/2$. To compare revenues, note that when $p_h - p_l < (v-r)/2$, the equilibrium in both auctions (with and without consolation prize) is (p_h, p_h) and the revenue to the seller is $2p_h$ in both cases. When $p_h - p_l > v/2$ the equilibrium in both auctions is (p_l, p_l) and the revenue to the seller is $2p_l$. Finally if $(v-r)/2 < p_h - p_l < v/2$ the equilibrium without consolation prize is (p_h, p_h) and the revenue $2p_h$ while the equilibrium with consolation prize is (p_l, p_l) and the revenue to the seller goes down to $2p_l$.

e) How would the equilibrium change if the two players choose their bids sequentially rather than simultaneously?

Nothing changes because the equilibrium is in dominant strategies.

Question 4. [25 points] There are four students, 1, 2, 3 and 4, and two schools, X and Y, that can each admit up to 2 students. The four students have different academic abilities with 1 being the most talented followed by 2, then 3, and then 4 being the least academically talented. The four students simultaneously apply to either school X or Y. If a school receives more than two applications it admits the best 2 applicants and the remaining must go to the other school.

First suppose that within each school the best student receives an A grade and the second best receives a B, and every student only cares about maximizing their grade.

a) Find a Nash equilibrium in the application game played by the students.

We can find a Nash equilibrium by studying the best response functions of the four students. First we notice that student 1 will be admitted and will receive a grade of A in any school she applies. Thus, regardless of the choices of the other three students, 1 is indifferent between applying to X or Y and they are both best responses. Student 2 will be admitted by any school she applies to and receives a B grade is she applies to the same school as 1 and an A grade if she applies to a school different from the school 1 is applying to. Thus it is a best response for 2 to apply to whatever school 1 is not applying to. In any (pure strategy) Nash equilibrium 1 and 2 apply to different schools. Finally, if 1 and 2 apply to different schools student 3 and 4, will always receive a B grade regardless of the school *they apply to. Thus applying to X and applying to Y are both best responses for 3 and 4 whenever 1 and 2 are applying to different schools. This implies that any strategy profile in which student 1 and student 2 apply to different schools is a Nash equilibrium.*

Next, suppose that both schools assign an A grade to every student, and that now each student cares only about having the brightest possible classmate.

b) Find a Nash equilibrium in the application game played by the students.

Student 1 and 2 will both be admitted in any school they apply to. It is a best response for 1 to apply to whatever school 2 is applying to and for 2 to apply to whatever school 1 is applying to. Thus in any pure strategy Nash equilibrium student 1 and 2 apply to the same school. If student 1 and 2 apply to the same school, regardless of their application strategy student 3 and 4 will end up in the same school, thus if 1 and 2 are applying to the same school, both applying to X and applying to Y are best responses for 3 and 4. This implies that any strategy profile in which student 1 and student 2 apply to the same school is a Nash equilibrium.

c) How would your answer to part a) and b) change if there are 100 students ranked from 1 to 100 and each school can admit 50 students? [For part a) assume that schools give to each of their 50 students a different grade reflecting their academic ability. For part b) assume that students care about the average academic ability rank of their classmates.]

For part a) student 1 will be the best in any school. Student 2 will be the best in the school where 1 did not apply, thus in a Nash equilibrium 1 and 2 apply to different schools. Given that 1 and 2 are in different schools, student 3 will be the second best in any school she applies to and student 4 will be the second best in the school 3 did not apply to and third best in the school 3 applied to. Thus in any Nash equilibrium 3 and four apply to different schools. The same logic apply the pair of students 5 and 6, and then 7 and 8, and so on. Thus any strategy profile in which student $2n$ and student $2n-1$ for $n = 1, 2, ..., 50$ apply to different schools is a Nash equilibrium (e.g. every odd student apply to X and every even student apply to Y.)

For part b) a strategy profile in which the top 50 students apply to the same school is a Nash equilibrium.