COMPETING MATCHMAKING

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Abstract
We study how matchmakers use prices to sort heterogeneous participants into competing matching markets and how equilibrium outcomes compare with monopoly in terms of prices, matching market structure, and sorting efficiency under the assumption of complementarity in the match value function. The role of prices to facilitate sorting is compromised by the need to survive price competition. We show that price competition leads to a high-quality market that is insufficiently exclusive. As a result, the duopolistic outcome can be less efficient in sorting than the monopoly outcome in terms of total match value in spite of servicing more participants. (JEL: C7, D4)

1. Introduction

Since the seminal work on network competition by Katz and Shapiro (1985) (see also Farrell and Saloner 1986; Fujita 1988), there has been a growing economic literature on competing marketplaces. This literature reflects the importance of network externalities in industries ranging from telecommunications to software platforms and to credit cards. In these industries, a competing marketplace is a network (platform) on which participants interact, and agents' network choices have external effects on each other's welfare. In most of the earlier works of the literature, the driving force is the "thick market" effect that a larger network provides a greater chance of finding a trading partner. This positive size effect favors the dominance of a single marketplace, and a central question is whether and when multiple marketplaces can coexist in equilibrium.[1] Although size effects

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1. Recently, research in this literature has increasingly focused on two-sided marketplaces, where participants are interested in matching with those on the other side. Ellison and Fudenberg (2003) and Ellison, Fudenberg, and Mobius (2004) reexamine the coexistence of multiple networks by allowing a negative size effect whereby agents prefer networks with fewer competitors; Caillaud and Jullien

are important in network competition, in many industries network participants also care about the identities of other participants in the same network. For example, in markets such as job search, real estate, and dating — where networks are intermediaries — participants are heterogeneous and networks differ not only in relative size but also in quality. In these markets, participants' network choices can have external effects on each other's welfare by changing the composition and hence the quality of the network pool. This type of "sorting externality" and its implications for price competition have been neglected in the literature on competing marketplaces, which focuses on the size effects and assumes either that agents are homogeneous or that agents' choice of network is independent of their type.[2]

This paper introduces a model of price competition among marketplaces in an environment where agents have heterogeneous qualities and where the expected quality of the pool of participants affects agents' decision of which marketplace to join. In our model, a marketplace is a random matching market or, more specifically, a meeting place where participants randomly match with each other. We have in mind a job market or a dating market, where agents have private information about their one-dimensional quality characteristics (type) and where the match value function exhibits complementarity between types. Because type information is private, agents self-select into matching markets based on the prices and their expectations of the quality of the pool in the matching market. Under the assumption of complementarity, how agents sort into the matching markets by type has implications to efficiency in terms of total match value. The random matching technology we adopt implies the absence of any size effect. This allows us to isolate the implications of the sorting externality that we introduce here from the consequences of the much studied size effects. We stress that our analytical framework applies equally well to one-sided intermediary markets, such as private schools that compete with tuition charges and country clubs that compete with membership fees. In these applications, instead of random pairwise match formation in each matching market (such as a school or a club), we can allow any form of interaction among the participants so long as the reduced form payoff function exhibits complementarity between the individual type and the average type. Hence this paper, by introducing price competition in an oligopoly model, also contributes to the literature on locational choices where the peer effect plays

(2001, 2003) and Rochet and Tirole (2003) analyze the "divide and conquer" strategy of subsidizing one side of the market while recovering the loss from the other side; and Armstrong (2006) studies the implications on price competition of "multi-homing", where participants on one side of the market can use multiple networks.

2. Ambrus and Argenziano (2004) modify the framework of Caillaud and Jullien (2001, 2003) and allow for heterogeneous preferences. Agents have the same quality but differ in terms of willingness to pay for participating in a larger network. In their model, the equilibrium distribution of participant types can be different across networks; however, the size effect remains the only externality.

a critical role (de Bartolome 1990; Epple and Romano 1998).[3]

In Section 2 we lay out the framework of duopoly price competition in a matching environment. In our model, matchmakers use prices (subscription fees) to induce agents to sort into different matching markets. We introduce the concept of matching market structure, which describes how agents sort into two matching markets given the two prices. We then provide a criterion for selecting a unique market structure for any price profile. Price competition in a matching environment with friction differs from the standard Bertrand models because prices also play the role of sorting heterogeneous agent types into different matching markets. Aside from the usual strategy of lowering price to steal rivals' market share, our selection criterion formalizes a pricing strategy called *overtaking* that is unique to the sorting role of prices. Overtaking a rival is achieved by charging a price just higher than the rival does and thus providing a market with a higher quality (average agent type). When the price difference is small enough, the rival's matching market loses all its customers because quality difference dominates.

Section 3 contains our main results. No pure-strategy equilibrium exists in the simultaneous-move pricing game because, for any price profile, at least one of the matchmakers has an incentive to drive its rival out of the market by using the overtaking strategy. We provide a sufficient condition for the two matchmakers to coexist in the equilibrium of the sequential-move version of the pricing game. This condition requires the type distribution to be sufficiently diffused so that the first mover can create a niche market for the low types in order to survive the overtaking strategy of the second mover, which in equilibrium serves the higher types. With the assumption of uniform type distribution, we show that — at the equilibrium outcome of the duopoly competition — the total market coverage is greater than the optimal total coverage under a monopolist that maximizes revenue from two matching markets, because the first mover must lower its price to prevent overtaking. However, the equilibrium outcome involves inefficient sorting compared to the monopoly outcome, because competition results in an insufficiently exclusive high-quality matching market. When the type distribution is tight, the matching market structure is less efficient overall under competition than under monopoly, as the loss from inefficient sorting outweighs the gain from greater coverage. We conclude our analysis with a brief discussion of the robustness of our main results when the type distribution is nonuniform, and when more than two matching markets are created. Section 4 provides further remarks on the existing literature and the implications of our results for regulatory policies in

3. As a model of a one-sided intermediated market, our paper is also related to the literature on demand externalities and pricing (Karni and Levin 1994; Rayo 2002) and to the literature on clubs (Cole and Prescott 1997). For example, in Rayo's model of a monopolist selling status goods, demand externalities arise from a complementarity in buyers' utility functions between the buyer's type and the average type of buyers who purchase the same status good.

intermediated markets. Proofs of all lemmas can be found in the Appendix.

2. A Duopoly Model of Competing Matchmakers

Consider a two-sided matching environment. Agents of the two sides have heterogeneous one-dimensional characteristics, called “types”. For simplicity, we assume that the two sides have the same size and the same type distribution function F, with a support $[a, b] \subseteq \mathbb{R}_+$ and a differentiable density function f. We assume that $a > 0$ and that b is finite; the following analysis carries through with appropriate modifications if $a = 0$ or $b = \infty$, and all our results hold without change.

Two matchmakers, unable to observe types of agents, use prices (entrance fees) to create two matching markets.[4] For each $i = 1, 2$, let p_i be the price charged by matchmaker i. Given p_1 and p_2, agents simultaneously choose one of three options: participate in matchmaker 1’s matching market, participate in 2’s market, or not participate. In each matching market, agents are randomly pairwise matched. Random matching means that the probability that a type x agent meets an agent from the other side whose type is in some set equals the proportion of matching market participants whose type belongs to that set. We assume that matching markets are costless to organize and that each matchmaker’s objective is to maximize the sum of entrance fees collected from participants.[5]

A match between a type x agent and a type y agent from the other side produces a value of xy to both of them. This match value function satisfies the standard complementarity condition (positive cross-partial derivatives), which implies that, in a frictionless matching environment, the total match value is maximized by matching equal types of agents. Let m_i, $i = 1, 2$, be the expected type (average quality) in the matching market created by matchmaker i; the qualities m_1 and m_2 are endogenously determined in equilibrium by p_1 and p_2 and by the participation choices of the agents. The utility of a type x agent from participating in matching market i is then $xm_i - p_i$. Unmatched agents get a payoff of 0 regardless of type.

4. Owing to the assumptions of symmetry and random pairwise meeting, in our model each participant in a matching market is matched with probability 1. If the meeting technology is such that the probability of forming a match is less than 1, then our model with subscription fees is equivalent to a model of usage fees, where subscribers are charged after a match is delivered but before the type of the match is revealed.

5. The same framework can be used to analyze the optimal pricing of a single matchmaker that competes with a free-access matching market; see our earlier paper, Damiano and Li (2007).

2.1. Matching market structures

First we examine the Nash equilibria of the simultaneous-move game played by the agents for given prices p_1 and p_2. For concreteness, we refer to each equilibrium as a "matching market structure". Since our model is symmetric with respect to the two sides, we restrict our attention to symmetric Nash equilibria, where each matching market hosts an equal number of participants with identical support from the two sides. For any $c, c' \in [a, b]$ with $c < c'$, let $\mu(c, c')$ be the mean type on the interval $[c, c']$ and denote $\mu(c, c) = c$.

DEFINITION 1. Given prices p_1 and p_2, singular matching market structure S_i, $i = 1, 2$, is a symmetric Nash equilibrium of the simultaneous-move game played by the agents, such that agents participate in matching market i only.

Singular matching market structure S_i is characterized by a participation threshold c_i for matching market i, determined by

$$\begin{cases} c_i\mu(c_i, b) = p_i & \text{if } p_i \in [a\mu(a, b), b^2], \\ c_i = a & \text{if } p_i \in [0, a\mu(a, b)). \end{cases} \tag{1}$$

The average quality m_i of matching market i is $\mu(c_i, b)$, and the threshold participation type is either a type c_i, which is indifferent between participating in matching market i and not participating (when $p_i \geq a\mu(a, b)$), or the lowest type a, which strictly prefers participation (when $p_i < a\mu(a, b)$). In both cases, all types higher than the threshold type strictly prefer to participate in matching market i.

DEFINITION 2. Given prices $p_i < p_j$ with $i \neq j = 1, 2$, dual matching market structure D_{ij} is a symmetric Nash equilibrium of the simultaneous-move game played by the agents, such that agents participate in both matching markets.

Dual matching market structure D_{ij} is characterized by two participation thresholds, c_i and c_j, with $a \leq c_i < c_j < b$, such that: either c_i and c_j satisfy

$$c_i\mu(c_i, c_j) = p_i \qquad \text{and} \qquad c_j(\mu(c_j, b) - \mu(c_i, c_j)) = p_j - p_i; \tag{2}$$

or $c_i = a$ and c_j satisfies

$$a\mu(a, c_j) > p_i \qquad \text{and} \qquad c_j(\mu(c_j, b) - \mu(a, c_j)) = p_j - p_i. \tag{3}$$

In both of these cases, the average quality of matching market j is $m_j = \mu(c_j, b)$ and the threshold type c_j is indifferent between the two markets. In the first case, the threshold type c_i is indifferent between participating in matching market i

with the average quality $m_i = \mu(c_i, c_j)$ and not participating at all; in the second case, type c_i is the lowest type a and it strictly prefers participating in matching market i with $m_i = \mu(a, c_j)$.

The assumption of complementarity in the match value function implies that participation decisions can be described by thresholds and that, in any dual matching market structure, higher types join the more expensive market. As a result, the singular matching market structures and the dual structures — together with the "null matching market structure", where agents participate in neither matching market — cover all possible equilibrium matching market structures.[6]

We now make an assumption that allows us to determine, for each $p_i \in [0, b^2]$, a price range $[\theta(p_i), \lambda(p_i)]$ for prices $p_j > p_i$ such that (a) the dual matching market structure D_{ij} cannot be supported for any $p_j < \theta(p_i)$ or $p_j > \lambda(p_i)$ and (b) there is a unique D_{ij} for any $p_j \in [\theta(p_i), \lambda(p_i)]$. The lower bound $\theta(p_i)$ is given by

$$\theta(p_i) = \begin{cases} p_i + \sqrt{p_i}(\mu(\sqrt{p_i}, b) - \sqrt{p_i}) & \text{if } p_i \geq a^2, \\ p_i + a(\mu(a, b) - a) & \text{if } p_i < a^2; \end{cases} \tag{4}$$

the upper bound $\lambda(p_i)$ is given by

$$\lambda(p_i) = \begin{cases} p_i + b(b - \mu(c_i, b)) & \text{if } p_i \geq a\mu(a, b), \\ p_i + b(b - \mu(a, b)) & \text{if } p_i < a\mu(a, b), \end{cases} \tag{5}$$

where c_i is uniquely determined by $c_i\mu(c_i, b) = p_i$ in the first case of (5).

Assumption 1. The density function f is non increasing.

This assumption leads to the following result.

Lemma 1. *Under Assumption 1, a unique dual matching market structure D_{ij} exists if and only if $p_j \in [\theta(p_i), \lambda(p_i)]$.*

Finally we make the following standard assumption of monotone hazard rate. Let $\rho(\cdot) = (1 - F(\cdot))/f(\cdot)$ be the inverse hazard rate function. We assume that $\rho'(\cdot) \leq 0$; this is equivalent to assuming that the right-tail distribution function $1 - F(\cdot)$ is log-concave, which implies that the conditional mean function $\mu(t, b)$ satisfies $\partial\mu(t, b)/\partial t \leq 1$ (An 1998). Hereafter, let μ_l and μ_r be the partial derivative of the conditional mean function with respect to the first and the second argument, respectively.

6. If $p_1 = p_2$ then, for the participation threshold c that satisfies (1), any strategy profile such that types above c join one of the two markets and $m_1 = m_2$ constitutes a Nash equilibrium. We assume that the two matchmakers evenly split the types above c; the analysis is unaffected by this assumption.

ASSUMPTION 2. The hazard rate function of F is non decreasing.

The uniform distribution and the exponential distribution are the two polar cases that satisfy Assumptions 1 and 2. The uniform distribution on $[a, b]$ has a constant density while the hazard rate is strictly increasing. The exponential distribution on $[a, \infty)$ has a strictly decreasing density while the hazard rate is constant.

2.2. Selection of matching market structures

Unlike in standard Bertrand price competition, in a matching environment the participation decisions of agents are not completely determined by prices: What an entrance fee buys for agents on one side of the matching market depends on participation decisions by agents on the other side of the market. Nash equilibrium alone does not pin down the matching market structure. It is possible to have multiple matching market structures for a given pair of prices. Indeed, from equations (1) it follows that, for any $p_1, p_2 \in [0, b^2]$, either of the two singular matching market structures S_1 and S_2 can be supported as equilibrium.

We adopt as our selection criterion the "stable set of equilibria" notion of Kohlberg and Mertens (1986). Their notion is a strengthening of trembling hand perfection in strategic-form games (Selten 1975) and is derived from robustness considerations in perturbed games where agents are constrained to non optimal participation decisions (trembles) with increasingly small probabilities. Loosely speaking, in our model a collection of matching market structures constitutes a stable set (in the sense of Kohlberg and Mertens) if it is a minimal collection with the property that every perturbed game has a Nash equilibrium close to some matching market structure in the collection. In the Appendix, we give a formal definition of a stable collection of matching market structures and prove the following result.[7]

LEMMA 2. *Assume $p_1 < p_2$. The unique stable collection of matching market structures is a singleton and contains: (i) S_2 if $p_2 < \theta(p_1)$; (ii) D_{12} if $\theta(p_1) < p_2 < \lambda(p_1)$; and (iii) S_1 if $p_2 > \lambda(p_1)$.*

Stability in the sense of Kohlberg and Mertens yields a unique selection of matching market structure, even though the concept is a set-based refinement — because, in general, different equilibria are needed to provide robustness against different perturbed games. To understand this strong result, let us consider case (i)

7. At the boundary between S_2 and D_{12} (when $p_2 = \theta(p_1)$), the two matching market structures are both stable; however, since they are outcome-equivalent, which one is selected is immaterial to our analysis. A similar observation applies to the boundary between D_{12} and S_1 (when $p_2 = \lambda(p_1)$).

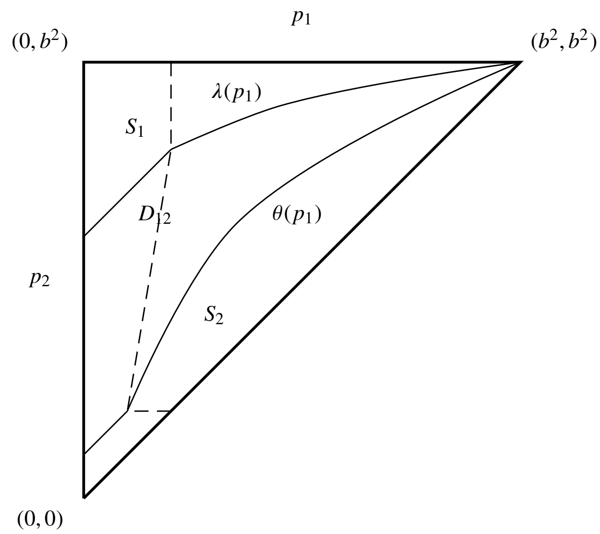

FIGURE 1. Stable matching market structures for $p_1 < p_2$

where $p_2 \in (p_1, \theta(p_1))$. In this case, the price difference is too small to support the dual matching market structure D_{12}, and the unique selection is the high-price singular market structure S_2. The low-price singular structure S_1 is not robust. This is because any perturbation in which high types are over represented in the high-price market would create a high quality there and, since the price difference is small, would further attract high types. As high types leave the low-price market, its quality decreases. This induces further deviations that unravel the low-price singular market structure.[8] In contrast, the high-price structure S_2 is robust. In any perturbation, the first types to deviate to the low-price market are the low types, which drives up the quality difference between the two markets and limits further deviations.

By a symmetric argument, a unique matching market structure is selected when $p_1 > p_2$. Figure 1 depicts the selected matching market structure for the case in which types are uniformly distributed on $[a, b]$. The dashed line represents the border between the region with high prices and full participation ($c_1 = a$) and the region with low prices and partial participation ($c_1 > a$). We refer to case (i) in Lemma 2 as matchmaker 2 "overtaking" matchmaker 1 and to case (iii)

8. A similar argument can be made if we adapt the concept of most likely deviating type from Banks and Sobel's (1987) theory of refinement in extensive games. In the low-price singular structure S_1, the type that is most likely to deviate to the high-price market is the highest type b. When $p_2 < \theta(p_1)$, type b agents would indeed want to deviate if they expect a sufficiently high quality in the high-price market.

as matchmaker 1 "undercutting" matchmaker 2. The strategy of overtaking is unique to the sorting role of prices. Overtaking a rival is achieved by charging an appropriately higher price than the rival does. This provides a higher quality market that induces a deviation from the rival's market by the highest types, which triggers further deviations by lower type agents and eventually drives out the rival. The overtaking strategy plays on the differences in willingness to pay for quality (average match type) between the highest and the lowest type agents participating in a market.

3. Duopolistic Sorting

In this section we analyze the equilibrium outcome under duopolistic competition. First we show that no pure-strategy equilibrium exists in the simultaneous-move pricing game. This result is established by showing that, by using the strategy of overtaking, each matchmaker can earn a revenue strictly greater than its competitor, which is impossible in a Nash equilibrium.

LEMMA 3. *There is no pure-strategy equilibrium in a simultaneous-move game.*

The non existence of pure-strategy equilibria in the simultaneous-move game points to a difference between competing matchmaking and the standard Bertrand price competition. As in Bertrand competition, payoff discontinuities exist in competing matchmaking because the matching market structure switches from one singular structure to the other when prices move from p_1 just below p_2 to p_1 just above p_2. Payoff discontinuities tend to homogenize prices in the absence of any asymmetry between the competitors. In Bertrand competition this leads to marginal cost pricing, but the same is not true in competing matchmaking because prices also play the role of sorting. If one matchmaker charges zero price, then the other matchmaker can charge a price in the region of dual market structure and earn a strictly positive revenue by sorting out the types willing to pay more for a higher match quality.

Rather than studying mixed-strategy equilibria in a simultaneous-move game, we look at pure-strategy (subgame perfect) equilibria in a sequential-move game.[9] We are able to identify intuitive conditions for a dual matching market structure to emerge in equilibrium as well as generate results about the sorting role of

9. Existence of a mixed-strategy equilibrium can be established using the concept of payoff security (Reny 1999). By charging a slightly higher price, each matchmaker can secure a payoff that is at worst only marginally lower against small perturbations of its rival's price. It follows that the mixed extension of our simultaneous-move game is "payoff secure" and hence a mixed strategy equilibrium in prices exists (see Reny 1999, Cor. 5.2).

prices and the implications for sorting efficiency of price competition. These insights would not be easily obtained in a mixed-strategy equilibrium analysis of the simultaneous-move game. Furthermore, although we do not claim that a sequential pricing setup is necessarily more realistic than a simultaneous-move setup, we believe the sequential setup is appropriate for an environment in which there is a natural price leader as a result of unmodeled factors such as incumbency or historic precedent.

In what follows, we consider a game where first matchmaker 1 picks a price p_1, and then, after observing p_1, matchmaker 2 chooses p_2. Matchmaker 2 has a clear advantage in this sequential-move game in that, for any price p_1, matchmaker 2 can overtake matchmaker 1 with a slightly higher price and earn a strictly greater revenue than matchmaker 1. Further, it turns out that when p_1 is very high, it is optimal for matchmaker 2 to undercut matchmaker 1, whereas overtaking is optimal for a very low price p_1. The best response function of matchmaker 2 is not monotone or continuous; in fact, it is undefined for some values of p_1. Hence, in this game the prices are neither strategic substitutes nor strategic complements.[10]

3.1. Surviving overtaking

Because of the overtaking strategy, matchmaker 2 has an advantage in the sequential-move game. We want to know whether this advantage is so overwhelming that matchmaker 1 cannot survive as a first mover. A possible strategy for matchmaker 1 to survive overtaking is to choose a price so low that matchmaker 2 finds it more profitable to create a more exclusive matching market than to overtake matchmaker 1 and drive it out of the competition. We say that the type distribution is "sufficiently diffused" if $\mu(a, b) > 3a/2$. Intuitively, when the type distribution is sufficiently diffused, there is room for two matchmakers to coexist because the lowest type's willingness to pay for a higher quality match is low relative to that of the higher type agents. When matchmaker 1 posts a sufficiently low price, overtaking effectively entails serving the entire market. The opportunity cost of overtaking is high, since matchmaker 2 could charge a much higher participation fee by focusing on a more exclusive matching market.

PROPOSITION 1. *If the type distribution is sufficiently diffused, then there exists a pure-strategy equilibrium with a dual matching market structure in a sequential-move game.*

Proof. Fix any $p_1 < a^2$. First, note that undercutting is dominated by overtaking for matchmaker 2. This is because, in both cases, matchmaker 2 will serve all

10. For a survey of recent developments in the literature on complementarities and the applications to oligopoly pricing, see Vives (2005).

types and overtaking generates a greater revenue with a higher price. It remains to show that, for p_1 sufficiently small, it is not optimal for matchmaker 2 to drive matchmaker 1 out of the market by overtaking. By equation (1), the singular matching market structure S_2 obtains for any $p_2 \in (p_1, \theta(p_1)]$. The threshold type of participation is $c_2 = a$ because $p_1 < a^2$ implies $\theta(p_1) < a\mu(a,b)$. Matchmaker 2's revenue from overtaking is simply p_2 for any $p_2 \in (p_1, \theta(p_1)]$, so the best overtaking price is $\theta(p_1)$. For any $p_2 \in (\theta(p_1), \lambda(p_1))$, the dual matching market structure D_{12} obtains. By equation (3), $c_1 = a$ and c_2 satisfies $c_2(\mu(c_2, b) - \mu(a, c_2)) = p_2 - p_1$. Consider how matchmaker 2's revenue in the dual matching market structure D_{12}, given by $p_2(1 - F(c_2))$, changes at $p_2 = \theta(p_1)$. Since $c_2 = a$ at $p_2 = \theta(p_1)$, the derivative of matchmaker 2's revenue with respect to p_2 at $\theta(p_1)$ is positive if and only if

$$\mu(a,b) - a + a\left(f(a)(\mu(a,b) - a) - \frac{1}{2}\right) > f(a)\theta(p_1). \tag{6}$$

As p_1 approaches 0, $\theta(p_1)$ approaches $a(\mu(a,b) - a)$. Thus, the derivative is positive at $p_2 = \theta(p_1)$ for p_1 approaching 0 if and only if $\mu(a,b) > 3a/2$. □

A sufficiently diffused distribution allows the first mover to survive the overtaking strategy of the second mover by focusing on a lower quality "niche" market. Note that the survival strategy of charging $p_1 < a^2$ for the first mover implies that all low types are served and that, with a relaxed participation constraint, some rents are left to the lowest type a. Also, the sufficient condition of Proposition 1 depends on the type distribution only through the unconditional mean $\mu(a,b)$. This is because, at the boundary between S_2 and D_{12}, the behavior of matchmaker 2's revenue is independent of the type distribution and is locally identical to that under the uniform type distribution.

3.2. Low-price niche market

By considering the second mover's incentives to overtake the first mover when the latter charges a sufficiently low price, Proposition 1 provides a sufficient condition for the two matchmakers to coexist in an equilibrium. The analysis leaves open the possibility that, in equilibrium, both matchmakers have positive market shares and the first mover charges a higher price. We now investigate this possibility.

We will need a result about the revenue function of a one-price monopolist, $(1 - F(c))p$, where c is determined by p via equation (1). In the Appendix we prove that the revenue function is quasi-concave in price p (Lemma A.1). Let $\hat{p}$ be the solution to the one-price monopolist's revenue maximization problem. We have the following result.

LEMMA 4. *Under the uniform type distribution, for any p_1 such that $\theta(p_1) > \hat{p}$, any best response p_2 of matchmaker 2 leaves zero revenue to matchmaker 1.*

Clearly, the optimal response of matchmaker 2 is $\hat{p}$ if it is a feasible undercutting or overtaking price (which occurs when $p_1 \in (\lambda(\hat{p}), b^2]$ or $p_1 \in (\theta^{-1}(\hat{p}), \hat{p})$ respectively). This leaves zero revenue to matchmaker 1. When $p_1 \in [\hat{p}, \lambda(\hat{p})]$, the maximum one-price monopolist revenue is not feasible. However, since $p_1 > \hat{p} > \theta^{-1}(\hat{p})$, overtaking matchmaker 1 dominates serving the higher quality market in a dual structure D_{12}. To see this, observe that any price $p_2 \in (\theta(p_1), \lambda(p_1))$ supporting D_{12} leads to a duopolist's revenue, which is lower than the one-price monopolist's revenue at the same price p_2 because the latter has a greater market share. Since $\theta(p_1) > \hat{p}$, the quasi-concavity of the revenue function of the one-price monopolist implies that this is, in turn, lower than the overtaking revenue reached by charging $\theta(p_1)$. The proof of Lemma 4 uses the assumption of uniform type distribution to rule out serving the lower quality market in a dual structure D_{21} by showing that this is dominated by either overtaking or undercutting.[11] In either case, matchmaker 1 gets zero revenue. The next result follows immediately from Lemma 4.

PROPOSITION 2. *Under the uniform type distribution, in any equilibrium with a dual matching market structure, the first mover serves the lower quality matching market.*

Proof. By Lemma 4, under the uniform type distribution we must have $p_1 \leq \theta^{-1}(\hat{p})$ in any equilibrium with a dual matching market structure. We claim that matchmaker 2's best response p_2 belongs to the interval $[\theta(p_1), \lambda(p_1))$. The proposition then immediately follows this claim, because either there is no equilibrium with a dual market structure if $p_2 = \theta(p_1)$ or else D_{12} is the equilibrium market structure. To establish the claim, note that charging $p_2 \in [\lambda(p_1), b^2]$ cannot be optimal because matchmaker 2 would have zero revenue. If instead matchmaker 2 chooses $p_2 \in [0, \theta(p_1))$, there are at most three possible scenarios. When the price pair (p_1, p_2) falls in the S_1 region, matchmaker 2 has no revenue. When the (p_1, p_2) falls in the S_2 region, matchmaker 2 is a monopolist. However, at price $p_2 = \theta(p_1)$, matchmaker 2 is also a monopolist but has a higher revenue because $\theta(p_1) \leq \hat{p}$ and because the revenue function of the monopolist is quasi-concave. Finally, when (p_1, p_2) falls in the D_{21} region, matchmaker 2's revenue is lower than the revenue of a one-price monopolist at the same price p_2, which is lower than the revenue generated by charging $p_2 = \theta(p_1)$ owing to the

11. In the range of p_1 for which there exists an overtaking price p_2 that yields a greater revenue than undercutting, matchmaker 2's best response does not exist. This is immaterial to our equilibrium construction and to all the results.

quasi-concavity. □

Proposition 2 shows that, when matchmaker 1 chooses a low price p_1 such that $\theta(p_1) \leq \hat{p}$, matchmaker 2's best response is either charging the maximum overtaking price $\theta(p_1)$ or serving the higher quality market in a dual structure D_{12}. Note that this result holds regardless of the type distribution. Lemma 4 (and hence the conclusion of Proposition 2) does depend on the assumption of the uniform type distribution, but the intuition is more general. By charging a very low price or a very high price, the first mover targets a niche market of few types and makes the overtaking strategy unappealing to the second mover. However, while a high price might invite undercutting, a low price is less vulnerable. Indeed, under the uniform type distribution, if the first mover charges a price high enough to deter overtaking then the second mover will find it optimal to undercut. Thus, in order to deter undercutting as well as overtaking, matchmaker 1 must find its niche market with low prices. See Section 3.6 for further discussion on the robustness of this result.

3.3. Market coverage

To study the effects on competition, we compare our duopoly model with a two-price monopoly matchmaker. This is a natural comparison because the number of potential matching markets is two in both cases. The monopolist's problem can be stated as choosing two participation thresholds, c_1 and c_2 with $c_1 \leq c_2$, to maximize total revenue:[12]

$$(1 - F(c_1))c_1\mu(c_1, c_2) + (1 - F(c_2))c_2(\mu(c_2, b) - \mu(c_1, c_2)). \tag{7}$$

Here we consider how competition affects the total matching market coverage — that is, the lower participation threshold.

Proposition 3. *In any equilibrium with the dual structure D_{12}, the market coverage is at least as large as in the optimal structure of a monopolist if matchmaker 2's revenue is quasi-concave in c_2 for any p_1.*

Proof. Rewrite the revenue of the monopolist (equation (7)) as

$$((1 - F(c_1))c_1 - (1 - F(c_2))c_2)\mu(c_1, c_2) + (1 - F(c_2))c_2\mu(c_2, b). \tag{8}$$

12. The proof of Lemma 6 in the Appendix shows that the monopolist will always choose prices such that a dual matching market structure obtains. Thus, applying the selection criterion introduced in Section 2, we can use participation thresholds (rather than prices) as choice variables for the monopolist's revenue maximization problem.

Since the first term in this expression can be made arbitrarily small with c_1 just below c_2, the optimal thresholds $\hat{c}_1$ and $\hat{c}_2$ satisfy

$$(1 - F(\hat{c}_1))\hat{c}_1 \geq (1 - F(\hat{c}_2))\hat{c}_2. \tag{9}$$

Differentiating (8) with respect to c_1 and assuming an interior $\hat{c}_1$, we find that

$$\frac{1 - F(\hat{c}_2)}{F(\hat{c}_2) - F(\hat{c}_1)}(\hat{c}_2 - \hat{c}_1) = \frac{\rho(\hat{c}_1)\mu(\hat{c}_1, \hat{c}_2) - \hat{c}_1^2}{\mu(\hat{c}_1, \hat{c}_2) - \hat{c}_1}.$$

If $\rho(\hat{c}_1) > \hat{c}_1$, then the right-hand-side of this equality is greater than $\hat{c}_1$, resulting in an inequality that contradicts (9). Thus, $\rho(\hat{c}_1) \leq \hat{c}_1$.

For duopolistic coverage, matchmaker 2 chooses c_2 to maximize its revenue,

$$(1 - F(c_2))(p_1 + c_2(\mu(c_2, b) - \mu(c_1, c_2))), \tag{10}$$

subject to

$$\begin{cases} c_1\mu(c_1, c_2) = p_1 & \text{if } c_1 > a, \\ a\mu(a, c_2) \geq p_1 & \text{if } c_1 = a. \end{cases} \tag{11}$$

It suffices to consider the case where c_1 is greater than a and is determined by (11) at some equilibrium price $p_1 = \tilde{p}_1$. Taking derivatives of (11) yields

$$\frac{dc_1}{dc_2} = -\frac{c_1\mu_r(c_1, c_2)}{\mu(c_1, c_2) + c_1\mu_l(c_1, c_2)}. \tag{12}$$

Since matchmaker 2's revenue function is quasi-concave in c_2, a necessary condition for equilibrium is that matchmaker 2's revenue increases with c_2 at the boundary between S_2 and D_{12} where $c_2 = c_1$ and $p_2 = \theta(\tilde{p}_1)$. At this point, equation (12) becomes $dc_1/dc_2 = -1/3$ because $\mu_l = \mu_r = 1/2$ at the boundary; by taking derivatives of (10) we find that this necessary condition is satisfied if and only if at some equilibrium lower threshold $\tilde{c}_1 > a$,

$$\rho(\tilde{c}_1)\left(\mu(\tilde{c}_1, b) - \frac{4}{3}\tilde{c}_1\right) > \tilde{c}_1^2. \tag{13}$$

Since $\mu_l(c, b) \leq 1$ by Assumption 2, for any c we have

$$\mu(c, b) - c \leq \rho(c). \tag{14}$$

Therefore, condition (13) can be satisfied only if $\rho(\tilde{c}_1) > \tilde{c}_1$. The proposition then follows immediately from Assumption 2. □

Without the assumption of quasi-concavity, condition (13) is generally not necessary for an equilibrium dual market structure. This is because matchmaker 2's revenue may decrease with c_2 at the boundary between S_2 and D_{12} and yet there is a price p_2 in the D_{12} region that dominates any overtaking price. In the Appendix we show that, under uniform type distribution, matchmaker 2's revenue is globally concave in c_2 for any p_1 in the D_{12} region (Lemma A.2). Moreover, by Proposition 2 it follows that, under the same assumption, any equilibrium dual market structure is D_{12} . Thus we have the following result.

COROLLARY 1. *If the type distribution is uniform then, in any equilibrium with a dual matching market structure, the market coverage is at least as large as in the optimal structure of a monopolist.*

The intuition behind this result is more general than implied by the uniform type distribution. Competition expands the total market coverage because of the first mover's need to survive price competition. Only by lowering its price sufficiently and catering to a low quality matching market can the first mover prevent overtaking.

3.4. Market differentiation

We now ask how the equilibrium market differentiation, in terms of how exclusive the high quality matching market is, compares with the optimal market differentiation that maximizes total revenue for the two-price monopolist.

DEFINITION 3. A dual matching market structure with participation thresholds $c_1 < c_2$ has a greater conditional market differentiation than another dual structure with thresholds $c'_1 < c'_2$ if $c_1 = c'_1$ and $c_2 > c'_2$.

Definition 3 limits our comparison of matching market structures to those with the same market coverage. We drop the qualifier "conditional" when there is no risk of confusion. Market differentiation here does not refer to the comparison in terms of the quality difference $m_2 - m_1$ between the two markets. Instead, it describes how exclusive the high quality matching market is: a dual matching market structure has a greater differentiation if $c_2(m_2 - m_1)$ is greater — in other words, if the participant with type equal to the higher threshold c_2 is willing to pay more for the quality difference between the two matching markets. Under Assumption 1, the quality difference $m_2 - m_1$ is non decreasing in c_2 for fixed c_1, so this interpretation of exclusivity coincides with Definition 3. Note that market differentiation in terms of exclusivity is what matters to revenue maximization for the two-price monopolist and also for the duopolist that serves the high quality matching market. We have the following comparison result.

PROPOSITION 4. *In any equilibrium with the dual structure D_{12}, the equilibrium outcome has less market differentiation than the optimal structure of a monopolist.*

Proof. The monopolist's differentiation problem is to choose c_2 to maximize (7), taking as given c_1 and subject to the constraint (11). The first order condition can be written as

$$\frac{\mu(\hat{c}_2, b) - \mu(c_1, \hat{c}_2)}{\hat{c}_2 - \mu(c_1, \hat{c}_2)} = \frac{1 - F(c_1)}{\rho(\hat{c}_2)} \frac{\hat{c}_2 - c_1}{F(\hat{c}_2) - F(c_1)}. \tag{15}$$

The right-hand side of (15) approaches 1 while the left-hand side becomes arbitrarily large when $\hat{c}_2$ takes on the value of c_1, and the opposite happens when $\hat{c}_2$ approaches b. Thus, for any c_1, there exists at least one $\hat{c}_2$ that satisfies (15). Furthermore, the right-hand side is increasing in $\hat{c}_2$ because Assumption 1 implies that $(\hat{c}_2 - c_1)/(F(\hat{c}_2) - F(c_1))$ increases with $\hat{c}_2$, and $\rho(\hat{c}_2)$ decreases with $\hat{c}_2$ by Assumption 2. The left-hand side decreases in $\hat{c}_2$ because $\mu_l(\hat{c}_2, b) \leq 1$ by Assumption 2. Thus, a unique $\hat{c}_2$ satisfies (15).

For duopolistic differentiation, matchmaker 2 chooses c_2 to maximize (10), taking as given p_1, and subject to (11). The first order necessary condition can be written as

$$\frac{(\mu(\tilde{c}_2, b) - \mu(c_1, \tilde{c}_2))}{(\tilde{c}_2 - \mu(c_1, \tilde{c}_2))/\rho(\tilde{c}_2)} = \frac{\tilde{c}_2(1 - F(c_1))}{F(\tilde{c}_2) - F(c_1)} \left(1 + \frac{\mu(c_1, \tilde{c}_2) - c_1}{\tilde{c}_2 - \mu(c_1, \tilde{c}_2)} \frac{dc_1}{dc_2}\right) + \frac{c_1 \mu(c_1, \tilde{c}_2)}{\tilde{c}_2 - \mu(c_1, \tilde{c}_2)},$$

where dc_1/dc_2 is given by (12). By Assumption 1, we have $\mu(c_1, \tilde{c}_2) - c_1 \leq \tilde{c}_2 - \mu(c_1, \tilde{c}_2)$. Moreover, $\mu_r(c_1, c_2) \leq 1/2 \leq \mu_l(c_1, c_2)$. Thus, the first order condition implies

$$\frac{(\mu(\tilde{c}_2, b) - \mu(c_1, \tilde{c}_2))}{(\tilde{c}_2 - \mu(c_1, \tilde{c}_2))/\rho(\tilde{c}_2)} > \frac{\tilde{c}_2(1 - F(c_1))}{F(\tilde{c}_2) - F(c_1)} - \frac{1 - F(c_2)}{F(\tilde{c}_2) - F(c_1)} \frac{\tilde{c}_2 c_1}{2\mu(c_1, \tilde{c}_2) + c_1}. \tag{16}$$

Comparing (16) and (15), we find that $\tilde{c}_2 < \hat{c}_2$ for any c_1 if

$$(2\mu(c_1, \tilde{c}_2) + c_1)(1 - F(c_1)) - (1 - F(\tilde{c}_2))\tilde{c}_2 \geq 0.$$

Observe that this inequality holds at $\tilde{c}_2 = c_1$. Further, the derivative of the left-hand side with respect to $\tilde{c}_2$ has the same sign as

$$2(1 - F(c_1)) \frac{\tilde{c}_2 - \mu(c_1, \tilde{c}_2)}{F(\tilde{c}_2) - F(c_1)} + \tilde{c}_2 - \rho(\tilde{c}_2).$$

This expression is strictly positive as $\tilde{c}_2$ approaches c_1 from above, and it is strictly increasing in $\tilde{c}_2$ because $(\tilde{c}_2 - \mu(c_1, \tilde{c}_2)/(F(\tilde{c}_2) - F(c_1))$ is weakly increasing in $\tilde{c}_2$ by Assumption 1 while $\rho(\tilde{c}_2)$ is weakly decreasing by Assumption 2. □

Definition 3 requires us to compare monopolistic and duopolistic differentiation for fixed market coverage. The proof of Proposition 4 establishes that, for any equilibrium coverage, at the optimal choice of differentiation $\tilde{c}_2$ of matchmaker 2, the monopolist's revenue is strictly increasing in c_2. Since it uses only the first order necessary condition, the proof does not require the assumption that matchmaker 2's revenue is quasi-concave. Under uniform type distribution we can strengthen the proposition as follows.

COROLLARY 2. *If the type distribution is uniform then, in any equilibrium with the dual matching market structure, the equilibrium outcome has less market differentiation than the optimal structure of a monopolist.*

Intuitively, when choosing its own price, matchmaker 2 does not internalize the cannibalization of the lower market. Rewrite the monopolist's revenue function (7) as

$$(F(c_2) - F(c_1))p_1 + (1 - F(c_2))(p_1 + c_2(\mu(c_2, b) - \mu(c_1, c_2))), \qquad (17)$$

and compare it with matchmaker 2's objective function (10). According to (11), c_1 either stays constant at a or increases as c_2 decreases, so the first term that appears in (17) but is absent from (10) means that duopolist matchmaker 2 has a greater incentive to lower c_2 than does a monopolist.[13] Such incentive exists regardless of the type distribution. The uniform distribution assumption is used to ensure that the equilibrium matching market structure is D_{12} by way of Proposition 2.

3.5. Welfare comparison

To complete the comparison between duopolistic matchmaking and monopolistic matchmaking, we now examine welfare in terms of the total match value, which is given by

$$(F(c_2) - F(c_1))\mu^2(c_1, c_2) + (1 - F(c_2))\mu^2(c_2, b) \qquad (18)$$

13. The proof of Proposition 4 is complicated by the fact that the constraint (11) has different implications for the monopolist and the duopolist matchmaker 2: the former chooses c_2 for fixed c_1 with p_1 determined by (11), whereas the latter chooses c_2 for fixed p_1 with c_1 determined by (11).

for any pair of participation thresholds c_1 and c_2 with $c_1 \leq c_2$. A useful benchmark for the comparison is the two-market planner's problem: choosing the efficient thresholds c_1^* and c_2^*, with $c_1^* \leq c_2^*$, to maximize the total match value (18).[14] We first compare optimal coverage $\hat{c}_1$ and efficient coverage c_1^*.

LEMMA 5. *Monopolistic market coverage is at most the efficient coverage.*

This result that the monopolist's matching markets are smaller and more selective than the planner's does not require the assumption of uniform type distribution. In particular, following the standard price discrimination literature, we can define "virtual type" of x as $x - \rho(x)$. As shown in Proposition 3. the monopolist will never serve agents of negative virtual types, and this establishes that the optimal coverage $\hat{c}_1$ satisfies $\hat{c}_1 \geq \rho(\hat{c}_1)$. In contrast, the planner will service additional low types so long as the benefit from the expansion of the market coverage is not outweighed by the loss due to the reduction in the average quality of the lower matching market. The proof of Lemma 5 shows instead $c_1^* < \rho(c_1^*)$ whenever $c_1^* > a$, implying that $\hat{c}_1 \leq c_1^*$ by Assumption 2. Next, we compare the optimal market differentiation $\hat{c}_2$ for the two-price monopolist with the efficient differentiation c_1^* for the two-market planner under any total coverage c_1.

LEMMA 6. *Monopolistic differentiation is efficient if the type distribution is uniform.*

For both the planner and the monopolist, increasing c_2 raises the quality in both matching markets at the expense of reducing the relative size of the higher quality market. The effect on the objective functions is generally different, because the monopolist is concerned with the change in the marginal type's willingness to pay whereas the planner cares about the change in the average expected type. Lemma 6 shows that the effect is the same for type distributions, including uniform and exponential distributions, with a linear conditional mean function $\mu(\cdot, b)$.

Since by Lemma 6 the monopolist and the planner have identical incentives for market differentiation under the uniform type distribution, Corollary 2 implies that competition between the two matchmakers induces a smaller (and less efficient) degree of market differentiation. On the other hand, Lemma 5 establishes that the monopolist has an inefficiently small market coverage; by Corollary 1, duopolistic matchmaking may correct this distortion. The trade-off between differentiation and coverage thus implies that the welfare comparison between duopolistic matchmaking and monopolistic matchmaking in terms of total

14. We implicitly assume that the planner is restricted to threshold participation strategies. This may be motivated by the assumption that the planner faces the same informational constraints.

match value can go either way. The comparison generally depends on how diffused the type distribution is. For the uniform type distribution, the degree of diffusion is determined by the value of a/b, with a lower value of this ratio corresponding to a more diffused distribution. We have the following result.

PROPOSITION 5. *If the type distribution is uniform, then duopolistic matchmaking generates a smaller total match value than monopolistic matchmaking if and only if the diffusion of the type distribution falls below a critical value.*

Proof. Under the uniform type distribution, the total match value (18) is given by

$$R(c_1, c_2) = \frac{1}{4(b-a)} \left((c_2 - c_1)(c_1 + c_2)^2 + (b - c_2)(c_2 + b)^2 \right).$$

Note that the comparison between the total match value $\tilde{R}$ under duopolistic matchmaking and $\hat{R}$ under monopolistic matchmaking depends on a only through its effects on the equilibrium thresholds $\tilde{c}_1$ and $\tilde{c}_2$ versus the monopolist's optimal thresholds $\hat{c}_1$ and $\hat{c}_2$.

Under the two-price monopolist, the optimal thresholds $\hat{c}_1$ and $\hat{c}_2$ can be solved from the first order conditions with respect to c_1 and c_2, derived from (7). This yields $\hat{c}_1 = \max\{a, h\}$ and $\hat{c}_2 = (\hat{c}_1 + b)/2$, with $h = b(2\sqrt{6} + 3)/15$. Since $h > b/2$, it follows that $\hat{c}_1$ and $\hat{c}_2$ are constant in a for any $a/b < 1/2$. The total match value $\hat{R}$ is then $R(\hat{c}_1, \hat{c}_2)$.

For duopolistic matchmaking, we distinguish three cases. In the first case, a/b lies between $\sqrt{19} - 4$ and $1/2$. By Propositions 1 and 2, an equilibrium with a dual market structure D_{12} exists. Since a necessary condition for an equilibrium with a lower threshold $\tilde{c}_1 > a$ is condition (13), which under the uniform distribution becomes $\tilde{c}_1 < (\sqrt{19} - 4)b$, the equilibrium satisfies $\tilde{c}_1 = a$. In this case, the equilibrium higher threshold $\tilde{c}_2$ can be computed explicitly by backward induction: the best response of matchmaker 2 to any p_1 is $c_2 = b/2 - p_1/(b-a)$; the equilibrium price for matchmaker 1 is $\tilde{p}_1 = (b - 2a)(b - a)/4$; and finally $\tilde{c}_2 = (b + 2a)/4$. The total match value $\tilde{R}$ for a/b between $\sqrt{19} - 4$ and $1/2$ is then $R(a, (b + 2a)/4)$. Comparing $\tilde{R}$ with $\hat{R}$, we find that there is a critical value of a/b between $\sqrt{19} - 4$ and $1/2$ such that $\tilde{R} < \hat{R}$ if and only if a/b is greater than this critical value.

In the second case, a/b is smaller than $\sqrt{19} - 4$. Explicit formulas for $\tilde{c}_1$ and $\tilde{c}_2$ are not easily obtained because $\tilde{c}_1$ may exceed a, but we make the following two observations. First, if $\tilde{c}_1 > a$ then $\tilde{c}_1$ and $\tilde{c}_2$ are both independent of a. This is because, under the uniform type distribution, equations (10) and (11) imply that matchmaker 2's best response does not depend on a, which in turn implies that matchmaker 1's problem of $\max_{p_1} (F(c_2) - F(c_1))c_1\mu(c_1, c_2)$ does not depend on a either. Second, the equilibrium thresholds $\tilde{c}_1$ and $\tilde{c}_2$ are continuous in the parameter a/b. Hence for some $a/b \leq \sqrt{19}-4$ we have $\tilde{R} = R(\tilde{c}_1, \tilde{c}_2) = R(a, (b+2a)/4)$, which is strictly greater than $\hat{R}$ by direct calculation. Since $\tilde{c}_1$ and $\tilde{c}_2$ do not depend on a, we have $\tilde{R} > \hat{R}$ for all a/b such that $\tilde{c}_1 > a$.

In the third case, $a/b \geq 1/2$. We prove in the Appendix that there is no equilibrium with a dual market structure (Lemma A.3). Then, there does exist a continuum of equilibria indexed by the price charged by the first mover. In any such equilibrium, the total match value does not exceed the value achieved by a one-market planner that solves $\max_c(1 - F(c))\mu^2(c, b)$. Under the uniform type distribution, the planner's solution is $c^* = a$. Thus, the maximum total match value $\tilde{R}$ under duopolistic matchmaking is $R(a, a)$. It is easily verified that $\hat{R} > R(a, a)$ for all $a/b \geq 1/2$. □

It is the intermediate values of a/b that best reveal why the equilibrium outcome is less efficient in sorting than the monopoly outcome when the type distribution is not too diffused. That is, when a/b lies between $\sqrt{19} - 4$ and $1/2$, it is efficient to serve all types (i.e. $c_1^* = a$). The monopolistic coverage is inefficiently small with $\hat{c}_1 > a$ even though its differentiation is efficient. In contrast, the equilibrium outcome has the efficient market coverage with $\tilde{c}_1 = a$ but suffers from inefficiently small market differentiation. When a/b is large in this range, the loss from insufficient coverage under monopoly is small relative to the gain in efficient differentiation because the optimal coverage becomes close to the efficient coverage. As a result, sorting is more efficient overall under monopoly than under competition. The intuition is similar for extreme values of diffusion of the type distribution. Indeed, the trade-off between coverage and differentiation disappears when a/b is sufficiently high (more precisely, if a/b is greater than h, defined in the preceding proof): the monopolistic coverage is efficient, while the duopolistic differentiation is nonexistent because the first mover cannot survive overtaking.

3.6. Discussion

Our results comparing duopolistic sorting and monopolistic sorting in terms of market differentiation and market coverage are obtained under the assumption of uniform type distribution, although both the non existence of pure-strategy equilibrium in the simultaneous pricing game (Lemma 3) and the sufficient condition for existence of dual market structure in the sequential pricing game (Proposition 1) hold more generally. Among non uniform distributions of particular interest is the exponential distribution, which has a density function $\exp(-(x - a)/\beta)/\beta$ with $a, \beta > 0$ and satisfies Assumptions 1 and 2. Since the support of the type distribution is unbounded, the upper bound function λ is undefined and undercutting is not feasible. As a result, matchmaker 1 can always survive overtaking by charging a price p_1 sufficiently high that it becomes more profitable for matchmaker

2 to serve low types in the dual market structure D_{21}. Indeed, we can show that if $a/\beta > 2$ then the equilibrium matching market structure is D_{21}.[15] Intuitively, serving low types is lucrative when a is great and the type distribution is tightly concentrated on these types (i.e. when β is small). In this case, the first mover would be overtaken by the second mover if it tried to compete for low types. This fact forces the first mover to serve a niche market of high types. In this kind of equilibrium, we expect differentiation to be greater under competition than under monopoly, because the second mover does not internalize the negative impact on the size of the first mover's market in increasing the lower participation threshold. Since the exponential distribution has a linear conditional mean function, our result about the efficiency of monopolistic market differentiation continues to hold (Lemma 6). Thus, duopolistic differentiation remains less efficient than under monopoly matchmaking.

A restriction in the present model of competing matchmaking is that each matchmaker is allowed to use only one price and thus to create only one matching market. We have made this assumption to simplify the analysis.[16] The results comparing the monopolist and the planner in terms of matching market coverage and differentiation turn out to be robust with respect to the restriction to two matching markets. In an earlier paper (Damiano and Li 2007), we consider the problem of a monopoly matchmaker that uses a schedule of entrance fees to sort different types of agents on the two sides of a matching market into exclusive markets wherein agents randomly form pairwise matches. That paper uses a more general setup than our model here of monopolistic sorting, and the former accomodates asymmetric type distributions and an unrestricted number of matching markets.[17] By the results of Damiano and Li (2007), our Assumption 2 here is sufficient to imply that the monopolist unconstrained in the number of matching markets has the same incentive as the planner to perfectly sort all participating types (i.e. one market for each participating type) and that the market coverage for the monopolist is at most as large as the efficient full coverage for the planner. It is also straightforward to establish that, under the uniform type distribution, for

15. This inequality is exactly the opposite of the condition in Proposition 1 for the exponential case. If $a/\beta > 2$ then, for any price p_1 below a^2, the maximum revenue for matchmaker 2 as a duopolist in D_{12} is reached at the boundary between D_{12} and S_2, leaving zero revenue for matchmaker 1. Moreover, in this parameter range, the one-price monopolist's optimal price is $\hat{p} = a(a + \beta)$ and so matchmaker 2's best response to $p_1 \in (a^2, \hat{p})$ is to overtake with price $p_2 = \hat{p}$ leaving zero revenue for matchmaker 1. Finally, for prices p_1 above $\hat{p}$, the best response of matchmaker 2 is either to be a duopolist in D_{21} with a price $p_2 < \theta^{-1}(p_1)$, or to overtake with a price p_2 just above p_1.

16. McAfee (2002) shows that most of the efficiency gains in sorting can be made with a total of just two matching markets. He does not consider the incentives of market participants.

17. Rayo (2002) studies how a monopolist can use price discrimination to sell status goods. This monopolist's problem can be interpreted as a special case of the matching model of Damiano and Li (2007) by assuming that the two sides have identical type distributions.

any finite number of matching markets that can be offered: total market coverage is at least as large for the planner as for the monopolist; and monopolistic market differentiation is efficient given the total market coverage. For price competition, it turns out that the result of inefficient sorting under competition (Corollary 2) is robust, but the extent of sorting inefficiency depends on the number of matching markets. In the extreme case when matchmakers can create an arbitrarily large number of matching markets — and hence perfect sorting of all agents is possible — price competition would not lead to inefficient sorting because the type distribution in each matching market is degenerate and so the overtaking strategy completely loses its power. This is likewise the case when there is free entry for matchmakers, even if each matchmaker can create only a single matching market. However, as long as types are not perfectly sorted, overtaking is possible and price competition interferes with sorting. When choosing their pricing structure, each matchmaker fails to internalize its effect on the market share of the competitors, and this leads to sorting inefficiency.

4. Concluding Remarks

Sorting of heterogeneous types is an essential ingredient in the literature on unintermediated matching markets, since the match formation decisions of participants in a matching market depend on the distribution of types in the market (Burdett and Coles 1997; Shimer and Smith 2000; Damiano, Li, and Suen 2005). However, as already discussed in Section 1, the existing literature on competing intermediaries in matching markets has so far ignored the issue of sorting and instead has focused exclusively on the size effects. More broadly, some recent papers on directed search and competitive search markets allow for type heterogeneity; however, since there is no complementarity in a buyer seller or worker firm matching market, prices do not play any sorting role (see Montgomery 1991; Mortensen and Wright 2002; Inderst 2005). By introducing type heterogeneity into a model of competing matchmakers, we have highlighted the role of prices in coordinating participants' market decisions and determining match qualities and have derived important implications of price competition for sorting efficiency. Our results on the potential inefficiencies of price competition do not suggest that competition is necessarily harmful or that monopoly is always desirable, but they do mean that regulatory policies in a matching environment should not be exclusively focused on enhancing price competition so as to expand market coverage. Attention must also be paid to how price competition interacts with the sorting of heterogeneous agents. The gain from expanding market coverage to additional low types may need to be weighed against the loss that results from less efficient sorting.

When perfect sorting is not feasible, price competition interferes with the sorting role of prices. How compelling is the assumption of imperfect sorting ultimately depends on how heterogeneous we think agents are. If only a few types of agents can be profitably distinguished, then perfect sorting is likely to be feasible and so the benefits of competition in terms of greater market coverage will tend to outweigh any sorting inefficiency. In contrast, when the type space is very rich, it is unlikely that sufficiently many matching markets can be created to perfectly sort all agents — either because the cost of market creation is too high or because the presence of some size effect makes thin markets unattractive. In this environment the benefits from sorting are large, and we may expect a monopolist to induce a more efficient matching market structure than competing matchmakers.

Appendix

A.1. Proof of Lemma 1

Assumption 1 implies that $\mu(t,x') - \mu(x,t)$ is a non decreasing function in t for any $t \in (x,x') \subset [a,b]$. To see this, first we note that

$$\mu_l(t,x') = \frac{f(t)(\mu(t,x') - t)}{F(x') - F(t)} \quad \text{and} \quad \mu_r(x,t) = \frac{f(t)(t - \mu(x,t))}{F(t) - F(x)}.$$

Observe that $\mu_l(t,x')$ converges to $1/2$ as x' approaches t. Further, the derivative of $\mu_l(t,x')$ with respect to x' has the same sign as $(t+x')/2 - \mu(t,x')$, which is non negative if $f'(\cdot) \leq 0$. Thus $\mu_l(t,x') \geq 1/2$ if $f'(\cdot) \leq 0$. Similarly, $\mu_r(x,t)$ converges to $1/2$ as x approaches t and is non decreasing in x if $f'(\cdot) \leq 0$, implying that $\mu_r(x,t) \leq 1/2$. Non increasing density is therefore sufficient to imply that $\mu(t,x') - \mu(x,t)$ is non decreasing in t as $\mu_l(t,x') \geq 1/2 \geq \mu_r(x,t)$.

Next we establish the sufficiency of the monotonicity condition on $\mu(t,x') - \mu(x,t)$. Consider first the case of D_{12} with $p_1 \geq a\mu(a,b)$. At $p_2 = \theta(p_1)$, equations (2) are satisfied by $c_1 = c_2 = \sqrt{p_1}$; under the monotonicity condition, as p_2 decreases c_2 decreases while c_1 increases, and thus there is no solution in c_1 and c_2 with $c_1 < c_2$ if $p_2 < \theta(p_1)$. Similarly, at $p_2 = \lambda(p_1)$, equations (2) are satisfied by $c_2 = b$ and c_1 such that $c_1\mu(c_1,b) = p_1$; under the monotonicity condition, there is no solution in c_1 and c_2 to equations (2) if $p_2 > \lambda(p_1)$. Finally, for $p_2 \in [\theta(p_1), \lambda(p_1)]$, the monotonicity condition implies that there is a unique pair of participation thresholds c_1 and c_2 satisfying equations (2). The cases of $p_1 < a^2$ and $p_1 \in [a^2, a\mu(a,b))$ can be similarly established, and a symmetric argument holds for D_{21}.

A.2. Kohlberg and Mertens Stability and Proof of Lemma 2

In this section we define the notion of a stable collection of matching market structures in the sense of Kohlberg and Mertens (1986) and also prove Lemma 2. Fix any p_1 and p_2 with $p_1 < p_2$, and consider the simultaneous-move game of agents choosing whether to participate in matching market 1 or 2, or not to participate at all. Let γ^ε be a perturbed game where, for each type x and each of the three participation choices, some fraction strictly between 0 and ε of type x agents is constrained to that choice.

DEFINITION A.1. A collection T of matching market structures is stable if: (i) for any $\eta > 0$, there exists an $\hat{\varepsilon} > 0$ such that, for all $\varepsilon < \hat{\varepsilon}$, any game γ^ε has a Nash equilibrium in which at most a fraction η of all agents makes a participation choice that differs from the one made in some matching market structure in T; and (ii) no strict subset of T satisfies property (i).

To prove Lemma 2, we consider only the case of $p_2 > \lambda(p_1)$ and show that the unique stable collection is a singleton that contains S_1; the other two cases can be similarly proved. First we demonstrate that, as ε becomes small, each perturbed game γ^ε has a Nash equilibrium that is arbitrarily close to S_1 in terms of participation decisions. Let m_1^ε and m_2^ε be the mean quality of the agents constrained to participating in matching market 1 and 2, respectively, in γ^ε. For each $i = 1, 2$, we use $\mu_i^\varepsilon(t, t')$ to denote the conditional mean in the interval (t, t') after excluding the agents constrained to not participating or to participating in market $j \neq i$. Let c_1^ε solve (1), with μ_1^ε in place of the conditional mean in market 1. Consider the strategy profile in which unconstrained agents of types lower than c_1^ε do not participate while all other unconstrained agents participate in matching market 1. Take any sequence of games $\{\gamma^\varepsilon\}_{\varepsilon\to 0}$. For any such sequence and any pair of thresholds $t < t'$, we have that $\mu_1^\varepsilon(t, t')$ converges to $\mu(t, t')$. For ε small, c_1^ε is close to the solution to (1). Since $m_2^\varepsilon \leq b$ and $p_2 > \lambda(p_1)$, it follows from the definition of $\lambda(p_1)$ that $b(m_2^\varepsilon - \mu_1^\varepsilon(c_1^\varepsilon, b)) + p_1 < p_2$ for ε sufficiently small. Then the proposed strategy profile is a Nash equilibrium of γ^ε, and it converges to S_1 as ε converges to 0.

Next we show that, for any sufficiently small ε, there exists some game γ^ε that does not have a Nash equilibrium close to any other matching market structure. Consider a sequence of games $\{\gamma^\varepsilon\}_{\varepsilon\to 0}$ where both m_1^ε and m_2^ε are close to b for all ε. (This is possible because we can make the probability of a tremble for the highest type converge to 0 infinitely slower than for all other types.) To rule out D_{12}, suppose that for each γ^ε there is a Nash equilibrium in which both matchmakers have a strictly positive market share, and let $c_1^\varepsilon < c_2^\varepsilon < b$ be the thresholds in this equilibrium. Then c_1^ε and c_2^ε must solve equations (2) or equations (3), with μ_1^ε and μ_2^ε in place of the conditional means in the two markets.

However, $p_2 > \lambda(p_1)$ and so, as ε becomes small, neither of systems (2) nor (3)has a solution—a contradiction. To rule out S_2, suppose there is a sequence of equilibria for ε approaching zero such that, in the limit, only matchmaker 2 has a positive market share. In such sequence, the marginal participating type c_2^ε in market 2 must converge to the solution to $c_2\mu(c_2, b) = p_2$. As ε becomes small, the quality of matching market 1 can be m_1^ε, or $\mu_1^\varepsilon(c_1^\varepsilon, c_2^\varepsilon)$ with c_1^ε converging to c_2^ε, or somewhere in between. Since $p_2 > \theta(p_1)$ it follows that, for ε sufficiently small, c_2^ε will strictly prefer joining market 1 — a contradiction. Finally, to rule out the null market structure, suppose there is a sequence of equilibria such that, in the limit, neither matchmaker has a positive market share. In any such sequence, market 1's quality is m_1^ε. Since m_1^ε is arbitrarily close to b and since $p_1 < b^2$, for ε sufficiently small the highest type agents will strictly prefer joining market 1 to not participating — a contradiction.

A.3. Proof of Lemma 3

First, note that only a dual matching market structure with strictly positive revenues for both matchmakers is a candidate for an equilibrium outcome. This is because, for any competitor's price, say p_1, the other matchmaker can earn a strictly positive revenue by either overtaking or undercutting.

Second, in any dual matching market structure, each matchmaker could use the overtaking strategy to earn a revenue strictly greater than its competitor. Therefore, no dual matching market structure can be the outcome of an equilibrium in pure strategies either. To see this point, without loss of generality suppose that $0 < p_1 \le p_2 < b^2$ and consider the dual matching market structure D_{12}. The participation thresholds c_1 and c_2 are determined by equations (2) or by equation (3) when $c_1 = a$,. If matchmaker 2 charges a price just above p_1 (say, $p_1 + \varepsilon'$) then, when $c_1 > a$, matchmaker 2 becomes a monopolist and the participation threshold c' is determined as in (1). Comparing the two equations $c'\mu(c', b) = p_1 + \varepsilon'$ and $c_1\mu(c_1, c_2) = p_1$, we conclude that $c' < c_1$ for some ε' slightly greater than zero. In the case of $c_1 = a$, matchmaker 2 becomes a monopolist by charging a price just above p_1, and the participation threshold $c' = a$. In either case, matchmaker 2 earns a strictly greater revenue in deviation than matchmaker 1 does in the dual matching market structure through a higher price and a larger matching market.

Similarly, given p_2, if matchmaker 1 overtakes with a price just above p_2 (say, $p_2+\varepsilon''$) then matchmaker 1 becomes a monopolist and the participation threshold c'' is determined as in (1). Since

$$c_2\mu(c_2, b) = p_2 + c_2\mu(c_1, c_2) - p_1 > p_2 + c_1\mu(c_1, c_2) - p_1 \ge p_2,$$

for some ε'' slightly greater than zero we have $c'' < c_2$. Thus, in the dual market structure, matchmaker 1 earns a strictly greater revenue in deviation than matchmaker 2.

A.4. Lemma A.1 and Proof

LEMMA A.1. *The revenue function of a one-price monopolist is quasi-concave in price p.*

Proof. Consider the equivalent problem of choosing a threshold $c \geq a$ to maximize $(1 - F(c))c\mu(c, b)$. If the optimal threshold $\hat{c}$ is interior then it satisfies the first order condition $\rho(\hat{c})\mu(\hat{c}, b) - \hat{c}^2 = 0$. Since $\mu_l(c, b) \leq 1$ and $\rho'(c) \leq 0$, the derivative of $\rho(c)\mu(c, b) - c^2$ is less than $\rho(c) - 2c$, which is less than 0 at any $\hat{c}$ that satisfies the first order condition. It follows that the monopolist's revenue function is quasi-concave in c. Since the revenue is simply p for $p < a\mu(a, b)$ and since there is a one-to-one relation (given by 1) between c and p for $p \geq a\mu(a, b)$, the revenue function is also quasi-concave in p. □

A.5. Proof of Lemma 4

We need only show that, for any $p_1 \in [\hat{p}, \lambda(\hat{p})]$, any best response of matchmaker 2 leaves zero revenue to matchmaker 1. For any such price p_1, matchmaker 2 has at most four viable options. (i) Matchmaker 2 can overtake by charging $p_2 \in (p_1, \theta(p_1)]$. By quasi-concavity, matchmaker 2's maximum overtaking revenue is $(1 - F(c_2))p_1$, with a price p_2 arbitrarily close to p_1 and with c_2 satisfying $c_2\mu(c_2, b) = p_1$ by (1). (ii) Matchmaker 2 can undercut by charging $p_2 \in [0, \lambda^{-1}(p_1)]$ (when $\lambda^{-1}(p_1)$ is defined). Since $p_1 \leq \lambda(\hat{p})$, quasi-concavity implies that the maximum undercutting revenue is $(1 - F(c_2))\lambda^{-1}(p_1)$, which is obtained by charging $p_2 = \lambda^{-1}(p_1)$ and with c_2 satisfying $c_2\mu(c_2, b) = \lambda^{-1}(p_1)$ by (1). (iii) Matchmaker 2 can allow the dual structure D_{12} by charging $p_2 \in [\theta(p_1), \lambda(p_1)]$. However, give quasi-concavity, this option is dominated by the option of overtaking. (iv) Matchmaker 2 can allow the dual structure D_{21} by charging $p_2 \in (\lambda^{-1}(p_1), \theta^{-1}(p_1))$.

We want to use the assumption of uniform type distribution to show that option (iv) is never optimal because it is dominated by either overtaking or undercutting. Observe that the maximum overtaking revenue decreases in p_1 while the maximum undercutting revenue increases in p_1. In addition — because, for fixed $p_2 < p_1$, as p_1 increases c_2 either decreases or does not change and c_1 increases (see equations (2) and equations (3), with the roles of the two matchmakers reversed) — matchmaker 2's maximum revenue in D_{21} is increasing in p_1.

The argument for ruling out $p_2 \in (\lambda^{-1}(p_1), \theta^{-1}(p_1))$ relies on two claims. The first claim is that there is a critical price $\bar{p}$ such that, for any $p_1 \geq \bar{p}$, matchmaker 2's maximum revenue as a duopolist is achieved at the boundary between

S_2 and D_{21}. Then the maximum revenue as a duopolist coincides with the maximum undercutting revenue for any $p_1 \geq \bar{p}$, with zero revenue for matchmaker 1. The second claim is that, at $p_1 = \bar{p}$, the maximum undercutting revenue is smaller than the maximum overtaking revenue. For any $p_1 < \bar{p}$, the maximum revenue as a duopolist is achieved in the interior of the D_{21} region. However, for fixed p_2, the revenue to matchmaker 2 in D_{21} is increasing in p_1 and so its maximum revenue is also increasing in p_1. Since the maximum overtaking revenue is decreasing in p_1, it follows from the second claim that the maximum revenue as a duopolist for any $p_1 < \bar{p}$ is smaller than the maximum overtaking revenue at the same p_1.

The derivation of $\bar{p}$ and the proof of the two claims depend on whether the price pair $(\bar{p}, \lambda^{-1}(\bar{p}))$ is located at the boundary between D_{21} and S_2 where $c_2 > a$ or $c_2 = a$. We will assume $c_2 > a$; the other case is similar. Consider the problem of choosing c_2 to maximize the revenue for matchmaker 2 in D_{21}; this revenue is given by $(F(c_1) - F(c_2))c_2\mu(c_2, c_1)$, where c_1 satisfies $p_1 = c_1(\mu(c_1, b) - \mu(c_2, c_1)) + c_2\mu(c_2, c_1)$. Under the uniform type distribution, the above relation becomes $p_1 = (c_1 b + c_2^2)/2$ and so $dc_1/dc_2 = -2c_2/b$. One can verify that matchmaker 2's revenue is concave in c_2 and hence that the optimal c_2 satisfies the first order condition

$$\frac{1}{2}c_1^2 - \left(\frac{2c_1}{b} + \frac{3}{2}\right)c_2^2 = 0. \tag{A.1}$$

Because $c_1 = b$ at the boundary between S_2 and D_{21}, there is a unique $\bar{p}_1 = (4/7)b^2$ such that (A.1) holds with equality at $p_2 = \lambda^{-1}(\bar{p}_1)$. Moreover, straightforward calculations reveal that, at $\bar{p}_1 = (4/7)b^2$, matchmaker 2's maximum undercutting revenue is smaller than its maximum overtaking revenue. Thus we have established both claims and so the lemma is proved.

A.6. Lemma A.2 and Proof

LEMMA A.2. *Under the uniform type distribution, matchmaker 2's revenue function is concave in c_2 for any p_1 in the D_{12} region.*

Proof. There are three cases, depending on p_1. In the first case $p_1 \leq a^2$, which implies $c_1 = a$. Under uniform type distribution, the derivative of (10) with respect to c_2 is proportional to $-p_1 + (b - a)(b - 2c_2)/2$. Thus, matchmaker 2's revenue is concave in c_2. In the second case $p_1 \geq a\mu(a, b)$, which implies that c_1 is greater than a and it satisfies $c_1\mu(c_1, c_2) = p_1$. Under uniform type distribution, using constraint (11) and differentiating (10) twice with respect to c_2, we find that the revenue function is concave in c_2 if

$$-\left(b - c_1 + \frac{c_1 c_2}{2c_1 + c_2}\right) + \frac{(b - c_2)c_1}{2c_1 + c_2} < 0.$$

This is equivalent to $-b(c_1+c_2)+2c_1^2-c_1c_2 < 0$, which is true because $c_1 \leq c_2 \leq b$. In the third case $p_1 \in (a^2, a\mu(a,b))$ and there is a critical value $\bar{c}_2$ satisfying $a\mu(a, \bar{c}_2) = p_1$ such that $c_1 > a$ for $c_2 < \bar{c}_2$ and $c_1 = a$ for $c_2 \geq \bar{c}_2$. By constraint (11), c_1 decreases in c_2 to the left of $\bar{c}_2$ and is constant to the right. It then follows from the revenue function (10) that the derivative with respect to c_2 jumps down at $\bar{c}_2$. Since the revenue function is concave to either side of the kink, it is globally concave in c_2. □

A.7. Proof of Lemma 5

We need only consider the case where the efficient c_1^* for the planner is interior. By differentiating the objective function (18) with respect to c_1, we find that the efficient thresholds c_1^* and c_2^* satisfy the first order condition $f(c_1^*)(\mu(c_1^*, c_2^*) - 2c_1^*) = 0$; it then follows from (14) that $\rho(c_1^*) > c_1^*$. Since $\rho(\hat{c}_1) \leq \hat{c}_1$ by Proposition 3, the lemma follows from Assumption 2.

A.8. Proof of Lemma 6

Using the identity

$$(F(c_2) - F(c_1))\mu(c_1, c_2) + (1 - F(c_2))\mu(c_2, b) = (1 - F(c_1))\mu(c_1, b), \quad \text{(A.2)}$$

we can rewrite the objective function of the planner (18) as

$$(1 - F(c_1))(\mu^2(c_1, b) + (\mu(c_1, b) - \mu(c_1, c_2))(\mu(c_2, b) - \mu(c_1, b)))$$

and the objective function of the monopolist (7) as

$$(1 - F(c_1))(c_1\mu(c_1, b) + (\mu(c_1, b) - \mu(c_1, c_2))(c_2 - c_1)).$$

Observe that, for any c_1, by adding a second market the monopolist can always increase its revenue. The first order condition with respect to c_2 is

$$\frac{\mu_r(c_1, c_2^*)}{\mu(c_1, b) - \mu(c_1, c_2^*)} = \frac{\mu_l(c_2^*, b)}{\mu(c_2^*, b) - \mu(c_1, b)} \quad \text{(A.3)}$$

for the planner's problem and

$$\frac{\mu_r(c_1, \hat{c}_2)}{\mu(c_1, b) - \mu(c_1, \hat{c}_2)} = \frac{1}{\hat{c}_2 - c_1} \quad \text{(A.4)}$$

for the monopolist's problem. It follows from comparing (A.3) to (A.4) that $c_2^* = \hat{c}_2$ if

$$\mu_l(c_2, b) = \frac{\mu(c_2, b) - \mu(c_1, b)}{c_2 - c_1}$$

for any c_2. This holds if $\mu(\cdot, b)$ is linear, as in the uniform type distribution.

In the proof of Proposition 4 we established that, for any c_1, there is a unique $\hat{c}_2$ satisfying the first order condition (A.4). It remains to show that there is a unique interior solution in c_2^* to the planner's problem. Using equation (A.2), we can rewrite the first order condition (A.3) as

$$f(c_2^*)(\mu(c_2^*, b) - \mu(c_1, c_2^*))(\mu(c_1, c_2^*) + \mu(c_2^*, b) - 2c_2^*) = 0.$$

For any c_1, there exists at least one c_2^* that satisfies this first order condition because $\mu(c_1, c_1) + \mu(c_1, b) \geq 2c_1$ and $\mu(c_1, b) + \mu(b, b) \leq 2b$. Such c_2^* is unique, too, since (by Assumption 2) we have $\mu_r(c_1, c_2) + \mu_l(c_2, b) \leq 3/2 < 2$ for any c_2.

A.9. Lemma A.3 and Proof

LEMMA A.3. *Under the uniform type distribution, there is no equilibrium with a dual market structure when $a/b \geq 1/2$.*

Proof. Condition (13) is necessary for an equilibrium with D_{12} and a matchmaker 1 price $\tilde{p}_1 > a^2$. This is because, for any such $\tilde{p}_1$, matchmaker 2 has the option of charging $p_2 = \theta(\tilde{p}_1)$ to overtake matchmaker 1, which leads to $c_1 > a$. Under uniform distribution, (13) becomes $c_1 < (\sqrt{19} - 4)b$; since this cannot be satisfied when $a/b \geq 1/2$, there is no equilibrium with a dual market structure in which $\tilde{p}_1 > a^2$. Furthermore, condition (6) is necessary for an equilibrium with $\tilde{p}_1 \leq a^2$ because, for any such $\tilde{p}_1$, matchmaker 2 can overtake matchmaker 1 by charging $p_2 = \theta(\tilde{p}_1)$, which leads to $c_1 = a$. This condition is violated for any $\tilde{p}_1$ if $\mu(a, b) \leq 3a/2$ or if $a/b \geq 1/2$ for the uniform distribution. □

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