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RESEARCH ARTICLE

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Price discrimination and efficient matching

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Abstract This paper considers the problem of a monopoly matchmaker that uses a schedule of entrance fees to sort different types of agents on the two sides of a matching market into exclusive meeting places, where agents randomly form pairwise matches. We make the standard assumption that the match value function exhibits complementarities, so that matching types at equal percentiles maximizes total match value and is efficient. We provide necessary conditions and sufficient conditions for the revenue-maximizing sorting to be efficient. These conditions require the match value function, modified to incorporate the incentive cost of eliciting private type information, to exhibit complementarities in types.

Keywords Complementarity · Subscription fees · Sorting structure · Random pairwise matching · Virtual match value

JEL Classification Numbers C7 · D4

1 Introduction

Many users of Internet dating agencies such as Match.com complain about the problem of misrepresentations and exaggerations by some users in the informa-

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tion they provide to the agencies.[1] This problem, and the perception of it among the public, is responsible for reducing the quality of Internet search and matching and for preventing many lonely people from fully utilizing the online dating services, in spite of the advantages in cost, safety, anonymity and breadth of the reach offered by the new technology compared to more traditional means of finding dates. Although Internet dating agencies rely on individual users to report information about themselves truthfully and have little capability or resource of directly validating the information, economic theory suggests price discrimination as a way of making the reported information credible and improving match quality.

In this paper we look at the theoretical problem of a monopoly matchmaker that uses a schedule of entrance fees to sort different types of agents on the two sides of a matching market into different "meeting places," in which agents are randomly pairwise matched. This problem is presented in Section 2. The monopoly matchmaker faces two constraints in revenue maximization. First, the matchmaker does not observe the one-dimensional characteristic ("type") of each agent. This information constraint means that the matchmaker must provide incentives in terms of match quality and fees for agents to self-select into the meeting places. We refer to the menu of meeting places created by the matchmaker as the "sorting structure." Second, the monopoly matchmaker faces a technology constraint that restricts match formation in each meeting place to random pairwise matching. This primitive matching technology allows us to focus on the impact of revenue-maximization on the sorting structure and matching efficiency. We make the standard assumption that the match value function exhibits complementarities between types. Under this assumption, the "perfect sorting," or matching types at equal percentiles with a continuum of meeting places, maximizes the total match value and is efficient. The goal of this paper is to understand when the perfect sorting is revenue-maximizing.

Our framework fits various two-sided market environments characterized by sorting or self-selection based on prices.[2] For example, online job search has become a major way to explore potential employer–employee relationships. However, existing job search services such as Monster.com are plagued by job intermediaries (head hunters) that post fake entries only to collect information from job applicants and positions and then profit from the information. The job market and dating market share a few common features that allow our framework to apply: match characteristics of market participants can be summarized in a one dimensional type; participants on one side of the market share the same preference ordering over matches with agents on the opposite side; and types are complementary in the match value function. Other two-sided matching markets where price-based intermediation can potentially play an important role include matching tenants to apartments, and matching loan applicants to bank loans. The results in the present paper show that a monopoly matchmaker can have the same incentive as a social planner to implement the efficient matching. In this case, the matchmaker makes directed search possible by creating one meeting place for each type and achieves the first best matching outcome, in spite of the technological constraint of random pairwise matching.

[1] See for example *The New York Times,* January 18, 2001, "On the Internet, Love Really is Blind."

[2] A limitation of this paper is the assumption of a monopoly matchmaker, as competition exists in most two-sided markets. See our companion paper Damiano et al. (2004) for an application of the present framework to issues of price competition.

In Section 3 we show how the matchmaker's problem of designing fee schedules and the corresponding sorting structure can be transformed into a problem of monopoly price discrimination. The assumption of complementarity in the match value function implies that the standard single-crossing condition in the price discrimination literature is satisfied for both sides of the market, and results in the incentive compatibility constraint that a higher type receives a higher match quality. The transformation is then achieved by combining this incentive compatibility constraint with the feasibility constraint that match qualities are generated in a two-sided matching environment where agents participate in at most one meeting place and are pairwise randomly matched in each meeting place. The outcome of the transformation is a sorting structure in which the efficient matching path in the type space (pairwise matching of types at equal percentiles) is partitioned into pooling intervals and sorting intervals: each pooling interval on the efficient matching path represents a meeting place with the corresponding intervals of types on the two sides being pooled together, while each sorting interval represents a continuum of meeting places with the types efficiently matched. This allows us to rewrite the objective function of the monopolist by using a "virtual match value function," which is the match value function adjusted for the incentive costs of eliciting private type information.

Unlike a standard price discrimination problem, the solution to the sorting structure design problem cannot be characterized by pointwise maximization, because match qualities are not produced according to some exogenous cost function but are instead constrained by the choice of sorting structure. In Section 4 we provide a necessary condition for the optimal sorting structure to be the perfect sorting. This condition requires that the virtual match value function have positive cross partial derivatives at equal percentiles. If at any percentile it is not satisfied, the monopoly matchmaker can increase revenue by pooling a small interval of types into a single meeting place. This condition is local in nature, and is not generally sufficient for the optimal sorting structure to be the perfect sorting, because a greater revenue may be generated by pooling a large set of types on the two sides. A sufficient condition for the perfect sorting to be optimal is that the virtual match value function is supermodular, i.e., has positive cross partial derivatives on the entire type space. Intuitively, the inability to observe the type of agents creates an incentive cost for the matchmaker to extract surplus because the matchmaker has to rely on self-selection by the agents. The perfect sorting structure maximizes revenue for the monopolist matchmaker if this incentive cost does not dominate the complementarities in the match value function. In this case, the only possible source of inefficiency is the exclusion policy that the monopoly matchmaker may want to adopt. For types that are matched by the monopolist, there is no distortion in match quality provision, in contrast to the standard result in the price discrimination literature that quality is under-provided for all types except the highest. Moreover, if the virtual match value function is supermodular, the matchmaker's revenue is increasing in the number of meeting places created. Hence, revenue-maximization always leads to improvement in matching efficiency even with technological limits on creation of meeting places.

A classical reference in the price discrimination literature is Maskin and Riley (1984) (see also Mussa and Rosen 1978). In both the standard price discrimination and our sorting structure design problems, the monopolist faces consumers with one-dimensional private information about their willingness to pay, and must provide incentives for self-selection. In a price discrimination problem, the monopolist controls the quality (or quantity) of the good provided. Consumers of different types self-select by choosing a price-quality combination from the schedule offered by the monopolist. In contrast, in our sorting structure design problem the monopolist chooses a partition of the market into meeting places in which agents randomly match, and the associated fee schedules. Besides the standard incentive compatibility and participation constraints, the monopolist also faces additional feasibility constraints because the pair of quality schedules must be consistent with the sorting structure.

The most closely related paper in the price discrimination literature is Rayo (2005, forthcoming). He considers the price-discrimination problem of a monopolist that sells a status good. In his benchmark model, there is no intrinsic quality dimension to different varieties of the good, and buyers of one variety care only about who else are buying the same variety. Our result in Section 3 implies that this is essentially the same price discrimination problem considered here if one restricts to a symmetric matching environment. His results on when providing different varieties to different types is optimal can therefore be obtained as a special case of our conditions for the perfect sorting to be optimal.

Inderst (2001) questions the classical result in the price discrimination literature that it is optimal for the monopolist to offer low types distorted contracts in order to extract more rents from higher types. His paper looks at contract design in a matching market environment with frictions and shows that the distortion result does not hold anymore. In particular, for low enough search frictions all contracts are free of distortion. The driving force of the result is that in a search and matching environment reservation values are type dependent as higher types will generally have more match opportunity and therefore higher reservation values. In contrast, our no-distortion result does not rely on type-dependent reservation values, and is generated by feasibility restrictions on match quality provision in a two-sided matching market.

Our paper is the first to investigate intermediation in two-sided markets with heterogeneous types and search frictions from the mechanism design point of view. In the existing literature on two-sided search, sorting of heterogeneous types occurs in equilibrium either because finding a good match takes time (Burdett and Coles 1997; Smith 2002), or because meeting a potential partner is costly (Morgan 1995). Unlike these models, our framework is static and we obtain sorting as a result of maximizing revenue by an intermediary. Building on the two-sided search literature, Bloch and Ryder (2000) analyze the problem of a monopolistic matchmaker that competes with a decentralized matching market with frictions. Unlike our paper, the matchmaker observes the types and can implement perfect sorting in exchange for a fee. Due to its information advantage, the only decision for the matchmaker is what types to service given that their reservation utilities are endogenously determined in the decentralized market.

The present paper grew out of our previous work on dynamic sorting (Damiano et al. 2005). The two papers share the same interest in efficiency of matching markets in the presence of search frictions. In both papers, search frictions are modeled by the primitive search technology of random meeting. In Damiano et al. (2005) dynamic sorting provides higher types more search opportunities and improves matching efficiency. In the present paper, price discrimination by the monopolist creates directed search markets and can achieve the efficient matching. In a companion paper (Damiano et al. 2004), we use a simplified framework of the present paper to study how competition among matchmakers can affect the sorting structure and matching efficiency.

2 The model

Consider a two-sided matching market. Without loss of generality, we assume that the two sides have the same size. Agents of the two sides, called men and women, have heterogeneous one-dimensional characteristics, called types. The type distribution is $F(\cdot)$ for men and $G(\cdot)$ for women. Both type distributions are assumed to have differentiable densities, denoted as f and g, respectively. The support is $[a_m, b_m]$ for men and $[a_w, b_w]$ for women, with both subsets of $\mathbb{R}_+$, and b_m and b_w possibly infinite. A match between a type x man and a type y woman produces value xy to both the man and the woman, so $2xy$ is the total match value for the pair.[3] We assume that all market participants are risk neutral and have quasi-linear preferences. They care only about the difference between the expected match value and the entrance fee they pay. An unmatched agent gets a payoff of 0, regardless of type. Section 5 discusses how our results can be extended when reservation utilities either differ for the two sides or are type-dependent.

An important assumption about the matching preferences that we have made above is that matching characteristics of each agent can be summarized in one-dimensional type. This simplification relative to the reality of matching markets, facilitates comparison with the existing literature, where the assumption of one-dimensional type is standard. Implicit in our specification of the matching preferences is that all agents on each side of the market have homogeneous preferences. For the same price, they all prefer the highest type agents on the other side. Clearly there are matching characteristics that are ranked differently by agents in real matching markets. For example, it is sometimes argued that not everyone wishes to date the smartest person. Rather, matching preferences may be single peaked. However, when the most desirable match differs across agents, the competition among agents is reduced and so are the incentives to misrepresent this kind of matching characteristics. Since the present paper is about how the monopoly matchmaker uses price discrimination to mitigate the problem of misrepresentation in a matching market, we focus on matching characteristics that all agents rank identically and compete for.[4]

Another important assumption about the matching preferences we have made is that types are complementary in generating match values. This standard assumption is embedded in the match value function xy: each agent's willingness to pay

[3] Given our later assumption of 0 payoff for unmatched agents, the payoffs are unchanged if matched couples bargain over the division of the total match value $2xy$ using the Nash bargaining solution.

[4] Users of online dating tend to segregate into services that cater groups that share the same preferences for non-competing characteristics. One such example is religious affiliation. Jdate.com attracts only Jewish users while Eharmony.com targets the Christian population.

for an improvement in match type increases with the type of the agent.[5] Under this assumption, matching types at equal percentiles maximizes the total value of pairwise two-sided matches and is efficient (Becker 1981). Formally, for each $x \in [a_m, b_m]$, let

$$s_m(x) = G^{-1}(F(x))$$

be the female type at the same percentile of the male type x. We refer to

$$\{(x, s_m(x))|x \in [a_m, b_m]\}$$

as the "efficient matching path." We adopt the specific match value function xy for analytical convenience. Since we allow the type distributions to be different for the two sides of the market, this specification is without loss of generality in so far as the match value function is multiplicatively separable and monotone in male and female types. To be precise, any match value function of the form $u(x)v(y)$, with u and v being positive-valued and monotone, can be transformed into the match value function xy by redefining types and changing the distribution functions appropriately. The separability assumption implies that each agent in a meeting place with pairwise random matching cares only about the average agent type on the other side, as opposed to the entire distribution. As a result, the monopolist problem of designing the sorting structure can be reduced to be a one-dimensional problem of match quality provision. The importance of this assumption will become clear in Section 3. We will briefly discuss the case of non-separable match value functions in Section 5.

A monopoly matchmaker, unable to observe types of men and women, can create a menu of meeting places with a pair of schedules of entrance fees p_m and p_w. Each man or woman participates in only one meeting place. We will restrict each meeting place to have equal measure of men and women. We assume that men and women in each meeting place form pairwise matches randomly, with the probability of finding a match equal to 1 for all agents, and that the probability a type x man meets a type y woman is given by the density of type y in that meeting place. In other words, the meeting technology in our model is random matching. For simplicity, we assume that meeting places cost nothing to organize. The objective function of the matchmaker is to maximize the sum of entrance fees collected from men and women.

The technology side of our framework is modeled on the motivating example of online dating. Imagine that each meeting place consists of two data bases, of men and women who have paid the corresponding subscription fees. Any man in the meeting place has access to the data base of women and can "search" it for a match. We have assumed that the probability of finding a match is 1 for all agents. This assumption rules out any size effect, which postulates a different probability of finding a match depending on the size of the market, and allows us to focus on the issue of price discrimination. The search technology in each meeting place, which is pairwise random matching, is admittedly primitive, compared to

[5] In online dating, a more attractive individual is more likely to have a successful first date than a less attractive individual, so even if both derive the same utility from a given potential match, the more attractive individual is willing to pay more for an improvement in the quality of the potential match.

the actual matching technology used by online dating services where agents can search according to the information available on the data base and exchange further information through anonymous email correspondence. We have adopted the pairwise random matching technology in order to focus on the misrepresentation problem, by implicitly assuming that any information volunteered by participants beyond what is signaled by their choices of meeting place is not credible and therefore cannot be used to improve matching efficiency. The importance and the implications of the assumption of random pairwise matching are discussed in Section 5. Similarly, we have ignored the possibility of verifying certain information by providers of online dating services. For example, claims of college education in principle can be verified. Verifiable information can help the monopolist extract surplus. In the extreme case where the type information is public, the monopolist can achieve perfect discrimination through the perfect sorting. In general, the way availability of verifiable information changes the conditions for the optimality of the perfect sorting depends on how conditioning on public information affects the type distributions. We focus on unverifiable information and misrepresentation.

We refer to a menu of meeting places as a sorting structure. Let ϕ_m be a set-valued function that maps any male type x in $[a_m, b_m]$ to a subset $\phi_m(x)$ of $[a_w, b_w]$. The set $\phi_m(x)$ represents the set of female types that the male type x men can hope to meet. We sometimes refer to $\phi_m(x)$ as type x men's "match set." We allow the possibility that male type x is excluded by the monopolist matchmaker, with $\phi(x) = \emptyset$. Define ϕ_w similarly, and denote $\phi = \langle \phi_m, \phi_w \rangle$. For any $X \subseteq [a_m, b_m]$, let

$$\Phi_m(X) = \{y | y \in \phi_m(x) \text{ for some } x \in X\}$$

represent the union of match sets of male types in X. Define Φ_w similarly.

Definition 2.1 A sorting structure ϕ is feasible if for any $x, \tilde{x} \in [a_m, b_m]$, $y, \tilde{y} \in [a_w, b_w]$, $X \subseteq [a_m, b_m]$ and $Y \subseteq [a_w, b_w]$, i) $y \in \phi_m(x)$ implies $x \in \phi_w(y)$, and $x \in \phi_w(y)$ implies $y \in \phi_m(x)$; ii) $\phi_m(x) \neq \phi_m(\tilde{x})$ implies $\phi_m(x) \cap \phi_m(\tilde{x}) = \emptyset$, and $\phi_w(y) \neq \phi_w(\tilde{y})$ implies $\phi_w(y) \cap \phi_w(\tilde{y}) = \emptyset$; and iii) $\Phi_m(X)$ has the same measure as $\{x | \phi_m(x) \subseteq \Phi_m(X)\}$, and $\Phi_w(Y)$ has the same measure as $\{y | \phi_w(y) \subseteq \Phi_w(Y)\}$.

Condition (i) is analogous to the standard symmetry condition for matching correspondences. It states that if type x men are participating in a meeting place where there are type y women, then type y women are participating in a meeting place where there are type x men, and vice versa. This condition is needed for a meeting place to have the interpretation of a matching market. Condition (ii) requires that each type participates in at most one meeting place. This simplifies the analysis. Condition (iii) requires that each meeting place consists of men and women of equal measures. This ensures that match probability is one for each agent in any meeting place, and helps us minimize the role of search technologies and focus on the impact of revenue-maximization on the sorting structure and matching efficiency.

3 Weak sorting

The monopolist's problem is to choose a sorting structure and the corresponding two fee schedules, one for males and one for females. A sorting structure assigns to each male type a set of potential female matches and to female types a set of potential male partners. The design problem appears multi-dimensional because what a type buys from the matchmaker is a type distribution on the other side of the meeting place. However, the assumption of a multiplicatively separable match value function allows us to reduce the problem to one dimension. Our first step of analysis is to substitute a pair of expected match types for each meeting place in the design problem, and transform the market design problem to a more familiar price discrimination problem.

A feasible sorting structure ϕ generates two schedules of expected match types, q_m and q_w. The function $q_m : [a_m, b_m] \to [a_w, b_w] \cup \{0\}$ assigns to each male type the expected value of his match; the function $q_w : [a_w, b_w] \to [a_m, b_m] \cup \{0\}$ is the corresponding function for female types. We refer to $q = \langle q_m, q_w \rangle$ as a pair of "quality schedules," given by[6]

$$q_m(x) = \mathrm{E}[y|y \in \phi_m(x)]; \quad q_w(y) = \mathrm{E}[x|x \in \phi_w(y)] \tag{1}$$

for all $x \in [a_m, b_m]$ such that $\phi_m(x) \neq \emptyset$ and $y \in [a_w, b_w]$ such that $\phi_w(y) \neq \emptyset$. We adopt the convention that if any type is excluded by the matchmaker, the match quality assignment is 0, which is the reservation utility. The lemmas in the remainder of this section refer to types that are served by the monopoly matchmaker. With the convention we have adopted, the lemmas can be easily restated to cover the excluded types.

As in a price discrimination problem, the monopolist does not observe agent types and must rely on self-selection of agents into their assigned expected match quality.[7] Given equations (1), we can now formally state the optimal mechanism design problem of the matchmaker. Let $p_m(x)$ be the participation fee for type x and define $p_w(y)$ similarly; denote $p = \langle p_m, p_w \rangle$. The monopolist chooses a feasible sorting structure ϕ and a pair of fee schedules p to maximize the revenue

$$\int_{a_m}^{b_m} p_m(x)\, dF(x) + \int_{a_w}^{b_w} p_w(y)\, dG(y),$$

subject to incentive compatibility constraints

$$xq_m(x) - p_m(x) \geq xq_m(\tilde{x}) - p_m(\tilde{x}); \quad yq_w(y) - p_w(y) \geq yq_w(\tilde{y}) - p_w(\tilde{y})$$

for all $x, \tilde{x} \in [a_m, b_m]$ and $y, \tilde{y} \in [a_w, b_w]$ respectively, and participation constraints

$$xq_m(x) - p_m(x) \geq 0; \quad yq_w(y) - p_w(y) \geq 0$$

for all $x \in [a_m, b_m]$ and $y \in [a_w, b_w]$ respectively, where q is given in (1).

[6] Without the restrictions of types participating in at most one meeting place, ϕ would not be sufficient to define q_m and q_w and we would need additional notation to specify the fraction of agents of a given type who participate in any given meeting place.

[7] Since the matching market is two-sided, self-selection involves a coordination problem that is absent in a standard price discrimination problem. We ignore such problem in this paper by assuming that the monopoly matchmaker can decide how agents self-select so long as the sorting structure is feasible and incentive compatible. See our companion paper (Damiano et al. 2004) for a discussion of how to resolve the coordination problem.

Under the complementarity assumed in the match value function, standard arguments imply that q_m being nondecreasing is necessary for the incentive compatibility constraints for men to be satisfied (see, e.g., Maskin and Riley 1984). Further, the associated indirect utility $U_m(x)$ of male type x, defined as

$$U_m(x) = xq_m(x) - p_m(x),$$

satisfies the envelope condition

$$U'_m(x) = q_m(x). \tag{2}$$

at every x such that $q_m(x)$ is continuous. Finally, condition (2) and the monotonicity of q_m together are sufficient for incentive compatibility. Similar observations hold for monotonicity of q_w and the indirect utility function U_w of women.

Unlike in a typical price discrimination problem, the monopolist can only choose schedules q_m and q_w consistent with some feasible sorting structure. Through a series of lemmas, whose proofs can be found in the Appendix, we show how the feasibility constraints on the sorting structure translate into direct restrictions on quality schedules. Monotonicity of any schedules q leads to the following definition.[8]

Definition 3.1 An interval $T_m \subseteq [a_m, b_m]$ is a maximal pooling interval under q_m if q_m is constant on T_m, and there is no interval $T'_m \supset T_m$ such that q_m is constant on T'_m.

Maximal pooling intervals T_w under q_w can be similarly defined. We say that $q = \langle q_m, q_w \rangle$ is "feasible" if there is a feasible $\phi = \langle \phi_m, \phi_w \rangle$ such that equations (1) are satisfied for almost all x and y. We call ϕ the "associated" sorting structure.

Lemma 3.2 *If q is feasible, then for any maximal pooling interval T_m under q_m and any associated sorting structure ϕ, $\Phi_m(T_m)$ is a maximal pooling interval under q_w.*

By symmetry, if a pair of nondecreasing schedules q is feasible, then $\Phi_w(T_w)$ is a maximal pooling interval under q_m for any maximal pooling interval T_w under q_w and any associated sorting structure ϕ. A corollary of Lemma 3.2 is thus $\Phi_w(\Phi_m(T_m)) = T_m$, and symmetrically $\Phi_m(\Phi_w(T_w)) = T_w$. Another implication is that for any associated sorting structure ϕ, and for any maximal pooling interval T_m under q_m, we have $q_m(x) = \mathrm{E}[y|y \in \Phi_m(x)]$ for all $x \in T_m$.[9] Symmetrically, for any maximal pooling interval T_w under q_w and for any $y \in T_w$, we have $q_w(y) = \mathrm{E}[x|x \in \Phi_w(T_w)]$.

Lemma 3.2 is the first step in showing that a pair of nondecreasing, feasible schedules q defines two sequences $\{T^l_m\}^L_{l=1}$ and $\{T^l_w\}^L_{l=1}$ of maximal pooling intervals in $[a_m, b_m]$ and $[a_w, b_w]$ respectively, with $T^l_w = \Phi_m(T^l_m)$ and $T^l_m = \Phi_w(T^l_w)$

[8] There is no need to specify whether a maximal pooling interval contains the two end points. The assignment of values of q_m and q_w to the end points does not affect the revenue function stated later in Proposition 3.6.

[9] In general, it is not true that $\phi_m(x) = \Phi_m(T_m)$ for all $x \in T_m$, as there can be more than one way of assigning match sets for x in T_m such that $q_m(x)$ is constant. However, by condition (iii) of Definition 2.1, we have $\int_{T_m} \mathrm{E}[y|y \in \phi_m(x)]\, dF(x)(F(\sup(T_m) - F(\inf(T_m)))\mathrm{E}[y|y \in \Phi_m(T_m)]$. Since $\mathrm{E}[y|y \in \phi_m(x)]$ equals $q_m(x)$ and is constant on T_m, it follows that $q_m(x) = \mathrm{E}[y|y \in \Phi_m(T_m)]$ for all $x \in T_m$.

for each l. The next step is to identify the end points of each maximal pooling interval.

Lemma 3.3 *If q is feasible, then for any maximal pooling interval T_m under q_m and any associated sorting structure ϕ, $s_m(\inf T_m) = \inf \Phi_m(T_m)$ and $s_m(\sup T_m) = \sup \Phi_m(T_m)$.*

Lemmas 3.2 and 3.3 completely characterize a nondecreasing, feasible q for x and y in maximal pooling intervals. It remains to characterize $q_m(x)$ and $q_w(y)$ for any x and y not in a maximal pooling interval, respectively.

Lemma 3.4 *If q is feasible, then $q_m(x) = s_m(x)$ for any $x \in [a_m, b_m]$ such that x does not belong to any maximal pooling interval under q_m.*

The following proposition summarizes the feasibility restrictions on incentive compatible quality schedules that we have derived from the restrictions imposed on feasible sorting structure (Definition 2.1). For any $a_m \le x < \tilde{x} \le b_m$, let

$$\mu_m(x, \tilde{x}) = \mathrm{E}[t | x \le t \le \tilde{x}]$$

be the mean male type on the interval $[x, \tilde{x}]$. Define $\mu_w(y, \tilde{y})$ similarly.

Proposition 3.5 *A pair of nondecreasing quality schedules $\langle q_m, q_w \rangle$ is feasible if and only if i) for any maximal pooling interval T_m under q_m and each $x \in T_m$, $q_m(x) = \mu_w(s_m(\inf T_m), s_m(\sup T_m))$ and $q_w(s_m(x)) = \mu_m(\inf T_m, \sup T_m)$; and ii) for any x not in any maximal pooling interval under q_m, $q_m(x) = s_m(x)$ and $q_w(s_m(x)) = x$.*

Proof Necessity of (i) and (ii) follow immediately from Lemmas 3.2–3.4. For sufficiency, fix any q that is nondecreasing and feasible. For each maximal pooling interval T_m under q_m, construct the set-valued function ϕ_m such that $\phi_m(x) = [s_m(\inf T_m), s_m(\sup T_m)]$ for any x in the closure of T_m, and correspondingly ϕ_w such that $\phi_w(y) = [\inf T_m, \sup T_m]$ for any $y \in [s_m(\inf T_m), s_m(\sup T_m)]$. For all other x, let $\phi_m(x) = \{s_m(x)\}$ and $\phi_w(s_m(x)) = \{x\}$. By conditions i) and ii) stated in the proposition, $\phi_m(x)$ and $\phi_w(y)$ are well-defined for all $x \in [a_m, b_m]$ and $y \in [a_w, b_w]$ respectively, and further, ϕ_m and ϕ_w satisfy equations (1) for almost all x and y. Thus, $\langle q_m, q_w \rangle$ is feasible. □

The above result can be viewed as a characterization of any feasible sorting structure associated with an incentive compatible, feasible pair of quality schedules. We refer to the characterization as "weak sorting." Since meeting places are mutually exclusive in type, if two types on the same side of the market participate in two different meeting places, the higher type not only has a higher average match type, but never gets a lower match.

We have completed transforming the design problem from choosing a feasible and incentive compatible sorting structure ϕ_m and ϕ_w to a problem of choosing a pair of nondecreasing quality schedules that satisfy Proposition 3.5. The advantage of this transformation is that from the mechanism design literature we know how to manipulate the incentive compatibility and individual rationality constraints associated with one-dimensional schedules q to rewrite the matchmaker's revenue. Define

$$J_m(x) = x - \frac{1 - F(x)}{f(x)}, \quad J_w(y) = y - \frac{1 - G(y)}{g(y)}$$

to be the "virtual type" for male type x and female type y, respectively. Virtual type of x takes into account the incentive cost of eliciting private type information from type x. These are familiar definitions from the standard price discrimination literature (e.g., Myerson 1981). Next, we combine the virtual types and define

$$K(x, y) = x J_w(y) + y J_m(x) \tag{3}$$

as the "virtual match value" for male type x and female type y. Virtual match value of types x and y is based on the match value between x and y with proper adjustment of the incentive costs of eliciting truthful information from the two types.

For the following proposition, note that for any q that is nondecreasing and feasible, there are at most countable many maximal pooling intervals. This is because for any maximal pooling interval T_m, the quality schedule q_m is discontinuous at $\inf\ T_m$ (unless $\inf T_m = a_m$) and $\sup T_m$ (unless $\sup T_m = b_m$). Since q_m is monotone, it can only have a countable number of discontinuities. Let L be the total number of maximal pooling intervals under q_m; note that we allow L to be infinite.

Proposition 3.6 *Fix a pair of nondecreasing and feasible quality schedules $\langle q_m, q_w \rangle$. Define $c_m = \inf\{x : q_m(x) > 0\}$, and let $\{T_m^l\}_{l=1}^L$ be the collection of all maximal pooling interval under q_m over $[c_m, b_m]$. The maximum revenue generated by $\langle q_m, q_w \rangle$ is*

$$\int_{[c_m, b_m] \setminus (\cup_{l=1}^L T_m^l)} K(x, s_m(x))\, \mathrm{d}F(x)$$

$$+ \sum_{l=1}^{L} \int_{\inf T_m^l}^{\sup T_m^l} \int_{s_m(\inf T_m^l)}^{s_m(\sup T_m^l)} \frac{K(x, y)\, \mathrm{d}G(y)\, \mathrm{d}F(x)}{F(\sup T_m^l) - F(\inf T_m^l)}. \tag{4}$$

Proof Incentive compatibility and feasibility of the quality schedules imply that the monopoly matchmaker's exclusion policy takes the form of a cutoff male type $c_m \in [a_m, b_m]$ such that male types $x < c_m$ and female types $y < s_m(c_m)$ are excluded. Using the definition of the indirect utility function U_m, we can write the total revenue from the male side as

$$\int_{c_m}^{b_m} (x q_m(x) - U_m(x))\, \mathrm{d}F(x).$$

After integration by parts we can use (2) and the definition of virtual type function J_m to rewrite the revenue from the male side as

$$-U_m(c_m)(1 - F(c_m)) + \int_{c_m}^{b_m} q_m(x) J_m(x)\, \mathrm{d}F(x). \tag{5}$$

The revenue from the female side can be similarly stated. The cutoff types c_m and $s_m(c_m)$ receive their reservation utility of 0 in any optimal price discrimination mechanism for the monopolist. The revenue formula (4) in Proposition 3.6 then follows from equation (3) and the characterization result of Proposition 3.5. □

Proposition 3.6 restates the original sorting structure design problem given at the beginning of this section as choosing quality schedules q. We note that there are two components in the restated maximization problem: one is the exclusion policy or choosing c_m, and the other is the optimal sorting problem for a given c_m. By assumption, the match value of any pair of types is positive and the reservation utility of each type is zero, implying that a social planner that maximizes the total match value will implement full market coverage. In contrast, the virtual match value of a pair of types need not be positive, and so the monopolist may find it optimal to exclude some types. The focus of the present paper is when the revenue-maximizing sorting structure is the perfect sorting, which can be studied independently of the exclusion policy. The conditions for the perfect sorting to be revenue-maximizing, derived in the next section, are a characterization of the optimal sorting problem for any given exclusion policy.

The objective function of the optimal sorting problem is given by the revenue formula (4). The formula contains two terms, corresponding respectively to the revenue from the types that are perfectly sorted and the revenue from a sequence of pooling regions. Note that the quality schedule q does not appear explicitly in the revenue formula; by Proposition 3.5, the feasibility constraint on q, together with the incentive compatibility constraint, has already pinned down q once the sequence of maximal pooling intervals $\{T_m^l\}_{l=1}^L$ is given. Thus, the monopolist's optimal sorting problem is reduced to choosing the break points of the maximal pooling intervals.[10] We can think of the monopolist partitioning the set of serviced male types $[c_m, b_m]$ into a sequence of pooling intervals and sorting intervals, with the set of serviced female types correspondingly partitioned. Since the revenue is written as sum of revenues from these intervals in the formula of Proposition 3.6, whether it is optimal to pool or to sort the types in one particular interval can be determined in isolation. This feature will be repeatedly used in the next section.

4 Perfect Sorting

Proposition 3.5 establishes weak sorting as the outcome of satisfying both the incentive compatibility constraint in self-selection and the feasibility constraint on the sorting structure. Weak sorting can take different forms, from pooling the entire population of agents into a single meeting place to segregating each type into separate meeting places. Due to the assumption of complementarity in the match

[10] By definition two sorting intervals cannot be adjacent to each other. However, it is possible, and may even be optimal, to have two pooling intervals adjacent to each other.

value function xy, the finer the agents are partitioned into separate meeting places, the higher the matching efficiency in terms of the total match value.[11] The question is whether the monopoly's revenue is also increased.

In this section, we use the revenue formula of Proposition 3.6 to study the implications of revenue maximization to the sorting structure and matching efficiency. We are particularly interested in the perfect sorting structure. If the perfect sorting maximizes the monopolist's revenue, then the monopolist has the same incentive to create meeting places as a social planner who maximizes the total match value. In this case, the incentive cost of eliciting private type information generates no distortion in terms of match quality provision. This is in contrast with the standard price discrimination result that quality is under-provided for all types but the very highest (Mussa and Rosen 1978; Maskin and Riley 1984). The standard result is commonly explained in terms of the tradeoff between "efficiency" and rent extraction: efficiency for a given type means a quality level that maximizes the trade surplus, defined as the type's utility of consuming the quality less the cost of producing it, but the price function that implements the efficient quality schedule leaves too much rent to types. This tradeoff is resolved by a downward distortion of the quality level for all types except the highest. Indeed, in the standard model the efficient quality schedule is never profit-maximizing, because marginally lowering the qualities while maintaining full separation of types will have a first order effect on information rent and only a second order effect on efficiency, and will thus increase the revenue. This argument does not work in our model where quality distortion can only be achieved by pooling types due to the feasibility constraints (equations 1) on the quality schedules. The choice of the sorting structure uniquely determines the monopolist's quality decisions, and hence the efficiency and the rent. Whether or not the monopolist optimally chooses the efficient quality schedules is equivalent to whether or not the perfect sorting is the solution to the monopolist's optimal sorting problem. First, we provide a necessary condition.

Proposition 4.1 *Suppose that a pair of quality schedules q is optimal and $q_m(x) = s_m(x)$ for all $x \in (x_1, x_2)$. Then, for all $x \in (x_1, x_2)$*

$$K_{12}(x, s_m(x)) = J_m'(x) + J_w'(s_m(x)) \geq 0. \tag{6}$$

Proof Fix some $x \in (x_1, x_2)$ and a small positive ϵ. Construct a new pair of quality schedules $q(\epsilon)$ by pooling the male types on the interval $[x, x+\epsilon]$ with the female types on the corresponding interval $[s_m(x), s_m(x+\epsilon)]$, while retaining the sorting structure outside the region $[x, x+\epsilon] \times [s_m(x), s_m(x+\epsilon)]$ and the quality schedules. Let $\Delta_x(\epsilon)$ be the revenue difference between the original quality schedules and $q(\epsilon)$. We note that $q(\epsilon)$ is nondecreasing by construction, and feasible because it satisfies Proposition 3.5. Thus, we can apply the revenue formula of Proposition

[11] Although this statement is intuitively obvious, we are not aware of a direct proof in the existing literature, except that McAfee (2002) shows that a relatively large efficiency gain can be made by optimally splitting one market into two. Proposition 4.4 below provides a general argument for the efficiency gain, if we replace K with the match value function xy in the proof of the proposition and note that by assumption the match value function satisfies supermodularity.

3.6 and write $\Delta_x(\epsilon)$ as:

$$\int_x^{x+\epsilon} K(t, s_m(t))\,\mathrm{d}F(t) - \int_x^{x+\epsilon} \int_{s_m(x)}^{s_m(x+\epsilon)} K(t, y) \frac{\mathrm{d}G(y)\,\mathrm{d}F(t)}{F(x+\epsilon) - F(x)}.$$

Consider the behavior of $\Delta_x(\epsilon)$ for $\epsilon \to 0$. Clearly, we have $\Delta_x(0) = 0$. Straightforward but tedious calculations[12] reveal that $\Delta'_x(0) = \Delta''_x(0) = 0$, while $\Delta'''_x(0) = \frac{1}{2}(J'_m(x) + J'_w(s_m(x))) f(x) s'_m(x)$. If (6) is not satisfied at x, then there exists an $\epsilon > 0$ such that the monopoly matchmaker can increase revenue by pooling male types in $[x, x+\epsilon]$ and corresponding female types in $[s_m(x), s_m(x+\epsilon)]$ into a single meeting place, instead of perfectly sorting these types. □

In a simple price-quality discrimination problem, where the trade surplus equals the product of the quality and the type less the cost of producing the quality, one presumes full separation of types, drops the monotonicity constraint on the quality schedule, and chooses a quality level to maximize the "virtual surplus" for each type, which is the trade surplus with the virtual type in place of the type. This pointwise maximization method cannot be applied in our mechanism design problem. This is because match qualities are not produced according to some exogenous cost function: in equation (5), pointwise maximization would imply exclusion (i.e. $q_m(x) = 0$) for any type x with negative virtual type $J_m(x)$, and unbounded quality if $J_m(x)$ is positive. Instead, quality provision for the two sides of the market is simultaneously determined by the choice of sorting structure through Proposition 3.5. This allows us to use the local approach to identify the necessary condition for the perfect sorting to be optimal.[13] Note that when the two sides of the market have the same type distributions, with $s_m(x) = x$, condition (6) boils down to the virtual type function being nondecreasing (although monotonicity of the virtual types on both male and female sides is clearly not required with asymmetric distributions.) In the simple price discrimination problem, monotonicity of the virtual type is necessary for a strictly increasing quality schedule to be optimal. Although our conclusion coincides with the standard monotonicity condition in the special case of identical type distributions for the two sides, the logic is different between the two models. In the price discrimination problem, monotonicity of the virtual type is equivalent to monotonicity of the solution to the pointwise maximization problem, and is necessary for the solution to be incentive compatible. In contrast, the necessity of the local condition (6) follows from a variational argument over the revenue formula (equation 4), which respects the feasibility constraint as well as the monotonicity constraint on the quality schedules.

The local necessary condition (6) in Proposition 4.1 does not impose any constraint on the behavior of the virtual match value function away from a small neighborhood of the efficient matching path. As a result, it fails to ensure that a greater revenue cannot be generated by pooling a large set of types. A sufficient condition for the perfect sorting to be optimal is that the virtual match value function

[12] The details are available from the authors upon request.

[13] Bergemann and Pesendorfer (2001) use similar techniques to answer the question of how much private information an auctioneer should allow the bidder to learn about his valuation. The analogy between our sorting structure design problem and theirs can be seen if one thinks of a partition element in an information structure in their paper as a pooling of types in our problem.

is supermodular on the entire type space, that is,

$$K_{12}(x, y) = J'_m(x) + J'_w(y) \geq 0 \tag{7}$$

for all $x \in (c_m, b_m)$ and $y \in (c_w, b_w)$.[14] If F and G are drawn from the large class of distribution functions that satisfy the standard condition of non-decreasing hazard rate, then K is supermodular.

Proposition 4.2 *If K satisfies (7) for any $x \in (x_1, x_2)$ and $y \in (s_m(x_1), s_m(x_2))$ with strict inequalities, then any nondecreasing, feasible pair of quality schedules with $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$ as the interior of a maximal pooling region is non-optimal.*

Proof Let $\hat{q} = \langle \hat{q}_m, \hat{q}_w \rangle$ be a pair of nondecreasing and feasible quality schedules, with $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$ as the interior of a maximal pooling region. Construct a pair of quality schedule $q^* = \langle q^*_m, q^*_w \rangle$ such that: i) q^*_m and q^*_w are identical to $\hat{q}_m$ and $\hat{q}_w$ outside the maximal pooling intervals that contain (x_1, x_2) and $(s_m(x_1), s_m(x_2))$, respectively; and ii) $q^*_m(x) = s_m(x)$ and $q^*_w(s_m(x)) = x$ for any x in the maximal pooling intervals that contain (x_1, x_2) and $(s_m(x_1), s_m(x_2))$, respectively. By construction, q^* is nondecreasing and feasible. Let Δ denote the revenue difference between q^* and $\hat{q}$, given by

$$\Delta = \int_{x_1}^{x_2} K(x, s_m(x))\, \mathrm{d}F(x) - \int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} K(x, y) \frac{\mathrm{d}G(y)\, \mathrm{d}F(x)}{F(x_2) - F(x_1)}. \tag{8}$$

The first term on the right-hand-side of the above can be also written as

$$\int_{x_1}^{x_2} K(x, s_m(x))\, \mathrm{d}F(x) = \int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} K(x, s_m(x)) \frac{\mathrm{d}G(y)\, \mathrm{d}F(x)}{F(x_2) - F(x_1)}.$$

With a change of variable $y = s_m(x)$, we can also write

$$\int_{x_1}^{x_2} K(x, s_m(x))\, \mathrm{d}F(x) = \int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} K(s_m^{-1}(y), y) \frac{\mathrm{d}G(y)\, \mathrm{d}F(x)}{F(x_2) - F(x_1)}.$$

In a similar way, after two changes of variable $x = s_m^{-1}(\tilde{y})$ and $y = s_m(\tilde{x})$, the second term on the right-hand-side of (8) can be written as

$$\int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} K(x, y) \frac{dG(y)\, dF(x)}{F(x_2) - F(x_1)} = \int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} K(s_m^{-1}(y), s_m(x)) \frac{dG(y)\, dF(x)}{F(x_2) - F(x_1)}.$$

Hence, Δ is equal to one-half of

[14] The proof of Proposition 4.2 below remains valid if $K(x, s_m(x)) + K(\tilde{x}, s_m(\tilde{x})) \geq K(x, s_m(\tilde{x})) + K(\tilde{x}, s_m(x))$ for all $x, \tilde{x} \in (x_1, x_2)$. Therefore, this weaker condition is also sufficient for the perfect sorting to be optimal. When the virtual match value function K is twice differentiable, supermodularity of K requires $\min_x J'_m(x) + \min_y J'_w(y) \geq 0$, while the weaker condition implies $\min_x J'_m(x) + J'_w(s_m(x)) \geq 0$. If the efficient matching path $s_m(x)$ is linear, including the special case where F and G are identical so that $s_m(x) = x$, then the minimum of $J'_m(x)$ and the minimum of $J'_w(y)$ are achieved at a point (x, y) on the efficient matching path and therefore supermodularity and the weaker condition coincide. A linear efficient matching path occurs only when the distribution of type on one side of the market is the same as the distribution of a linear transformation of type on the other side. In general K can satisfy the weaker condition but fail to be supermodular. Examples with simple distribution functions that illustrate the difference between the two concepts are available upon request.

$$\int_{x_1}^{x_2} \int_{s_m(x_1)}^{s_m(x_2)} \big(K(x, s_m(x)) + K(s_m^{-1}(y), y) - K(x, y) - \\ K(s_m^{-1}(y), s_m(x))\big) \frac{dG(y)\, dF(x)}{F(x_2) - F(x_1)},$$

which is strictly positive because K satisfies (7) with strict inequalities. □

The idea of the proof comes from the revenue formula in Proposition 3.6. The revenue from perfectly sorting the types in the region $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$ is the integral of the virtual match value function K along the segment of the efficient matching path $\{(x, s_m(x)) | x \in (x_1, x_2)\}$, while the revenue from pooling the types is the integral of K over the entire region. By changes of variables we can write the revenue difference as the integral of a function which has positive values because K is supermodular. Proposition 4.2 immediately implies a sufficient condition for the perfect sorting to be optimal.

Corollary 4.3 *If K satisfies (7) for any $x \in (c_m, b_m)$ and $y \in (c_w, b_w)$, then the perfect sorting is optimal.*

Proposition 4.2 suggests that when the virtual match value function K satisfies (7) for all x and y in some region $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$, breaking up the region into sufficiently many small pooling regions generates more revenue than pooling all types in the region together. The next proposition establishes that under supermodularity of K we in fact have a stronger result that every division of the market into meeting places increases the revenue. This is useful in practice because it implies that setting up a new meeting place always strictly increases revenue.

Proposition 4.4 *Let q^* be a pair of nondecreasing, feasible quality schedules with $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$ as the interior of a maximal pooling region. If K is strictly supermodular on $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$, then for any $t \in (x_1, x_2)$, any pair of nondecreasing, feasible quality schedules $\hat{q}$ such that $\hat{q}$ is identical to q^* outside $[x_1, x_2] \times [s_m(x_1), s_m(x_2)]$ and $\hat{q}$ has $(x_1, t) \times (s_m(x_1), s_m(t))$ and $(t, x_2) \times (s_m(t), s_m(x_2))$ as the interiors of two maximal pooling regions generates strictly more revenue than q^*.*

Proof Let the revenue difference between $\hat{q}$ and q^* be Δ. Using the revenue formula from Proposition 3.6 we can show that Δ is proportional to

$$\int_{x_1}^{t} \int_{s_m(x_1)}^{s_m(t)} K(x, y)\, dF_l(x)\, dG_l(y) + \int_{t}^{x_2} \int_{s_m(t)}^{s_m(x_2)} K(x, y)\, dF_h(x)\, dG_h(y) \\ - \int_{x_1}^{t} \int_{s_m(t)}^{s_m(x_2)} K(x, y)\, dF_l(x)\, dG_h(y) - \int_{t}^{x_2} \int_{s_m(x_1)}^{s_m(t)} K(x, y)\, dF_h(x)\, dG_l(y), \quad (9)$$

where for any $x \in (x_1, t)$ and $\tilde{x} \in (t, x_2)$

$$F_l(x) = \frac{F(x)}{F(t) - F(x_1)}, \quad F_h(\tilde{x}) = \frac{F(\tilde{x})}{F(x_2) - F(t)},$$

and for $y \in (s_m(x_1), s_m(t))$ and any $\tilde{y} \in (s_m(t), s_m(x_2))$

$$G_l(y) = \frac{G(y)}{F(t) - F(x_1)}, \quad G_h(\tilde{y}) = \frac{G(\tilde{y})}{F(x_2) - F(t)}.$$

Next, apply the change of variables $F_h(x) = F_l(\tilde{x})$ to x in the second integral and in the fourth integral, and $G_h(y) = G_l(\tilde{y})$ to y in the second integral and in the third integral. Then, Δ is proportional to the double integral of

$$K(x, y)+K(F_h^{-1}(F_l(x)), G_h^{-1}(G_l(y)))-K(x, G_h^{-1}(G_l(y)))-K(F_h^{-1}(F_l(x)), y),$$

which is strictly positive because $F_h^{-1}(F_l(x)) > x$, $G_h^{-1}(G_l(y)) > y$ and K is strictly supermodular on $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$. □

The revenue difference between sorting the types in $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$ into two meeting places and pooling all types in the region, is the integral of a function with x varying between x_1 and t and correspondingly y between $s_m(x_1)$ and $s_m(t)$. This function has positive values due to supermodularity of K. Note that other changes of variables would also work. For example, one can define new integration variables by setting $F_h(\tilde{x}) = 1 - F_l(x)$ and $G_h(\tilde{y}) = 1 - G_l(y)$. The proof of the proposition proceeds in a similar fashion; the only change is that the revenue difference is the integral of a different function, which is still positively valued due to supermodularity of K.

Equation (9) implies that a necessary condition for $(x_1, x_2) \times (s_m(x_1), s_m(x_2))$ to be the interior of a maximal pooling region is that there does not exist a point $(t, s_m(t))$ on the efficient matching path contained in the region such that the virtual match value function is "on average" supermodular at the point. An implication of this result is that if the matchmaker can create at least two meeting places, it would never be optimal to pool all men and women into a single market. This follows because regardless of the type distributions, the virtual type functions J_m and J_w eventually become increasing towards the end of the efficient matching path and reach their respective maximum at the end. This in turn implies that there is always a point $(t, s_m(t))$ such that the virtual type functions satisfy $\min_{x \geq t} J_m(x) \geq \max_{x \leq t} J_m(x)$ and $\min_{y \geq s_m(t)} J_w(y) \geq \max_{y \leq s_m(t)} J_w(y)$, and therefore the virtual match value function is supermodular on average at $(t, s_m(t))$. At this point it would increase the revenue to split the market into two pools.

To conclude this section, we explicitly calculate the fee schedules under the perfect sorting. Using condition (2) and the perfect sorting condition $q_m(x) = s_m(x)$, we have

$$p_m(x) = x s_m(x) - \int_{c_m}^{x} s_m(t)\, dt.$$

A similar expression holds for the female fee schedule p_w. Note that p_m and p_w are continuous. This property holds only when the perfect sorting is optimal. In general any pooling will make the quality schedule discontinuous. Since the indirect utility functions are necessarily continuous, the fee schedules will be discontinuous at the boundaries of each maximal pooling region. Finally, the sum of revenues from a pair of male and female types on the efficient sorting path is

$$p_m(x) + p_w(s_m(x)) = 2x s_m(x) - U_m(x) - U_w(s_m(x)).$$

This implies that the rate of increase of the sum of revenues is

$$p'_m(x) + p'_w(s_m(x))s'_m(x) = s_m(x) + xs'_m(x),$$

which is one-half of the rate of increase of the sum of match values $2xs_m(x)$. Thus, even though it is optimal for the monopoly matchmaker to implement the socially efficient sorting structure, the monopolist cannot implement perfect price discrimination and does not extract all the surplus.

5 Discussions

So far we have considered conditions for the perfect sorting to be optimal under two substantive assumptions about the reservation utility. First, we have assumed that the reservation utility is the same for the two sides of the market. This assumption can be easily dispensed without affecting our results. Given any exclusion policy (i.e., cutoff types c_m and c_w such that $c_w = s_m(c_m)$), the solution to the optimal sorting problem is independent of the reservation utilities, because the only change to the objective function (the revenue formula 4) is the addition of two constant terms $-\underline{U}_m(1 - F(c_m))$ and $-\underline{U}_w(1 - G(c_w))$, where $\underline{U}_m$ and $\underline{U}_w$ are the reservation utility for men and for women, respectively. Thus, the conditions for the perfect sorting to be optimal will not change.[15]

The second assumption is that the reservation utility is type independent. However, higher types may have better outside options. This can be captured by assuming that men and women excluded from the monopolist's mechanism can randomly match among each other for free. In this case the reservation utility of a type is the type's expected payoff from joining the free pool, and is endogenously determined by the exclusion policy of the monopolist. Under any feasible, incentive compatible market structure, the types that participate in the free pool are determined by a cutoff rule, with only men and women below the respective cutoff types participating in the free pool. This is because the free pool corresponds to a participation fee of zero, so it cannot be optimal for the monopolist to create a meeting place with a quality lower than the quality of the free pool. Further, as in the case of exogenous type-independent reservation utility, the fees for the types served by the matchmaker are determined by the usual incentive compatibility constraints, rather than by the participation constraint that these types have to get as much utility from the matchmaker as from the free pool, even though higher types receive more utility from the free pool. This latter claim follows from the fact the indirect utility of a type x above the cutoff increases at the rate of $q_m(x)$ (equation 2), while the utility from the free pool increases at the rate of the conditional mean of female types below the cutoff, which is lower than $q_m(x)$. Thus, for any exclusion policy or a pair of cutoff types c_m and c_w, the introduction of the free pool (with the utility for unmatched agents remaining zero) changes the objective function (the revenue formula 4) by adding two constants $-c_m\mu_w(a_w, c_w)(1 - F(c_m))$ and $-c_w\mu_m(a_m, c_m)(1 - G(c_w))$. This means that the solution to the optimal sorting

[15] Asymmetric reservation utilities will in general change the optimal exclusion policy. For example it might be optimal to charge a negative price to the cutoff type on the side with a higher reservation utility in order to induce greater participation and extract more rent from the other side.

problem does not change as a result of endogenous reservation utility, and the conditions for the perfect sorting to be optimal remain unchanged.[16]

An important assumption in our model is that the match value function is multiplicatively separable. Without this assumption, the payoff to an agent from a random pairwise matching in a meeting place generally depends on the entire type distribution of participants from the other side. This means that the monopolist problem of designing the sorting structure ϕ cannot be reduced to be a one-dimensional problem of choosing quality schedules q. In place of equations (1), the monopolist has to choose a pair of "match schedules" α_m and α_w, with $\alpha_m(x)$ representing the distribution of female types on the match set $\phi_m(x)$ for male type x. The key to the weak sorting result of Proposition 3.5 is the monotonicity condition on the quality schedule, but there is no counterpart to this ordering with a non-separable match value function because $\alpha_m(x)$ is a multi-dimensional object. Thus, we cannot further reduce the monopolist problem of designing the sorting structure to choosing the break points of maximal pooling intervals. However, if one is willing to assume weak sorting, then we can derive an analogous expression for the revenue formula of Proposition 3.6, and identify necessary and sufficient conditions for the perfect sorting to be optimal in the same way as in Section 4.

An assumption complementary to multiplicative separability of the match value function is that agents are randomly matched within each meeting place. Without the assumption of random matching, the expected quality of a match in a meeting place may be type dependent and may depend on the entire distribution of types in the meeting place. In this case, the expected payoffs from joining a meeting place would not be multiplicatively separable even if the match value function is, and this would create the same kind of analytical difficulties as discussed earlier. For example, if instead of one round of random matching we have sequential search as in Burdett and Coles (1997) or in Damiano et al. (2005), the expected match quality for any type in a meeting place depends on which "class" the type belongs to. Moreover, the class structure is endogenously determined by the type distributions in the meeting place. How to incorporate sequential search into the framework of price discrimination is an interesting and challenging topic that deserves further research.

6 Appendix

Proof of Lemma 3.2 Suppose $\Phi_m(T_m)$ is not a maximal pooling interval under q_w. There are two cases.

Case 1 Suppose that q_w is not constant on $\Phi_m(T_m)$. Then, we can find $y, \tilde{y} \in \Phi_m(T_m)$ such that $q_w(y) < q_w(\tilde{y})$. It follows from condition ii) in Definition 2.1 that $\phi_w(y) \cap \phi_w(\tilde{y}) = \emptyset$ and $\Phi_m(\phi_w(y)) \cap \Phi_m(\phi_w(\tilde{y})) = \emptyset$. Since T_m is a maximal pooling interval and $y, \tilde{y} \in \Phi_m(T_m)$, we have $\mathrm{E}[t|t \in \Phi_m(\phi_w(y))] =$

[16] Endogenous reservation utilities will affect the monopolist's exclusion policy. For example, when the type distributions are symmetric and the common virtual type function $J(t)$ crosses zero only once, one can show that a free pool forces the matchmaker to increase market coverage. This follows because to counter the competition by the free pool, the matchmaker needs to admit more types at the bottom of the distribution so as to reduce the outside option for the participating types.

$\mathrm{E}\left[t | t \in \Phi_m(\phi_w(\tilde{y}))\right]$, which is possible only if $\inf \Phi_m(\phi_w(\tilde{y})) < \sup \Phi_m(\phi_w(y))$. Then, there exist $y_1 \in \Phi_m(\phi_w(y))$ and $\tilde{y}_1 \in \Phi_m(\phi_w(\tilde{y}))$ such that $y_1 > \tilde{y}_1$. It follows that $q_w(y_1) = q_w(y) < q_w(\tilde{y}) = q_w(\tilde{y}_1)$, which contradicts the assumption that q_w is nondecreasing.

Case 2 Suppose that there is a $W \supset \Phi_m(T_m)$ such that q_w is constant on W. By a symmetric argument as in Case 1, we can show that q_m is constant on $\Phi_w(W)$. Since $W \supset \Phi_m(T_m)$, we can write $\Phi_w(W) = \Phi_w(\Phi_m(T_m)) \cup \Phi_w(W \backslash \Phi_m(T_m))$. We claim that $\Phi_w(\Phi_m(T_m)) \supseteq T_m$: if $x \in T_m$, then there exists $y \in \Phi_m(T_m)$ such that $y \in \phi_m(x)$, which by condition i) of Definition 2.1 implies that $x \in \phi_w(y)$, and therefore $x \in \Phi_w(\Phi_m(T_m))$. Further, $\Phi_w(W \setminus \Phi_m(T_m)) \neq \emptyset$ because $W \supset \Phi_m(T_m)$, and q_w is constant and different from 0 on W. Finally, $\Phi_w(W \setminus \Phi_m(T_m)) \cap T_m = \emptyset$, because $y \notin \Phi_m(T_m)$ implies that $\phi_w(y) \cap T_m \neq \emptyset$ by condition i) of Definition 2.1. It follows that $\Phi_w(W) \supset T_m$. Since q_m is constant over $\Phi_w(W)$, we have reached a contradiction to the assumption that T_m is a maximal pooling interval under q_m. □

Proof of Lemma 3.3 We first establish that if $x < \inf T_m$ then $\sup \phi_m(x) \leq \inf \Phi_m(T_m)$. To prove this claim by contradiction, suppose that there exists $y > \inf \Phi_m(T_m)$ such that $y \in \phi_m(x)$. Since T_m is a maximal pooling interval and $x \notin T_m$, we have $\phi_m(x) \cap \Phi_m(T_m) = \emptyset$. By Lemma 3.2, $\Phi_m(T_m)$ is an interval and hence $y \geq \sup \Phi_m(T_m)$. If $\inf \phi_m(x) \geq \sup \Phi_m(T_m)$, then since $q_m(x) = \mathrm{E}[y | y \in \phi_m(x)]$ and $x < \inf T_m$, we have a contradiction to the assumption that q_m is nondecreasing. If instead $\inf \phi_m(x) \leq \inf \Phi_m(T_m)$, then there exists $\tilde{y} \in \phi_m(x)$ such that $\tilde{y} \leq \inf \Phi_m(T_m)$. By condition ii) of Definition 2.1 and the definition of q_w we have $q_w(y) = q_w(\tilde{y})$. Monotonicity of q_w implies that q_w is constant on $[\tilde{y}, y] \supset \Phi_m(T_m)$ therefore $\Phi_m(T_m)$ is not a maximal pooling interval under q_w, contradicting Lemma 3.2.

It follows from the above claim that $\phi_m(x) \subseteq \Phi_m([a_m, \inf T_m))$ for any $x < \inf T_m$, and hence $[a_w, \inf \Phi_m(T_m)] \supseteq \Phi_m([a_m, \inf T_m))$. Thus, $G(\inf \Phi_m(T_m)) \geq \int_{\Phi_m([a_m, \inf T_m))} dG$. By condition iii) of Definition 2.1, $\int_{\Phi_m([a_m, \inf T_m))} dG = \int_{\{x | \phi_m(x) \subseteq \Phi_m([a_m, \inf T_m))\}} dF$, which implies that $G(\inf \Phi_m(T_m)) \geq F(\inf T_m)$.

By a symmetric argument, we have $\sup \phi_w(y) \leq \inf T_m$ for any $y < \inf \Phi_m(T_m)$. Hence, $[a_m, \inf T_m] \supseteq \Phi_w([a_w, \inf \Phi_m(T_m)))$ and $F(\inf T_m) \geq G(\inf \Phi_m(T_m))$. Then, we have $G(\inf \Phi_m(T_m)) = F(\inf T_m)$, which by the definition of s_m implies that $s_m(\inf T_m) = \inf \Phi_m(T_m)$. The argument for $s_m(\sup T_m) = \sup \Phi_m(T_m)$ is symmetric. □

Proof of Lemma 3.4 Fix any sorting structure ϕ associated with q. We first show that, if x does not belong to any maximal pooling interval, $\phi_m(x)$ is a singleton. Suppose $y, \tilde{y} \in \phi_m(x)$ for some $y < \tilde{y}$. By condition ii) of Definition 2.1, $\phi_w(y) = \phi_w(\tilde{y})$, and $q_w(y) = q_w(\tilde{y})$. Since q_w is monotone, it must be constant on the interval $[y, \tilde{y}]$. Therefore, there exists a maximal pooling interval $W \supseteq [y, \tilde{y}]$. By construction, x belongs to $\Phi_w(W)$ which is a maximal pooling interval by Lemma 3.2; a contradiction.

Let $\phi_m(x) = \{y_x\}$. Since $q_m(x) = \mathrm{E}[y | y \in \phi_m(x)]$, we can write $q_m(x) = y_x$. By monotonicity of q_m, if $\tilde{x} < x$ and $\tilde{x}$ does not belong to a maximal pooling interval then $y_{\tilde{x}} \leq y_x$ where $\phi_m(\tilde{x}) = \{y_{\tilde{x}}\}$. Together with Lemma 3.3, this implies that $\Phi_m[a_m, x] \subseteq [a_w, \phi_m(x)]$. Clearly, y_x does not belong to a maximal pooling interval under ϕ_w and $\phi_w(y_x) = \{x\}$, so an identical argument yields $\Phi_w[a_w, y_x] \subseteq$

$[a_m, x]$. Then, by condition iii) in Definition 2.1, we have $F(x) = G(y_x)$, or $q_m(x) = y_x = s_m(x)$. □

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