Informative Voting in Large Elections

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Abstract. Recounting introduces multiple pivotal events in two-candidate elections. In addition to determining which candidate is elected, an individual's vote is pivotal when the vote margin is just at the levels that would trigger a recount. In large elections, the motive to avoid recount cost can become the dominant consideration for rational voters, inducing them to vote informatively according to their private signals. In environments where elections without recount fail to aggregate information efficiently, a modified election rule with small recount cost can produce asymptotically efficient outcomes with a vanishing small probability of actually invoking a recount. In environments where efficient information aggregation obtains in elections without recount, a modified election rule with recount cost can increase the speed at which the information efficient outcome is approximated.

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1. Introduction

More than two centuries ago Condorcet (1875) first articulated the idea that voting groups with diverse information about their alternatives make a better choice the larger the group size. This celebrated Condorcet jury theorem is a statistical proposition based on an early application of the law of large numbers. It is an important result that gives confidence to our belief that large elections can resolve conflicts due to dispersed information and produce good collective decisions.

Although intuitively appealing, the presumed sincere-voting behavior by the electorate has been re-examined by economists who study this topic. Austen-Smith and Banks (1996) first point out how *informative voting*, that is voting according to one's own private signal, is generally inconsistent with rationality (see also Feddersen and Pesendorfer, 1996). Since a non-pivotal vote does not affect the outcome and is thus payoff-irrelevant, rational voting behavior requires conditioning one's vote on the information inferred from the vote being pivotal as well as on one's own private information. In a large election, the information inferred from being pivotal can overwhelm one's own private information. Thus, informative voting generally fails in a large election except for a small fraction of informed voters.[1]

The failure of informative voting notwithstanding, Feddersen and Pesendorfer (1997) show that in a large two-candidate election the outcome is information-efficient in the sense that almost surely it would remain the same even if all the private information about the candidates became common knowledge. Under any election rule, the outcome in a large election would be determined by the corresponding decisive voter's preference if the private information were perfectly aggregated. For example, under the simple majority rule, the decisive voter has the median preference in the electorate. Similarly, under strategic voting, votes are cast as if the election is close and the decisive voter is indifferent between the two candidates. Even though the fraction of voters whose vote depends on their private signals is small in a large election, their number goes to infinity. It is these voters that determine the election outcome, ensuring that the outcome is information efficient.

[1] In a two-candidate election model with a continuous payoff state, Feddersen and Pesendorfer (1997) show that the fraction of agents who vote informatively vanishes as the election size becomes arbitrarily large.

Nevertheless, there are environments in which information efficiency fails under strategic voting even though it could obtain had all informed voters voted informatively. One such environment involves "aggregate uncertainty," where there are partisan voters who randomly split their votes between the two candidates, resulting in uncertainty in realized vote shares even when the number of voters becomes arbitrarily large (Feddersen and Pesendorfer, 1997). Another environment involves conflicting preferences, where the same change in the public belief about a candidate can increase his appeal to some voters but lower his appeal to other voters (Bhattacharya, 2013).

In this paper, we resurrect informative voting as an equilibrium strategy in large two-candidate elections by introducing other pivotal events in addition to the standard one that determines the eventual winner. Although there are many ways to introduce additional pivotal events, we adopt a model of costly recounting. An election rule in this model is characterized by three thresholds of vote shares for a given candidate and a positive recount cost. If the vote share for the candidate exceeds the largest threshold then that candidate is declared an outright winner; and symmetrically, if the vote share falls below the smallest threshold then the the opposing candidate is declared an outright winner. If the vote share falls between the smallest and the largest thresholds, a "recount" takes place after each voter incurs the recount cost. The candidate is declared the winner if the vote share upon recount is above the middle threshold, and the opposing candidate wins otherwise. We study information aggregation in an environment with finitely many states and conditionally independent private signals. We explicitly model the presence of aggregate uncertainty, while we leave the description of strategic voters preferences as general as possible, to include the broadest set of environments including models such as Bhattacharya 2013's where preferences are non-monotone in the state. Aggregate uncertainty is modeled by the presence of non-strategic uninformed voters (who do not receive private signals); the fraction of uninformed voters voting for a given candidate remains random even in large elections.

Our main result establishes that, whenever the efficient information aggregation is possible, it is achieved, asymptotically, by a sequence of equilibria with recounting and every agent vote informatively. In our model, corresponding to the middle threshold is the standard pivotal event that votes for the two candidates are tied. Costly recounting creates two additional pivotal events: corresponding to the largest threshold is the pivotal event when one more vote for the given candidate would make him an outright winner and one more vote for the opponent would trigger a costly recounting but would not change the winner, and corresponding to the smallest threshold is the symmetric pivotal event. Although the probabilities of the three pivotal events conditional on the state all vanish in the limit, one of them becomes dominant because its conditional probability goes to zero at the slowest rate. This is a consequence of the theory of large deviations, which studies the limit behavior of rare events.[2] In our equilibrium construction, the pivotal events where a recounting can be triggered always dominate the third. At these pivotal events, the desire to save the recounting cost is the only motive, thus voters incentives are entirely aligned. They each vote for one candidate or the other depending on which of the two pivotal events is more likely, as a function of their private signal.

Not only our election rule with recounting aggregate information efficiently whenever that is possible, which include environments where a standard election rule would fail to do so. Furthermore, we show that the probability of recounting and thus incurring the cost in equilibrium is negligible in large elections, thus the improvement in information aggregation is achieved at no cost.

Finally, we show that in environments where a standard election without recounting aggregates information efficiently, recounting still improves the outcome by increasing the speed at which the information efficient outcome is approximated.

2. A Model of Elections with Recounting

We study an election with a large number $n+1$ of voters to choose between two candidates: $\mathcal{R}$ and $\mathcal{L}$. Denote the share of votes for $\mathcal{R}$ as V. An "election rule" consists of three thresholds $v_{\mathcal{L}}$, $v_{\mathcal{C}}$ and $v_{\mathcal{R}}$, satisfying $v_{\mathcal{L}} < v_{\mathcal{C}} < v_{\mathcal{R}}$, and specifies:

1. candidate $\mathcal{R}$ is elected if $V > v_{\mathcal{R}}$;
2. candidate $\mathcal{L}$ is elected if $V < v_{\mathcal{L}}$;
3. a "recount" is triggered at an additive payoff loss of $\delta > 0$ to each voter if $V \in [v_{\mathcal{L}}, v_{\mathcal{R}}]$; and after the recount, candidate $\mathcal{R}$ is elected if $V \geq v_{\mathcal{C}}$ and candidate $\mathcal{L}$ is elected otherwise.

[2] See, for example, Dembo and Zeitouni (1998) for a textbook treatment.

Note that a standard election rule without recounting can be represented as a special case of election rules defined above, with $\delta = 0$. We assume that there is no error in the initial vote count stage or in the the recount stage. Therefore the vote share for $\mathcal{R}$ in the recount stage will be exactly the same as that recorded in the initial count. We do not consider unanimity rules; both $v_{\mathcal{R}}$ and $v_{\mathcal{L}}$ are assumed to be strictly between 0 and 1.

Voters are independently drawn from a large population of potential voters. A fraction $1 - \alpha \in (0, 1]$ of potential voters are informed voters; the rest are uninformed. There is a finite number, M, of payoff relevant states of the world $S = \{s_1, \ldots, s_M\} \in 0, 1^M$. States are ordered with $0 \leq s_1 < \cdots < s_M \leq 1$, and the voters' common prior beliefs over S is described by the distribution $\mu = (\mu_1, \ldots, \mu_M)$, with μ_i being the probability the state is s_i. Each informed voter observes a conditionally independent signal $\sigma \in \Sigma = \{\sigma_1, \ldots, \sigma_J\}$ informative of the realized state. The signal's conditional distributions $\beta(\cdot|s)$, satisfy the property (MLRP)

Assumption 1.

$$\frac{\beta(\sigma|s)}{\beta(\sigma'|s)} > \frac{\beta(\sigma|s')}{\beta(\sigma'|s')} \quad \textit{for all} \quad \sigma > \sigma' \textit{ and } s > s'. \tag{1}$$

An immediate implication of property (1) is that the posterior distributions over S after observing a signal realization, $\{\mu^{\sigma}\}_{\sigma \in \Sigma}$, are ordered with respect to first order stochastic dominance, so that the higher the signal observed, the more an informed agent revises his expectation about the realized state upward.

Uninformed voters are introduced to preserve uncertainty about the realized vote share in each given state s in large elections. They are non-strategic; a fraction θ of them vote for candidate $\mathcal{R}$ and the remaining fraction $1 - \theta$ vote for $\mathcal{L}$.[3] The fraction θ is a random variable distributed on $[\underline{\theta}, \overline{\theta}] \subseteq [0, 1]$, with a continuous and positive density function f and corresponding distribution function F. The aggregate uncertainty state θ is independent of the payoff state s.

Informed voters are heterogeneous with respect to a preference type $t \in T$, and the heterogeneity of preferences types among the population of potential informed

[3]The uninformed voters are partisan in the sense that they have preferences between the two candidates that cannot be swayed by any evidence. Otherwise, they may optimally choose to abstain from voting. See Feddersen and Pesendorfer (1996).

voters is described by a probability measure P over T. The payoff to an informed voter depends on the outcome of the election, the realized payoff relevant state and her preference type t. A payoff function

$$\pi : \{\mathcal{L}\mathcal{R}\} \times \mathcal{S} \times \mathcal{T}\mathcal{B}\lfloor\lfloor\mathcal{R},$$

describes the payoff, without a recount, to an informed voter as a function of the candidate elected, the realized payoff relevant state, and the voter's type. The voter's payoff is reduced by δ if the same election outcome is achieved after a recount. We make the following joint assumption over the payoff function and preference types:

Assumption 2. *The payoff difference function $u(s,t) \equiv \pi(\mathcal{R},s,t) - \pi(\mathcal{L},s,t)$, and the distribution of preference types P satisfy*

$$P(\{t \in T \,|u(s,t) > 0\}) > P(\{t \in T \,\big|u(s',t) > 0\}) \quad \textit{for all} \quad s > s'.$$

Under the above assumption, in a large election with only informed voters and perfect information, the unique equilibrium outcome in un-dominated strategies would be monotone in the state. That is, the assumption is a requirement that the full-information outcome has a "threshold" structure, with the winner changing at most once as a function of the realized payoff relevant state. A sufficient condition for Assumption 2 is the commonly used requirement (e.g. Federsen and Pesendorfer (1997)) of "state-monotone preferences" that u is increasing in s. Monotone preferences are not necessary and Assumption 2 is also satified, for example, in Bhattacharya (2013)'s model where, for a majority of voters, u is increasing in s and for a minority of voters the opposite is true.

2.1. Strategy and equilibrium

For a given n, we consider a voting game with $n+1$ voters, Γ^n, described by: i) the election rule $\{v_{\mathcal{L}}, v_{\mathcal{C}}, v_{\mathcal{R}}\}$ and δ; ii) the payoff relevant states S, the preference type space T and the payoff function $u : S \times T \to \mathbb{R}$; and iii) the prior belief μ over S, the probability measure P over T, the distribution function over the aggregate uncertainty state F, as well as the set of signals Σ and the conditional probability distributions over Σ, $\{\beta(\cdot|s)\}_{s\in S}$. These are all common knowledge. Ultimately we are interested voting games with large n, so we will ignore all integer problems. The solution concept is Bayesian Nash equilibrium and we restrict attention to symmetric equilibria.

Fix some informed voter. Denote as v the number of votes for $\mathcal{R}$ from all voters other than this voter, divided by the total number of votes n from these voters. Let $g^n(\cdot|s)$ represent the probability distribution of v in state s. The function $g^n(\cdot|s)$ is derived from the strategies adopted by all other voters. Since no voter observes the identity of the other n voters, and the payoff state s is independent of the uncertainty state θ, the function $g^n(\cdot|s)$ depends neither on the preference type t nor on the private signal σ observed by the informed voter. Further, the presence of uninformed voters guarantees that $g^n(\cdot|\cdot)$ is a strictly positive function.

Upon observing a private signal realization $\sigma \in \Sigma$, an informed voter's private belief that the state is s becomes

$$\mu^\sigma(s) = \frac{\mu(s)\beta(\sigma|s)}{\mu(s)\beta(\sigma|s) + \sum_{s' \neq s} \mu(s')\beta(\sigma|s')}.$$

There are three events in which his vote is pivotal:

1. $v = v_{\mathcal{L}}$: Regardless of the state, voting $\mathcal{R}$ instead of $\mathcal{L}$ triggers a recount, incurring a cost of δ.
2. $v = v_{\mathcal{C}}$: Voting $\mathcal{R}$ instead of $\mathcal{L}$ tilts the election outcome (after the recount) to $\mathcal{R}$. In state s this changes the voter's payoff by $u(s,t)$.
3. $v = v_{\mathcal{R}}$: Regardless of the state, voting $\mathcal{R}$ instead of $\mathcal{L}$ determines the outcome of the election immediately, saving the recount cost δ.

Therefore, upon observing a private signal σ, voting for $\mathcal{R}$ yields a larger expected payoff than voting for $\mathcal{L}$ if

$$\sum_{s \in S} \mu^\sigma(s) \left(g^n(v_{\mathcal{R}}|s)\delta + g^n(v_{\mathcal{C}}|s)u(s,t) - g^n(v_{\mathcal{L}}|s)\delta \right) \geq 0 \tag{2}$$

A strategy profile must describe the probability the an informed voter casts a vote in favor of $\mathcal{R}$ as a function of both her preference type t and the realization of her private signal σ. Thus a strategy profile is a function

$$k : T \times \Sigma \to [0,1].$$

Definition 1 (BNE). *A Nash equilibrium of the Bayesian game Γ^n is a strategy profile k^n*

such that, for all $t \in T$ and $\sigma \in \Sigma$

$$\left(\sum_{s\in S}\mu^{\sigma}(s)\left(g^{n}(v_{\mathcal{R}}|s;\kappa^{n})\delta+g^{n}(v_{\mathcal{C}}|s;\kappa^{n})u(s,t)-g^{n}(v_{\mathcal{L}}|s;\kappa^{n})\delta\right)\right)k^{n}(t,\sigma)\geq 0 \quad \textit{and}$$
$$\left(\sum_{s\in S}\mu^{\sigma}(s)\left(g^{n}(v_{\mathcal{R}}|s;\kappa^{n})\delta+g^{n}(v_{\mathcal{C}}|s;\kappa^{n})u(s,t)-g^{n}(v_{\mathcal{L}}|s;\kappa^{n})\delta\right)\right)(1-k^{n}(t,\sigma))\leq 0.$$

The above definition simply states the mixed-strategy Nash equilibrium requirement that a strategic voter must cast with probability one a vote that yields a strictly larger expected payoff, where expectations are taken with respect to the probability distributions $g^n(\cdot|\cdot;k^n)$ obtained from the primitive of the game and the equilibrium strategy profile k^n.

More structure on the preference types and payoff functions would yield additional properties of the equilibrium strategy profile. For example, when T is an interval of the real line and $u(s,t)$ strictly increasing in t, as in Federsen and Pesendorfer (1997), an equilibrium strategy profile must have an ordered "threshold" structure and can be described by a set of thresholds $\{k_\sigma\}_{\sigma\in\Sigma}$, with all types $t > k_\sigma$ voting for $\mathcal{R}$ and all types $t < k_\sigma$ voting for $\mathcal{L}$ after observing a private signal σ. In Bhattacharya (2013) the type space has two dimensions $T = [0,1] \times \{M, m\}$. The utility function of the majority voters $u(s, x, M)$ is increasing in x, while the utility function of minority voters, $u(s, x, m)$ is decreasing in x. In this case two sets of thresholds, $\{k^M_\sigma, k^m_\sigma\}_{\sigma\in\Sigma}$, are sufficient to describe the equilibrium behavior of majority and minority voters respectively.

As for the two examples above, in most economic applications the type space T, and its associated probability space, as well as the payoff function $u(\cdot,\cdot)$, will naturally have additional structure. Our results do not depend on the properties of T or u, so we only impose the minimal requirement (which is already implicit in Assumption 2) that the function $u(s,\cdot)$ is T-measurable for each state $s \in S$. We also restrict the strategy space to those functions k such that $k(\cdot,\sigma)$ is T-measurable for each σ, so that the integral of $k(\cdot,\sigma)$ with respect to the probability measure P is well defined. For convenience of notation we denote such integral as

$$H(\sigma;\kappa) \equiv \int k(\cdot,\sigma)\mathrm{d}P.$$

Given a strategy profile k, the function $H(\cdot;\kappa)$ describes the probability that a randomly drawn informed voter casts a vote for $\mathcal{R}$ as a function of the private signal σ she observes.

For any strategy profile κ, we are now ready to define with $z(s,\theta;\kappa)$ the probability that a randomly drawn voter casts a vote for candidate $\mathcal{R}$ in the payoff state s and aggregate uncertainty state θ. This is given by

$$z(s,\theta;\kappa) = (1-\alpha)\sum_{\sigma\in\Sigma} H(\sigma;\kappa)\beta(\sigma|s) + \alpha\theta. \tag{3}$$

We will refer to $z(s,\theta;\kappa)$ as the vote share for candidate $\mathcal{R}$ in state (s,θ) given the strategy profile κ.

Given $z(s,\theta;\kappa)$, from the perspective of each individual informed voter, the probability of a vote share v for candidate $\mathcal{R}$ conditional on the payoff state s and the aggregate uncertainty state θ is then given by

$$g^n(v|s,\theta;\kappa) = \binom{n}{nv} z(s,\theta;\kappa)^{nv}(1-z(s,\theta;\kappa))^{n(1-v)}, \tag{4}$$

and thus

$$g^n(v|s;\kappa) = \int_{\underline{\theta}}^{\overline{\theta}} g^n(v|s,\theta;\kappa)f(\theta)\,\mathrm{d}\theta. \tag{5}$$

To avoid dealing with integer problems, when using the above expression of $g^n(\cdot|s;\kappa)$ in Definition 1 to derive the explicit constraints for a strategy profile k^n to be a Bayesian Nash equilibrium of the game Γ^n, we implicitly assume that $nv_{\mathcal{L}}, nv_{\mathcal{C}}$ and $nv_{\mathcal{R}}$ are integers. Given our focus on the limit case for n large, it is equivalent to assuming that the voting rule thresholds are rational.

2.2. Ranking of pivotal events

Fix a strategy profile κ and the implied vote share functions $z(s,\theta;\kappa^n)$. In a large election, the probability that the actual vote share equals a particular value v is vanishingly small. A key observation of this paper is that the *rates* at which the probabilities of different pivotal events go to zero are different, so that in the limit some pivotal events are infinitely more likely to occur than others. Calculating the rate of convergence is therefore an important part of the analysis of large elections with multiple pivotal events.

If voters knew the aggregate uncertainty state θ, then the probability that the vote share equals v is given by the binomial probability $g^n(v|s,\theta;\kappa)$ in equation (4). Using Stirling's approximation formula for the binomial coefficient, we have

$$g^n(v|s,\theta;\kappa) = \frac{\phi_v^n}{\sqrt{2\pi v(1-v)n}} I\left(v; z(S,\theta;\kappa)\right)^n, \tag{6}$$

where

$$I(v;z) = \left(\frac{z}{v}\right)^v \left(\frac{1-z}{1-v}\right)^{1-v},$$

and $\lim_{n\to\infty}\phi_v^n = 1$. The function $-\log I(v;z)$ is known as the "rate function" or "entropy function" in the theory of large deviations. It determines the rate at which the probability $g^n(v|s,\theta;\kappa)$ goes to zero. In particular, if there are two events v and v' such that $I(v;z(s,\theta;\kappa)) > I(v';z(s,\theta;\kappa))$, then

$$\lim_{n\to\infty} \frac{g^n(v'|s,\theta;\kappa)}{g^n(v|s,\theta;k)} = \left(\frac{I(v';z(s,\theta;\kappa))}{I(v;z(s,\theta;\kappa))}\right)^n = 0.$$

In our model an informed voter does not know the aggregate uncertainty state θ. The probability that she assigns to a certain pivotal event v to occur in the payoff relevant state s, $g^n(v|s;k)$ is the integral of $g^n(v|s,\theta;k)$ over all possible aggregate uncertainty states. The following lemma shows that in determining the rate of convergence, only the θ which maximizes the function $I(v;z(s,\theta;\kappa))$ matters.

Lemma 1. *Let $\theta(v,s) \equiv \arg\max_{\theta\in[\underline{\theta},\overline{\theta}]} I(v;z(s,\theta;\kappa))$. For any v, v' and any two payoff relevant states s, s',*

$$\lim_{n\to\infty} \frac{g^n(v|s;\kappa)}{g^n(v'|s';\kappa)} = \frac{f(\theta(v,s))}{f(\theta(v',s'))} \lim_{n\to\infty} \frac{g^n(v|s,\theta(v,s);\kappa)}{g^n(v'|s',\theta(v',s');\kappa)}.$$

Proof. The function $I(v;z)$ is increasing in z for $z < v$ and decreasing in z for $z > v$, attaining a maximum at $z = v$. Since $z(s,\theta;\kappa^n)$ is strictly increasing in θ, $I(v;z(s,\theta;\kappa^n)$ is decreasing in θ for $\theta < \theta(v,s)$ and increasing in θ for $\theta > \theta(v,s)$. Let $B_\epsilon(v,s) \subset [\underline{\theta},\overline{\theta}]$ be a small interval of width ϵ that contains $\theta(v,s)$. Specifically, if $\theta(v,s) = \underline{\theta}$, choose $B_\epsilon(v,s) = [\underline{\theta},\overline{b})$ where $\overline{b} = \underline{\theta} + \epsilon$; and if $\theta(v,s) = \overline{\theta}$, choose $B_\epsilon(v,s) = (\underline{b},\overline{\theta}]$ where $\underline{b} = \overline{\theta} - \epsilon$. If $\theta(v,s)$ is interior, choose $B_\epsilon(v,s) = (\underline{b},\overline{b})$ such that $\overline{b} - \underline{b} = \epsilon$ and $I(v;z(s,\underline{b};\kappa^n)) = I(v;z(s,\overline{b};\kappa^n))$. Denote $B^c_\epsilon(v,s) = [\underline{\theta},\overline{\theta}] \backslash B_\epsilon(v,s)$

to be the complement of $B_\epsilon(v,s)$. Note that $I(v; z(s,\theta;\kappa^n)) > I(v; z(s,\theta';\kappa^n))$ for any $\theta \in B_\epsilon(v,s)$ and $\theta' \in B^c_\epsilon(v,s)$.

For any pivotal event v and any state s, we have

$$\int_{B^c_\epsilon(v,S)} g^n(v|s,\theta) f(\theta)\,\mathrm{d}\theta < g^n(v|s,\theta'_n)\Pr[\theta \in B^c_\epsilon(v,s)],$$

where θ'_n is equal to $\underline{b}$ or $\overline{b}$.

Continuity of $g^n(v|s,\cdot)$ also implies that there is a unique $\hat{\theta}_n \in B_\epsilon(v,s)$ such that

$$\int_{B_\epsilon(v,S)} g^n(v|s,\theta) f(\theta)\,\mathrm{d}\theta = g^n(v|s,\hat{\theta}_n)\Pr[\theta \in B_\epsilon(v,s)].$$

We further claim that $\lim_{n\to\infty} \hat{\theta}_n = \theta(v,s)$. To see this, note that by definition

$$\lim_{n\to\infty} \int_{B_\epsilon(v,s)} \frac{g^n(v|s,\theta)}{g^n(v|s,\hat{\theta}_n)} f(\theta)\,\mathrm{d}\theta = \Pr[\theta \in B_\epsilon(v,s)],$$

which is only possible if $\hat{\theta}_n$ converges to $\theta(v,s)$ because from the fact that $\theta(v,s)$ maximizes $I(v; z(s,\theta;\kappa^n))$, we must have $\lim_{n\to\infty} g^n(v|s,\theta)/g^n(v|s,\theta(v,s)) = 0$ for all $\theta \neq \theta(v,S)$.

From the two conditions above, we obtain that for any ϵ positive,

$$\lim_{n\to\infty} \frac{\int_{B^c_\epsilon(v,S)} g^n(v|s,\theta) f(\theta)\,\mathrm{d}\theta}{\int_{B_\epsilon(v,S)} g^n(v|s,\theta) f(\theta)\,\mathrm{d}\theta} \leq \lim_{n\to\infty} \frac{g^n(v|s,\theta'_n)\Pr[\theta \in B^c_\epsilon(v,s)])}{g^n(v|s,\hat{\theta}_n)\Pr[\theta \in B_\epsilon(v,s)]} = 0, \tag{7}$$

where the equality follows because $\lim_{n\to\infty} g^n(v|s,\theta')/g^n(v|s,\theta) = 0$ whenever $\theta' \in B^c_\epsilon(v,s)$ and $\theta \in B_\epsilon(v,s)$, and because $\hat{\theta}_n$ is bounded away from θ'_n.

For any v, v' and s, s',

$$\lim_{n\to\infty} \frac{g^n(v|s)}{g^n(v'|s')} = \lim_{n\to\infty} \frac{\int_{\underline{\theta}}^{\overline{\theta}} g^n(v|s,\theta) f(\theta)\,\mathrm{d}\theta}{\int_{\underline{\theta}}^{\overline{\theta}} g^n(v'|s',\theta) f(\theta)\,\mathrm{d}\theta} = \lim_{n\to\infty} \frac{\int_{B_\epsilon(v,S)} g^n(v|s,\theta) f(\theta)\,\mathrm{d}\theta}{\int_{B_\epsilon(v',S')} g^n(v'|s',\theta) f(\theta)\,\mathrm{d}\theta},$$

where the last equality follows from (7). The above holds for any ϵ positive and thus

$$\lim_{n\to\infty} \frac{g^n(v|s)}{g^n(v'|s')} = \lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{\int_{B_\epsilon(v,s)} g^n(v|s,\theta) f(\theta)\,\mathrm{d}\theta}{\int_{B_\epsilon(v',s')} g^n(v'|s',\theta) f(\theta)\,\mathrm{d}\theta}.$$

Reversing the limit order and calculating the inner limit using l'Hopital's rule, we obtain:

$$\lim_{n\to\infty} \frac{g^n(v|s)}{g^n(v'|s)} = \lim_{n\to\infty} \frac{g^n(v|s,\theta(v,s)) f(\theta(v,s))}{g^n(v'|s,\theta(v',s')) f(\theta(v',s'))}. \qquad \blacksquare$$

Lemma 1 implies that given a sequence of strategy profiles $\{k_n\}_{n=1,2,\ldots}$, for any pair of pivotal events v, v' and any pair of payoff states s, s', the ratio $g^n(v|s;\kappa_n)/g^n(v'|s';\kappa_n)$ can have a limit different from zero or infinity only if

$$\lim_{n\to\infty} I\left(v; z(s,\theta(v,s);\kappa_n)\right) = \lim_{n\to\infty} I\left(v'; z(s',\theta(v',s');\kappa_n)\right). \tag{8}$$

We refer to the above property as the "equal-rate condition." Furthermore, since $I(v;z)$ is increasing in z for $z < v$ and decreasing in z for $z > v$, and since $z(s,\theta;\kappa)$ is increasing in θ, we have

$$\theta(v,s) = \arg\min_{\theta\in[\underline{\theta},\overline{\theta}]} |z(s,\theta;\kappa) - v| .$$

For example, when $z(s,\overline{\theta};\kappa) < v \le v' < z(s',\underline{\theta};\kappa)$, then $\theta(v,s) = \overline{\theta}$ and $\theta(v',s') = \underline{\theta}$. In this case, the ratio $g^n(v|s;k_n)/g^n(v'|s';k_n)$ can have a limit different from zero or infinity only if $I(v;z(s,\overline{\theta};\kappa_n)) - I(v';z(s',\underline{\theta};\kappa_n))$ converges to zero.

The following definition is a useful comparison of the rates at which two pivotal events vanish in a given payoff relevant state.

Definition 2. *Given a sequence of strategy profiles $\{\kappa_n\}$, a pivotal event v* **dominates** *another pivotal event v' in state s, if*

$$\lim_{n\to\infty} \frac{g^n(v'|s;k_n)}{g^n(v|s;k_n)} = 0$$

From Lemma 1, $\mathrm{I}(v;z(s,\theta(v,s);\kappa_n)) - I\left(v';z(s,\theta(v',s);\kappa_n\right) > \epsilon > 0$ for all n sufficiently large suffices for v to dominate v' in state s.

3. Information efficient equilibria

The central question of the paper is whether recounting can help achieve the same outcome as in an election with just informed voters and common knowledge of the payoff relevant state. In such an election and without recounting (i.e. $\delta = 0$), by Assumption 1 there is some $s^* \in \{1,\ldots,M+1\}$ such that candidate $\mathcal{R}$ would receive a share larger that v_C of informed votes for all payoff relevant states $s \ge s^*$, and would fail to do so, leading to candidate $\mathcal{L}$ being elected, for $s < s^*$. We further assume that $2 \le s^* \le M$, so that the full information outcome is not common knowledge, and efficiently aggregating the voters information is necessary to achieve it.[4] The full information outcome selects, in each state s, candidate $\mathcal{R}$ if a $v_{\mathcal{C}}$-majority of voter favors it in state s, and $\mathcal{L}$ otherwise.[5] The following definition of "full information equivalence," adapted from Feddersen and Pesendorfer (1997), reflects the presence of aggregate uncertainty in our model, and requires that the election outcome is not affected by the aggregate uncertainty state realization.

Definition 3. *A sequence of strategy profiles achieves "full information equivalence" if for all $\epsilon > 0$, there is an N such that for $n > N$ and for any realization of the uncertainty state, candidate $\mathcal{L}$ is chosen with probability greater than $1 - \epsilon$ when the payoff relevant state is $s < s^*$, and candidate $\mathcal{R}$ is chosen with probability greater than $1 - \epsilon$ if $s \geq s^*$.*

In the presence of aggregate uncertainty, full information equivalence might not be possible. While the aggregate information available to informed voters would always be sufficient to identify the payoff relevant state in a large election, the noise introduced in the voting outcome by the behavior of uninformed voters (modeled as the aggregate uncertainty state), might be large enough that no sequence of strategy profiles ever satisfy the conditions in Definition 3. The following definition describes a necessary and sufficient condition for full information equivalence to be possible. Its failure implies that no electoral rule can ever yield the outcome preferred by the $v_{\mathcal{C}}$-majority of informed voter for every payoff relevant state.

Definition 4. *For a given electoral rule, full information equivalence is "achievable" if there exists a strategy profile k such that*

$$z(s, \overline{\theta}; k) < v_{\mathcal{C}} < z(s', \underline{\theta}; k) \quad \text{for all } s < s^* \text{ and } s' \geq s^*. \tag{9}$$

The share of uninformed votes for $\mathcal{R}$ is largest in the aggregate uncertainty state $\overline{\theta}$ and smallest in the aggregate uncertainty state $\underline{\theta}$. Full information equivalence requires that voting by informed voters generates a sufficiently large spread of $\mathcal{R}$'s vote share between "high states" (i.e. $s \geq s^*$) and "low states" (i.e. $s < s^*$), so that the election outcome is determined by informed voters only and not the aggregate

[4]The first part of the Proposition 1 remains valid for $s^* \in \{1, M+1\}$, but the equilibrium construction does not satisfy (11).

[5]When preferences are monotone in the preference type as well as in the state, as in Feddersen and Pesendorfer (1997), this coincide with the outcome preferred by the $v_{\mathcal{C}}$-median voter.

uncertainty state. Note the (9) can only be satisfied if a non negligible measure of types vote "informatively" i.e. change their vote as their private signal varies. Finally, it is worth remarking that whether full information equivalence is achievable depends jointly on the outcome-determining threshold $v_{\mathcal{C}}$ together with the informativeness of the agents signals and the distribution of aggregate uncertainty. It does not depend, however, on whether the election has recounting (i.e. $\delta > 0$), or the values of the recounting thresholds $v_{\mathcal{L}}$ adn $v_{\mathcal{R}}$.

Next we introduce a class of strategy profiles that are "monotone in signals," and we show that whether a monotone strategy profile can distinguish between the threshold state s^* and its immediately preceding states s^*_- is a sufficient test for the achievability of full information equivalence.

Definition 5. *A strategy profile k is "monotone" if, for all $t \in T$,*

$$k(t,\sigma) > 0 \Rightarrow k(t,\sigma') = 1 \quad \text{for all } \sigma' > \sigma; \text{ and}$$
$$k(t,\sigma) < 1 \Rightarrow k(t,\sigma') = 0 \quad \text{for all } \sigma' < \sigma.$$

Given a monotone strategy profile, each type randomizes its vote for at most one signal realization and the probability of casting a vote for $\mathcal{R}$ is non-decreasing in the signal realization. The following lemma provides the intuitive result that full information equivalence is achievable if and only if it is achievable in monotone strategies.

Lemma 2. *For any electoral rule, full information equivalence is achievable if and only if there exists a monotone strategy profile k such that*

$$z(s^*_-,\overline{\theta};k) < v_{\mathcal{C}} < z(s^*,\underline{\theta};k). \tag{10}$$

Proof. The fact that (10) is sufficient is immediate after noting that, by Assumption 1, $z(s,\theta,k)$ is increasing in the payoff relevant state for every monotone strategy profile k. The only if part of the statement follows because any strategy profile k satisfying (9) can be changed into a monotone strategy satisfying the same properties by the following transformation. For each t, let $\tilde{k}(t,\cdot)$ be the monotone strategy such that

$$\sum_{\sigma\in\Sigma} \beta(\sigma|s^*)\tilde{k}(t,\sigma) = \sum_{\sigma\in\Sigma} \beta(\sigma|s^*)k(t,\sigma).$$

By construction, $z(s^*,\theta,\tilde{k}) = z(s^*,\theta,k)$. By Assumption 1, $z(s^*_-,\theta,\tilde{k}) \leq z(s^*_-,\theta,k)$. ∎

For monotone strategies, Assumption 1 implies that candidate $\mathcal{R}$'s vote share, $z(s,\theta;k)$, is an increasing function of the state. This observation, together with Lemma 1, immediately imply the following result, which will be critical in our equilibrium construction.

Lemma 3. *Let $\{k_n\}$ be a sequence of monotone strategy profiles such that*

$$z(s^*_-, \overline{\theta}, k_n) < v_{\mathcal{L}} < v_{\mathcal{R}} < z(s^*, \underline{\theta}, k_n) \quad \textit{for all } n \textit{ sufficiently large.} \tag{11}$$

Then, the pivotal event $v_{\mathcal{L}}$ dominates $v_{\mathcal{C}}$ and $v_{\mathcal{R}}$ for all $s \leq s^*_-$, and the pivotal event $v_{\mathcal{R}}$ dominates $v_{\mathcal{C}}$ and $v_{\mathcal{L}}$ for all $s \geq s^*$.

Note that a sequence of strategy profiles satisfying the condition in Lemma 3 would also achieve full information equivalence. The opposite is not true in general, but any sequence of monotone strategy profiles that satisfies (9) would also satisfy the conditions of Lemma 3 if the recounting thresholds $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ are close to $v_{\mathcal{C}}$.

Our main result will show that, whenever full information equivalence is achievable, in an election with recounting there is a sequence of equilibrium strategy profiles that does so for all $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ sufficiently close to $v_{\mathcal{C}}$. The equilibrium construction will also satisfy (11). This implies that not only full information equivalence obtains, but also a costly recounting is never triggered in the limit, thus the election with recounting achieves the first best efficient outcome in the limit.

Proposition 1. *Suppose full information equivalence is achievable for an electoral rule. If $\delta > 0$ and $v_{\mathcal{L}}$, $v_{\mathcal{R}}$ are sufficiently close to $v_{\mathcal{C}}$, there exists a sequence of monotone strategy profiles $\{k_n\}$ that achieves full information efficiency, and such that k_n is an equilibrium of the game Γ^n for each n. Further, (11) holds.*

Proof. Since full information efficiency is achievable, by a construction similar to that in the proof of Lemma 2, there is a type independent monotone strategy profile, k, such that $z(s^*_-, \overline{\theta}; k) < v_{\mathcal{C}} < z(s^*, \underline{\theta}; k)$. Thus (11) also holds for any $v_{\mathcal{L}} \in (z(s^*_-, \overline{\theta}; k), v_{\mathcal{C}})$ and $v_{\mathcal{R}} \in (v_{\mathcal{C}}, z(s^*, \underline{\theta}; k))$.

A type independent and monotone strategy can be described by a single variable $\psi \in [0, J]$. The unique strategy profile associated to ψ is given by

$$k(t, \sigma_j) = \begin{cases} 0 & \text{if } j \leq \psi \\ 1 & \text{if } j \geq \psi + 1 \\ j - \psi & \text{otherwise} \end{cases}$$

and any type independent monotone strategy profile can be described by a single variable ψ. The integer part of ψ describes the highest signal for which all types vote for $\mathcal{L}$, the decimal part of ψ describes the probability of voting for $\mathcal{L}$ when observing the next signal, and for all higher signals all types vote for $\mathcal{R}$. Note that $z(s,\theta;\psi)$ is continuous and strictly decreasing in ψ. Thus, there are values $\underline{\psi} < \overline{\psi}$ such that $z(s^*_-,\overline{\theta},\overline{\psi}) = v_{\mathcal{L}}$ and $z(s^*,\underline{\theta},\underline{\psi}) = v_{\mathcal{R}}$.

The following lemma establishes that, in any sufficiently large election, the best response of any type to a strategy profile that satisfies (11) is a monotone strategy. Best responses are "almost type-independent" meaning that there is a unique signal realization such that the best responses of any two types may differ.

Lemma 4. *Let k be a strategy profile such that (11) is satisfied and $\sigma' > \sigma$. For all n sufficiently large, and for all $t, t' \in T$,*

$$\sum_{s\in S} \mu^{\sigma}(s)\left(g^n(v_{\mathcal{R}}|s;k)\delta + g^n(v_{\mathcal{C}}|s;k)u(s,t) - g^n(v_{\mathcal{L}}|s;k)\delta\right) \geq 0 \Rightarrow$$
$$\sum_{s\in S} \mu^{\sigma'}(s)\left(g^n(v_{\mathcal{R}}|s;k)\delta + g^n(v_{\mathcal{C}}|s;k)u(s,t') - g^n(v_{\mathcal{L}}|s;k)\delta\right) > 0.$$

Proof. Rewrite the first inequality as

$$\sum_{s\geq s^*} \mu^{\sigma}(s)\left(\frac{g^n(v_{\mathcal{R}}|s;k)}{g^n(v_{\mathcal{R}}|s^*;k)}\delta + \frac{g^n(v_{\mathcal{C}}|s;k)}{g^n(v_{\mathcal{R}}|s^*;k)}u(s,t) - \frac{g^n(v_{\mathcal{L}}|s;k)}{g^n(v_{\mathcal{R}}|s^*;k)}\delta\right) g^n(v_{\mathcal{R}}|s^*;k) \geq$$
$$\sum_{s< s^*} \mu^{\sigma}(s)\left(-\frac{g^n(v_{\mathcal{R}}|s;k)}{g^n(v_{\mathcal{R}}|s^*_-;k)}\delta - \frac{g^n(v_{\mathcal{C}}|s;k)}{g^n(v_{\mathcal{R}}|s^*_-;k)}u(s,t) + \frac{g^n(v_{\mathcal{L}}|s;k)}{g^n(v_{\mathcal{R}}|s^*_-;k)}\delta\right) g^n(v_{\mathcal{L}}|s^*_-;k).$$

For n large, by Lemma 3 the left-hand-side becomes arbitrarily close to $\mu^{\sigma}(s^*)g^n(v_{\mathcal{R}}|s^*;k)\delta$ and the right-hand-side arbitrarily close to $\mu^{\sigma}(s^*_-)g^n(v_{\mathcal{R}}|s^*_-;k)\delta$. The claim follows from observing that $\frac{\mu^{\sigma}(s^*)}{\mu^{\sigma}(s^*_-)}$ is strictly increasing in σ by (1). ■

An equilibrium strategy profile that satisfies (11) must be monotone and "almost type independent", meaning that how types vote can differ for at most one signal realization. Consider one such strategy, k, and let σ_j be the signal realization for which the strategy is type dependent. The probability that a randomly drawn informed voter with a private signal σ_j would vote for $\mathcal{R}$ is

$$\int_T k(t,\sigma_j)\mathrm{d}P(t).$$

Now consider the type independent strategy profile $\Psi(k) = j - \int_T k(t, \sigma_j)\mathrm{d}P(t)$. The strategy profiles k and $\Psi(k)$ are identical for every signal realization $\sigma < \sigma_j$ (every type votes for $\mathcal{L}$) and $\sigma > \sigma_j$ (every type votes for $\mathcal{R}$). They might differ for the realization σ_j, however the probability that a randomly drawn type with signal σ_j votes for $\mathcal{R}$ is the same for the two profiles. By construction, k and $\Psi(k)$ generate the same distribution of vote shares for $\mathcal{R}$, that is

$$g^n(v|s;k) = g^n(v|s;\Psi(k)) \quad \text{for all vote shares } v \text{ and } s \in S.$$

Since, k and $\Psi(k)$ generates the same distribution of vote shares, the set of best responses to k and to $\Psi(k)$ coincide. The following result, which is an immediate implication of this observation, will allow us to construct a fixed point mapping from which we can establish the existence of an equilibrium that satisy (11).

Lemma 5. *Let ψ be a type independent monotone strategy profile that satisfies (11). For n sufficiently large, if k is a best response to ψ and further $\Psi(k) = \psi$, then k is an equilibrium of Γ^n.*

Finally we construct a correspondence B, from the space of type independent monotone strategies, $[0, J]$ into itself as follows:

$$B(\psi) = \begin{cases} J & if\ \psi < \underline{\psi} \\ \{\Psi(k) \text{ s.t } k \in BR(\psi)\} & if\ \psi \in [\underline{\psi}, \overline{\psi}] \\ 0 & if\ \psi > \overline{\psi} \end{cases}$$

Since $g^n(v|s;\psi)$ is a continuous function of ψ for all pivotal events and states, for n large enough, for each type the best response correspondence is u.h.c. in ψ and monotone in signal for all $\psi \in [\underline{\psi}, \overline{\psi}]$. Thus, $B(\cdot)$ is u.h.c. for $\psi \in [\underline{\psi}, \overline{\psi}]$. Further, since $\frac{g^n(v_\mathcal{L}|s^*_-;\underline{\psi})}{g^n(v_\mathcal{R}|s^*;\underline{\psi})}$ and $\frac{g^n(v_\mathcal{R}|s^*;\overline{\psi})}{g^n(v_\mathcal{L}|s^*_-;\overline{\psi})}$ go to zero for n large, every type's best response to $\underline{\psi}$ is to vote for $\mathcal{R}$ regardless of the signal and the best response to $\overline{\psi}$ is to vote for $\mathcal{L}$ regardless of the signal. Thus $B(\underline{\psi}) = J$ and $B(\overline{\psi}) = 0$, and B is u.h.c. on $[0, J]$. The fact the B has a fixed point is an application of Kakutani's fixed point theorem. Let ψ^* be a fixed point of $B(\cdot)$. Then there exists a best response to ψ^*, k^* such that $\Psi(k^*) = \psi^*$. By Lemma 5, k^* is an equilibrium of Γ^n. ■

Since our equilibrium construction satisfies (11), the pivotal events $v_\mathcal{L}$ dominates for $s < s^*$, and $v_\mathcal{R}$ dominates for $s \geq s^*$. Further, it must also be the case that

$\frac{g^n(v_{\mathcal{L}}|s^*_-;k_n)}{g^n(v_{\mathcal{R}}|s^*;k_n)}$ neither goes to 0 nor to ∞, along the sequence of equilibrium strategy profiles. Otherwise, eventually it becomes a unique best response for all types to vote for the same candidate regardless of the signal observed, which is not a fixed point of $B(\cdot)$. This implies that along any sequence of equilibrium strategy profiles that satisfies (11)

$$\lim_{n\to\infty} I(v_{\mathcal{R}}, z(s^*, \underline{\theta}; k_n)) = \lim_{n\to\infty} I(v_{\mathcal{L}}, z(s^*_-, \overline{\theta}; k_n)). \tag{12}$$

Note that since for any monotone and type independent profile ψ, the vote share function $z(v, z(s, \theta; \psi))$ is strictly decreasing in ψ, the "equal I" condition (12) is satisfied by a unique $\psi^I \in [\underline{\psi}, \overline{\psi}]$. This means that, while the existence result of Proposition 1 does not exclude that there might be multiple equilibria for each game Γ^n, all equilibrium strategy profiles generate, in the limit, the same distribution over vote shares. A result that we formally state in the next proposition.

Proposition 2. *Let $\{k_n\}$ be a sequence of strategy profile satisfying (11) and such that k_n is an equilibrium of Γ^n for each n. Then,*

$$\lim_{n\to\infty} \Psi(k_n) = \psi^I.$$

Our main result of Proposition 1 establishes that recounting can improve the information efficiency of the electoral outcome whenever full information efficiency is achievable yet, without recounting, there is no sequence of equilibria that achieves it. It is possible to construct examples where this happens. When full information equivalence obtains in the limit of equilibria without recounting, adding the recounting thresholds can still improve on the equilibrium outcome. Our final result shows that recounting improves the "informativeness" of the equilibrium strategy profile. That is, recounting generates a larger spread of the expected vote share for $\mathcal{R}$ across the two critical states s^* and s^*_-. This property, which explains why, with recounting, full information equivalence is more robust to aggregate uncertainty, also implies that the information efficient outcome will be approximated at a faster rate by a small modification of the electoral rule. This result is obtained, in the following proposition, by comparing a sequence of equilibria of a game Γ^n_δ, where it is assumed that the recounting cost is $\delta > 0$, with a sequence of equilibria of a game Γ^n_0 which differs from Γ^n_δ only by the absence of recounting cost (i.e. $\delta = 0$).

The key property explaining this result is that, in the equilibrium construction of Proposition 1, at the dominant pivotal events a vote does not change the identity of the winner, thus the incentives to vote are type independent. This allows to construct an equilibrium where all type vote informatively (i.e. their vote changes across signals). Without recounting, at the only pivotal event, $v_{\mathcal{C}}$, the vote determines the election outcome. Thus a voter's belief over payoff relevant states and her preference type matter for her voting decision, and if an individual's private signal does not change significantly her beliefs conditional on the pivotal event being realized, her vote must be uninformative. Under the following assumption, there is always a positive mass of types who vote un-informatively in an equilibrium without recounting.

Assumption 3. *For any probability distribution μ over S, there is a subset of types $T' \subseteq T$, such that $P(T') > 0$ and for all $t \in T'$*

$$\left(\sum_{s\in S} \mu^{\sigma}(s)u(s,t)\right)\left(\sum_{s\in S} \mu^{\sigma'}(s)u(s,t)\right) > 0 \quad \textit{for all } \sigma, \sigma' \in \Sigma.$$

If μ is the posterior belief conditional on the pivotal event $v_{\mathcal{C}}$ in an equilibrium without recounting, all types in T' vote un-informatively in equilibrium as their expected payoff difference between voting $\mathcal{R}$ and voting $\mathcal{L}$ does not change sign with the voter's private information. Assumption 3 is both a requirement that preferences vary enough with types, and that private signals are not too informative. It is satisfied in several strategic voting models such as, for example, Feddersen and Pesendorfer (1997) and Bhattacharya (2013).

Proposition 3. *Let $\{k_n\}$ and $\{k_n^{FP}\}$ be two sequences of monotone strategy profiles such that: i) both achieve full information equivalence; ii) the sequence $\{k^n\}$ satisfies (11); and iii) k_n is an equilibrium of Γ_{δ}^{n} and k_n^{FP} is an equilibrium of Γ_0^n for all n. Then, for all $v_{\mathcal{R}}$ and $v_{\mathcal{L}}$ sufficiently close to $v_{\mathcal{C}}$*

$$\lim_{n\to\infty} z(s_-^*, \overline{\theta}; k_n) \leq \lim_{n\to\infty} z(s_-^*, \overline{\theta}; k_n^{FP}) < \lim_{n\to\infty} z(s^*, \underline{\theta}; k_n^{FP}) \leq \lim_{n\to\infty} z(s^*, \underline{\theta}; k_n), \tag{13}$$

with all strict inequalities if Assumption 3 holds.

Proof. With recounting, in the limit the equilibrium must satisfy the "equal I" condition that

$$\lim_{n\to\infty} I(v_{\mathcal{L}}, z(s_-^*, \overline{\theta}; k_n)) = \lim_{n\to\infty} I(v_{\mathcal{R}}, z(s^*, \underline{\theta}; k_n)).$$

Similarly, the limit equilibrium with recounting must satisfy an analogous "equal I" condition

$$\lim_{n\to\infty} I(v_{\mathcal{C}}, z(s^*_-, \overline{\theta}; k_n^{FP})) = \lim_{n\to\infty} I(v_{\mathcal{C}}, z(s^*, \underline{\theta}; k_n^{FP})).$$

Otherwise, either the ratio $\frac{g^n(v_{\mathcal{C}}|s;k_n^{FP})}{g^n(v_{\mathcal{C}}|s^*;k_n^{FP})}$ goes to 0 for all $s \neq s^*$, or the ratio $\frac{g^n(v_{\mathcal{C}}|s;k_n^{FP})}{g^n(v_{\mathcal{C}}|s^*_-;k_n^{FP})}$ goes to 0 for all $s \neq s^*_-$. In either case, every type's best response to k_n^{FP} is independent of his signal, as conditional on the realization of the pivotal event the state is known, which contradicts the hypothesis that the $\{k_n^{FP}\}$ achieves full information equivalence.

For all recounting thresholds sufficiently close to $v_{\mathcal{C}}$, if $z(s^*_-, \overline{\theta}; k_n^{FP}) < z(s^*_-, \overline{\theta}; k_n)$, the two "equal I" conditions can be satisfied only if $z(s^*, \underline{\theta}; k_n^{FP}) > z(s^*, \underline{\theta}; k_n)$. To prove the claim is than sufficient to show that $z(s^*_-, \overline{\theta}; k_n^{FP}) \leq z(s^*_-, \overline{\theta}; k_n)$ always implies $z(s^*, \underline{\theta}; k_n^{FP}) \leq z(s^*, \underline{\theta}; k_n)$ and it implies $z(s^*, \underline{\theta}; k_n^{FP}) < z(s^*, \underline{\theta}; k_n)$ if Assumption 3 holds. To see this, first notice that given the strategy profile k_n^{FP} we can costruct a type independent strategy profile $\hat{\psi}$ such that $z(s^*_-, \overline{\theta}; k_n^{FP}) = z(s^*_-, \overline{\theta}; \hat{\psi})$. By (1), we have $z(s^*, \underline{\theta}; k_n^{FP}) \leq z(s^*, \underline{\theta}; \hat{\psi})$. The first claim follows from observing that: i) from the equilibrium construction in Proposition 1, there is a type independent strategy profile $\Psi(k_n)$ such that $z(s, \theta; k_n) = z(s, \theta; \Psi(k_n))$ for all s, θ; and ii) the vote share function $z(s, \theta; \psi)$ is increasing in ψ.

The second part of the claim is obtained by constructing a strategy profile $\tilde{k}$ such that: i) $\tilde{k}(\sigma, t) = k_n^{FP}(\sigma, t)$ for all types that vote un-informatively in k_n^{FP}; ii) $\tilde{k}(\sigma, t) = \tilde{k}(\sigma, t')$ for all types t, t' that vote informatively in k_n^{FP}; and iii) $z(s^*_-, \overline{\theta}; k_n^{FP}) = z(s^*_-, \overline{\theta}; \tilde{k})$. By (1), it is still the case that $z(s^*, \underline{\theta}; k_n^{FP}) \leq z(s^*, \underline{\theta}; \tilde{k})$. However, since there is a positive mass of types who vote un-informatively and the others use a type independent strategy,

$$z(s^*_-, \overline{\theta}; \tilde{k}) \leq z(s^*_-, \overline{\theta}; \Psi(k_n)) \Rightarrow z(s^*, \underline{\theta}; \tilde{k}) < z(s^*, \underline{\theta}; \Psi(k_n)),$$

which establishes the second part of the claim. ■

Proposition 3 establishes that the probability of a "mistake," meaning an election outcome different from the full information outcome, is smaller in both states s^*_- and in state s^*, when the electoral rules mandates a costly recounting exercise for sufficiently tight election. Since the probability of mistakenly electing $\mathcal{R}$ is larger in state s^*_- than in any other smaller state, and the probability of mistakenly electing

$\mathcal{L}$ is larger in state s^* than any other higher state, the rate of convergence to the full information outcome only depends on the probability of a mistake in the two threshold states s^*_- and s^*. Thus, as an implication of Proposition 3, whenever an electoral rule without recounting achieves full in information equivalence, adding recounting is still beneficial by providing a faster rate of convergence to the same informationally efficient outcome.

Finally, Proposition 3 provides sufficient conditions for recounting to improve the rate of convergence to the informationally efficient outcome, and recounting may still be beneficial when some or all of the assumptions are violated. For example, the requirement that the equilibrium without recounting be in monotone strategies can be weakened. The result still holds if the equilibrium strategies are monotone "on average." That is, if the average probability (across type) of voting for $\mathcal{R}$ is increasing in the signal observed. The result also holds when equilibrium strategies are "thresholds." That is, for every type there is at most one signal that induce randomization over votes. This is the case, for example, in Bhattacharya (2013)'s model.

4. Discussion

4.1. Two rounds of voting

Suppose the election rule is that candidate $\mathcal{R}$ is the outright winner if his vote share in the first round of voting is greater than $v_{\mathcal{R}}$, and candidate $\mathcal{L}$ is the outright winner if his vote share is greater than $1 - v_{\mathcal{L}}$. When neither candidate is an outright winner, there will be a second round of voting with a standard election rule $v_{\mathcal{C}}$ after imposing a second-round voting cost δ to each voter. We claim that under this alternative specification of the election rule $\{v_{\mathcal{L}}, v_{\mathcal{C}}, v_{\mathcal{R}}\}$, the equilibrium construction for our model of election with recounting can be replicated as an equilibrium in a model with two rounds of voting.

The equilibrium construction in such a model with two voting rounds poses some additional complications. First, there is a continuum of pivotal events because any realized first-round vote share for $\mathcal{R}$ between $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ might in principle lead to a different continuation equilibrium. However, if the first round strategy profile satisfies the full information equivalence condition (11), then it follows from Lemma 1 that for n sufficiently large the only probabilistically relevant pivotal events in the first round are that $v = v_{\mathcal{L}}$ in state s^*_- and that $v = v_{\mathcal{R}}$ in state s^*. All other pivotal events are dominated by one of these two events. A second complication in replicating our equilibrium construction arises because, at the two relevant pivotal events, the vote of an informed voter will change the timing of the election resolution—as in the recounting model—but might also change the election outcome. However, if the first round strategy profile satisfies (11), for n large the belief that the state is s^* is arbitrarily close to 1 at the pivotal event $v_{\mathcal{R}}$ and the belief that the state is s^*_- is arbitrarily close to 1 at the pivotal event $v_{\mathcal{L}}$. As long as in the continuation equilibrium the probability that $\mathcal{R}$ is elected approaches 1 (respectively, 0) when every informed voter's belief that the state is R is close to 1 (respectively, 0) then at the pivotal events $v_{\mathcal{L}}$ and $v_{\mathcal{R}}$ the vote affects the election outcome (i.e., which candidate wins) with a vanishing probability. In other words, the dominant consideration in the first round of voting is to avoid the cost δ incurred in a second round of voting, and our equilibrium construction for the recount model is replicated in a model with two rounds of voting.

4.2. Recount cost

Our model of election with recount does not depend on the magnitude of the recount cost δ. We only assume that δ is positive and fixed as n goes to infinity. This restriction can be further relaxed by assuming that a recount costs a fixed amount of $\Delta > 0$ and that in an election with $n+1$ voters, each voter bears a cost of $\delta^n = \Delta/(n+1)$.

When $z(s^*, \underline{\theta}; \kappa_n) > v_{\mathcal{R}}$, Lemma 1 and monotone strategies implies that $\frac{g^n(v|s;k_n)}{g^n(v_{\mathcal{R}}|s^*;k_n)}$ goes to 0 as n goes to infinity for $v = v_{\mathcal{L}}, v_{\mathcal{C}}$ and every $s \geq s^*$. Similarly, $\frac{g^n(v|s;k_n)}{g^n(v_{\mathcal{L}}|s^*_-;k_n)}$ goes to 0 as n goes to infinity for $v = v_{\mathcal{R}}, v_{\mathcal{C}}$ and every $s \leq s^*_-$. Moreover, these ratios go to 0 at an exponential rate because the rate functions of the different pivotal events are ranked. From the proof of Proposition 1, the voting incentives of an agent observing a signal realization σ are now described by the inequality

$$\sum_{s \geq s^*} \mu^{\sigma}(s) \left(\frac{g^n(v_{\mathcal{R}}|s;k)}{g^n(v_{\mathcal{R}}|s^*;k)} \delta^n + \frac{g^n(v_{\mathcal{C}}|s;k)}{g^n(v_{\mathcal{R}}|s^*;k)} u(s,t) - \frac{g^n(v_{\mathcal{L}}|s;k)}{g^n(v_{\mathcal{R}}|s^*;k)} \delta^n \right) g^n(v_{\mathcal{R}}|s^*;k) \geq$$

$$\sum_{s < s^*} \mu^{\sigma}(s) \left(-\frac{g^n(v_{\mathcal{R}}|s;k)}{g^n(v_{\mathcal{R}}|s^*_-;k)} \delta^n - \frac{g^n(v_{\mathcal{C}}|s;k)}{g^n(v_{\mathcal{R}}|s^*_-;k)} u(s,t) + \frac{g^n(v_{\mathcal{L}}|s;k)}{g^n(v_{\mathcal{R}}|s^*_-;k)} \delta^n \right) g^n(v_{\mathcal{L}}|s^*_-;k).$$

For n large, by Lemma 3 the left-hand-side still becomes arbitrarily close to $\mu^{\sigma}(s^*) g^n(v_{\mathcal{R}}|s^*;k)\delta^n$ and the right-hand-side arbitrarily close to $\mu^{\sigma}(s^*_-) g^n(v_{\mathcal{R}}|s^*_-;k)\delta^n$. This is because, even though the recount cost δ^n goes to 0 as n goes to infinity, it goes to 0 only at the rate $1/n$. The remainder of the proofs of Propositions 1 goes through with no change.

4.3. Counting errors

Our model does not allow for counting errors, so that the vote count in the initial stage is identical to the vote count in the recount stage. There are different ways to introduce counting errors. We consider two alternatives.

In the first version of a model with counting error, we assume that each vote for candidate $\mathcal{R}$ has an independent probability $\zeta < 1/2$ of being miscounted as a vote for candidate $\mathcal{L}$, and likewise each vote for $\mathcal{L}$ has an independent probability ζ of being miscounted as a vote for $\mathcal{R}$. Further assume that if there is a recount, all the counting errors are corrected. Under these assumptions, if the true vote share for candidate $\mathcal{R}$ is v, the initial vote count for $\mathcal{R}$ will be

$$v_e = (1-\zeta)v + \zeta(1-v).$$

Note that $v_e > v$ if and only if $v < 1/2$, which is due to regression to the mean. Define

$$v'_{\mathcal{L}} \equiv \frac{v_{\mathcal{L}} - \zeta}{1-2\zeta}, \qquad v'_{\mathcal{R}} \equiv \frac{v_{\mathcal{R}} - \zeta}{1-2\zeta}.$$

Then, under the election rule $\{v_{\mathcal{L}}, v_{\mathcal{C}}, v_{\mathcal{R}}\}$, the election would go into the recount stage if the true vote share v for $\mathcal{R}$ is between $v'_{\mathcal{L}}$ and $v'_{\mathcal{R}}$.

With recounting errors, whether full information efficiency is achievable now dependens on the specifics of the electoral rule as well as the probability of miscounting. Precisely, there must exist a strategy profile, k, such that

$$z(s,\overline{\theta};k) < \min\{v'_{\mathcal{R}}, v_{\mathcal{C}}\} \leq \max\{v'_{\mathcal{L}}, v_{\mathcal{C}}\} < z(s,\underline{\theta};k) \quad \text{for all } s < s^* \text{ and } s' \geq s^* \tag{14}$$

The conditions (4) and (14) coincide whenever

$$v'_{\mathcal{L}} < v_{\mathcal{C}} < v'_{\mathcal{R}}. \tag{15}$$

Unless $v_{\mathcal{C}} = 1/2$, in which case (15) holds for any pair of $(v_{\mathcal{L}}, v_{\mathcal{R}})$, it can be the case that (15) is violated. In fact, it will always be violated for $v_{\mathcal{L}}, v_{\mathcal{R}}$ sufficiently close to $v_{\rfloor} \neq 1/2$.

To replicate the equilibrium construction of Proposition 1 condition (15) is necessary. Further, we need to be able to find a strategy profiles k such that

$$z(s,\overline{\theta};k) < v'_{\mathcal{L}} < v'_{\mathcal{R}} < z(s,\underline{\theta};k) \quad \text{for all } s < s^* \text{ and } s' \geq s^*. \tag{16}$$

In the proof of Proposition 1, the requirement (16) is satisfied by taking $v_{\mathcal{L}}$ and $v\mathcal{R}$ sufficiently close to $v_{\mathcal{C}}$. In the presence of recounting errors this is not possible, as too tight recounting thresholds might lead to inefficiencies. However, for any pair $(v_{\mathcal{L}}, v\mathcal{R})$, if the recounting error, ζ, is sufficiently small (15) holds. Further, for all pairs $(v_{\mathcal{L}}, v\mathcal{R})$ sufficiently close to $v_{\mathcal{C}}$, for sufficiently small recounting errors if a strategy profile satisfies (9), it also satisfies (16). Thus, the equilibrium construction of Proposition 1 can be replicated provided the recounting error is small enough. We can summarize this discussion in the following statement

Proposition 4. *Suppose full information equivalence is achievable for an electoral rule. If $\delta > 0$, for all $v_{\mathcal{L}}$, $v_{\mathcal{R}}$ sufficiently close to $v_{\mathcal{C}}$ there is a $\overline{\zeta}(v_{\mathcal{L}}, v_{\mathcal{R}}) > 0$ such that, for all miscounting probabilities $\zeta < \overline{\zeta}(v_{\mathcal{L}}, v_{\mathcal{R}})$, there exists a sequence of monotone strategy profiles $\{k_n\}$ that achieves full information efficiency, and such that k_n is an equilibrium of the game Γ^n for each n.*

Our second model of counting errors assumes system-wide errors instead of independent mistakes in counting each ballot. For example, such correlated errors may occur when a certain counting protocol (how to deal with hanging chads, etc.) is not properly followed, so that all the votes in the same polling station or even the entire election are miscounted is a specific way. To model these errors, we assume that if the true vote share for candidate $\mathcal{R}$ is v, then upon the initial count the vote share is recorded as

$$v_e = \begin{cases} 1 & \text{if v+u>1;} \\ 0 & \text{if v+u<0;} \\ \text{v+u} & \text{otherwise.} \end{cases}$$

In the above, u is a random variable with positive and continuous density on the support $[\underline{u}, \overline{u}]$. Upon recounting, all errors are detected so that the election outcome is based on the true vote share v. The effect of the systematic counting error u is very similar to the effect of aggregate uncertainty θ, except that u only influences the initial vote share but not the final tally. Specifically, if $z(s,\underline{\theta};\kappa^n) + \underline{u} > v_{\mathcal{R}}$, then in state s the pivotal event $v_e = v_{\mathcal{R}}$ dominates the other pivotal events $v = v_{\mathcal{C}}$ and

$v_e = v_{\mathcal{L}}$ for sufficiently large n. Proposition 1 continues to hold if there exists a strategy profile, k, such that

$$z(s^*_-, \overline{\theta}; k) + \overline{u} < v_{\mathcal{C}} < z(s^*, \underline{\theta}; k) + \underline{u}. \tag{17}$$

While (17) is stronger than the requirement that full information efficiency is achievable, it is implied by the latter whenever the distribution of the systemwide error is sufficiently concentrated (i.e. $\overline{u} - \underline{u}$ is sufficiently small). Thus, similarly to the first model of counting errors, the result of Proposition 1 is robust to the introduction of small recounting errors. However, it is also worth noting that (17) is not necessary for full information efficiency to be achievable in the presence of recounting errors, which only requires that the counting error is not so large to induce the wrong election outcome without recounting, or

$$z(s^*_-, \overline{\theta}; k) + \overline{u} < v_{\mathcal{R}} \quad \text{and} \quad v_{\mathcal{L}} < z(s^*, \underline{\theta}; k) + \underline{u}.$$

4.4. Uncertain size of electorate

The analysis presented here can be generalized to the case with an uncertain electorate size if we assume that the number of voters is N, with N being a Poisson random variable with mean n. Myerson (1998; 2000) develops the tools to study such Poisson games.

Recall that from Stirling's approximation to the binomial probability in equation (6), the rate at which the pivotal probability that the vote share equals v goes to 0 is given by:

$$\lim_{n \to \infty} \frac{\log g^n(v|s, \theta; k_n)}{n} = \log I(v; z(s, \theta; \kappa_n)).$$

In contrast, Myerson (2000) shows that in a Poisson model, the corresponding rate is:

$$\lim_{n \to \infty} \frac{\log g^n(v|s, \theta; k_n)}{n} = I(v; z(s, \theta; \kappa_n)) - 1.$$

Since $\log I$ and $I - 1$ are positive transformation of one another, given any v, s and κ_n, the θ that maximizes $\log I$ in the model with no population uncertainty also maximizes $I - 1$ in the Poisson model. Lemma 1 then implies that if $z(s^*, \underline{\theta}; \kappa_n) > v_{\mathcal{R}}$, then the event $v = v_{\mathcal{R}}$ dominates the events $v = v_{\mathcal{C}}$ and $v = v_{\mathcal{L}}$ in every state $s \geq s^*$. Likewise, if $z(s^*_-, \overline{\theta}; \kappa_n) < v_{\mathcal{L}}$, then the event $v = v_{\mathcal{L}}$ dominates the events $v = v_{\mathcal{C}}$ and $v = v_{\mathcal{R}}$ in al states $s \leq s^*_-$. All the results in the current paper remains intact in the Poisson model.

5. Concluding Remarks

This paper is an outgrowth of our earlier papers (Damiano, Li and Suen, 2010; 2013) that use costly delay to improve information aggregation in a two-agent negotiation problem, and to study the design of deadline in negotiations. Here, we introduce multiple pivotal events to resurrect informative voting in large elections. The key to our equilibrium construction relies on the fact that while the probabilities of different pivotal events are all vanishingly small in large elections, the rate at which they go to zero can be ranked. Since the desire to avoid recount cost is preference-independent, and since pivotal events triggering a recount dominate the pivotal event involving a tie between the candidates, we demonstrate how informative voting by all types can be an equilibrium in large elections with recount, producing asymptotically information efficient outcomes which may otherwise be infeasible in standard elections. The analysis of elections with multiple pivotal events also features in Razin (2003) in the context of signaling policy preference by voters, and in Bouton and Castanheira (2012) and Ahn and Oliveros (2012) in models of multi-candidate and multi-issue voting.

In this paper we have considered the Condorcet jury theorem in large elections. In a jury setting, Austin-Smith and Banks (1996) and Feddersen and Pesendorfer (1998) have shown that the Condorcet jury theorem fails due to strategic voting. In particular, Feddersen and Pesendorfer (1998) show that a unanimous conviction rule in jury decisions may lead to higher probability of false conviction as well as false acquittal than the simple majority rule, and the probability of convicting an innocent defendant may increase with the size of the jury. More relevant to the present paper is a recent literature that asks whether the Condorcet jury theorem continues to hold when acquiring information is costly to individual agents. Mukhopadhaya (2005) shows that in a symmetric mixed strategy equilibrium, as the number of committee members increases, each member chooses to collect information with a smaller probability. He finds examples in which, using the majority rule, a larger committee makes the correct decision with a lower probability than does a smaller one. Koriyama and Szentes (2009) consider a model in which agents choose whether or not to acquire information in the first stage, and then the decision is made according to an ex post efficient rule in the second stage. They show that there is a maximum group size such that in smaller groups each member will choose to collect evidence, and the Condorcet jury theorem fails for larger groups. However, in a model with the quality of information as a continuous choice variable, Martinelli (2006) shows that if the marginal cost of information is near zero for nearly irrelevant information, then there will be effective information aggregation despite the fact that each individual voter will choose to be very poorly informed. In a recent paper, Krishna and Morgan (2012) show that when participation in an election is costly but voluntary, those who choose to participate will vote informatively even in a standard election. However the fraction of participating voters is vanishingly small in a large election, rendering asymptotic information efficiency difficult to achieve if there is aggregate uncertainty in the model.

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